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RESEARCH ARTICLE

On the mean square of the Epstein zeta-function

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Abstract. We consider the second power moment of the Epstein zeta-function and construct the asymptotic formula in special case, when $\varphi_0(u,v) = u^2 + Av^2$, $A > 0$, $A \equiv 1,2 (mod\, 4)$ and $\varphi_0(u,v)$ belongs to the one-class kind G_0 of the quadratic forms of discriminant $-4A$.

1. Introduction and statement of result

Let $\zeta(s)$ be the Riemann zeta-function. In 1926 Ingham [7] proved the relation

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt = \frac{T}{2\pi^2} \log^4 T + O(T \log^3 T)$$

In series this result was improved. In 1979 Heath-Brown [6] proved that

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt = T \sum_{j=0}^{4} a_j \log^j T + E_2(T),$$

where $E_2(T) = O(T^{7/8+\epsilon})$.

A.İvič [9] calculated the coefficients a_j, $j = 1,2,3,4$. Heath-Brown's bound for $E_2(T)$ was improved to

$$E_2(T) = O(T^{2/3} \log^c T), \ (c > 0)$$

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in [10] İvič and Motohashi.

In this paper we shall consider the second power moment of the Epstein zeta-function.

The function of divisor $d(n)$ and the function $r_\varphi(n)$ (number of representations of n by the positive quadratic form $\varphi(u,v)$) are close. Therefore we can expect that their Dirichlet series have like the mean value.

Let $\varphi(u,v)$ denotes positive definite quadratic form

$$\varphi(u,v)=au^2+2buv+cv^2,\quad a,b,c\in\mathbb{Z},(a,b,c)=1,D=ac-b^2>0.$$

For real numbers $\alpha,\beta,\gamma,\delta$ and a complex variable s, define the Epstein zeta-function for $Res>1$

$$Z_\varphi(\begin{vmatrix}\alpha&\beta\\ \gamma&\delta\end{vmatrix};s)=\sum_{\substack{(u,v)\in\mathbb{Z}^2\\(u,v)\neq(-\gamma,-\delta)}}e(\alpha u+\beta v)(\varphi(u+\gamma,v+\delta))^{-s}.$$

It is known that this function possesses an analytic continuation to the whole complex plane, with the possible exception of a simple pole with residue $\frac{\pi}{\sqrt{D}}$ at $s=1$ which occurs if and only if $(\alpha,\beta)\in\mathbb{Z}^2$ (see Epstein [5]). Moreover, one has a functional equation

$$Z_\varphi(\begin{vmatrix}\alpha&\beta\\ \gamma&\delta\end{vmatrix};s)=$$

$$=e(-\alpha\gamma-\beta\delta)\left(\frac{\pi}{\sqrt{D}}\right)^{-1+2s}\frac{\Gamma(1-s)}{\Gamma(s)}Z_\psi(\begin{vmatrix}-\gamma&-\delta\\ \alpha&\beta\end{vmatrix};1-s).\qquad(1)$$

Let $r_\varphi(\lambda)$ be the number of the representations λ in the form $\lambda=\varphi(u+\gamma,v+\delta)$, and let $r_\varphi(\lambda;\alpha,\beta)=\sum\limits_{\varphi(u+\gamma,v+\delta)=\lambda}e(\alpha u+\beta v)$.

We denote $\psi(u,v)=cu^2-2buv+av^2, A=B=\frac{\sqrt{D}}{\pi}$,

$$a_n=\sum_{\substack{u,v\in\mathbb{Z}\\ \varphi(u+\gamma,v+\delta)=\lambda_n}}e(\alpha u+\beta v),\ b_n=e(-\alpha\gamma-\beta\delta)\sum_{\substack{u,v\in\mathbb{Z}\\ \varphi(u+\alpha,v+\delta)=\mu_n}}e(-\gamma u-\delta v),$$

$0<\lambda_1<\lambda_2<\dots,\ 0<\mu_1<\mu_2<\dots.$

By (1) we have $A^s\Gamma(s)\Phi(s)=B^{1-s}\Gamma(1-s)\Psi(1-s)$, where

$$\Phi(s)=\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n^s}=Z_\varphi(\begin{vmatrix}\alpha&\beta\\ \gamma&\delta\end{vmatrix};s),$$

$$\Psi(s)=\sum_{n=1}^{\infty}\frac{b_n}{\mu_n^s}=e(-\alpha\gamma-\beta\delta)Z_\varphi(\begin{vmatrix}-\gamma&-\delta\\ \alpha&\beta\end{vmatrix};s).$$

We are now prepared to formulae our results.

Theorem 1. *Let* $0 \le Res = \sigma \le 1$, $|Ims| = |t| \ge 10$, $1 \le x, y$, $xy = \left(\frac{t\sqrt{D}}{\pi}\right)^2$. *Then the approximate functional equation*

$$Z_\varphi(\left|\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right|; s) = \sum_{\lambda_n \le x} \frac{a_n}{\lambda_n^s} + \chi_\varphi(s) \sum_{\mu_n \le y} \frac{b_n}{\mu_n^{1-s}} + R_\varphi(s, x)$$

holds, with

$$\chi_\varphi(s) = \left(\frac{\sqrt{D}}{\pi}\right)^{1-2s} \frac{\Gamma(1-s)}{\Gamma(s)};$$

$$R_\varphi(s, x) \ll |t|^{1/2} x^{-\sigma} \min(1, \frac{x}{|t|}) \log|t| \log(\frac{|t|\sqrt{D}}{x} + \frac{x}{|t|\sqrt{D}}) +$$

$$+ x^{1-\sigma}(|t|\sqrt{D})^{-1}(1 + \frac{|t|\sqrt{D}}{x}) \min(x^\epsilon + \log|t|,\ y^\epsilon + \log|t|).$$

Theorem 2. *Let* $r_\varphi(n)$ *denotes the number of the representations of* n *by form* $\varphi(u, v)$. *Then for any positive* ϵ

$$\int_0^T |Z_\varphi(\left|\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right|; \frac{1}{2} + it)|^2\, dt = 2T \sum_{n \le \frac{T\sqrt{D}}{\pi}} \frac{r_\varphi^2(n)}{n} - \frac{2\pi}{\sqrt{D}} \sum_{n \le \frac{T\sqrt{D}}{\pi}} r_\varphi^2(n) +$$

$$+ 2 \sum_{mn \le \frac{T^2 D}{\pi^2}} \frac{r_\varphi(m) r_\varphi(n)}{\sqrt{mn}} \left(\frac{m}{n}\right)^{it} \left(i \log\frac{m}{n}\right)^{-1} + O((T\sqrt{D})^{1/2+\epsilon}).$$

Theorem 3. *Let* $l, q \in \mathbb{N}$, $(l, q) = 1$. *Then*

$$\int\limits_{\substack{0 \\ Re\, s = \frac{1}{2}}}^{T} |\frac{1}{q^{2s}} \sum_{\substack{l_1, l_2 (mod\, q) \\ \varphi(l_1, l_2) \equiv l (mod\, q)}} Z_\varphi(\left|\begin{array}{cc} 0 & 0 \\ \frac{l_1}{q} & \frac{l_2}{q} \end{array}\right|; s) - \sum_{(u,v) \in \mathcal{B}} \varphi(u, v)^{-s}|^2\, dt \ll \frac{(T\sqrt{D})^{1+\epsilon}}{q^{1-\epsilon}},$$

where $\mathcal{B}$ *denotes the set of points* (u, v) *for which* $\varphi(u, v) \equiv l (mod\, q)$ *and* $0 < \varphi(u, v) < 2q$.

Theorem 4. *Let* $\varphi_0(u, v) = u^2 + Av^2$, $A > 0$, $A \equiv 1, 2 (mod\, 4)$ *and let* $\varphi_0(u, v)$ *belongs to the one-class kind* G_0 *of the quadratic forms of discriminant* $-4A$. *Then for any* $\epsilon > 0$

$$\int_0^T |Z_{\varphi_0}(\frac{1}{2} + it)|^2\, dt = E_0 T \log^2 T + E_1 T \log T + E_2 T + O(T^{7/8+\epsilon}),$$

where $E_0 > 0, E_1$ *are the computable constants which depends on* A.

We shall use the following notation. The Vinogradov symbol $X \ll Y$ means $X = O(Y)$. We use ϵ for a positive exponent which may be taken arbitrary close to zero; the constant implied by $\ll$(or O) may be depend on ϵ. $\exp(x) = e^x$, $e(x) = e^{2\pi ix}$, $e_q(x) = e(\frac{x}{q})$ for $x \in \mathbb{R}$; $(\frac{-A}{d})$ is symbol Jacoby; $\Gamma(z)$ is Gamma function.

2. Proof of theorem 1 and theorem 2

Assume first that $\sigma > 1$. We shall evaluate the integral

$$I = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{sx^{w-s}}{w(s-w)} Z_\varphi(\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}; w)dw, \quad (1 < c < \sigma)$$

in two ways.

In the above integral we replace $Z_\varphi(\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}; w)$ by the series $\sum\limits_{n=1}^{\infty} \frac{a_n}{\lambda_n^w}$. We then integrate termwise and move the line of integration to $Re\, w = -\infty$ if $\lambda_n \le x$, and to $Re\, w = +\infty$ if $\lambda_n > x$. By the theorem of residues we obtain

$$\sum_{\lambda_n \le x} \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{sx^{w-s}}{w(s-w)} \frac{a_n}{\lambda_n^w} dw = x^{-s} \sum_{\lambda_n \le x} a_n,$$

$$\sum_{\lambda_n > x} \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{sx^{w-s}}{w(s-w)} \frac{a_n}{\lambda_n^w} dw = \sum_{\lambda_n > x} \frac{a_n}{\lambda_n^s}. \tag{2}$$

Hence,

$$I = x^{-s} \sum_{\lambda_n \le x} a_n + \sum_{\lambda_n > x} \frac{a_n}{\lambda_n^s} = Z_\varphi(\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}; s) - \sum_{\lambda_n \le x} \frac{a_n}{\lambda_n^s} + x^{-s} \sum_{\lambda_n \le x} a_n. \tag{3}$$

In the second evaluation of the integral I we appeal to the analytic continuability and the functional equation of the function $Z_\varphi(\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}; s)$. We move the line of integration to $Re\, w = -b$ $(0 < b < \frac{1}{2})$, set $z = 1 - w$, and use the functional equation (1):

$$I = \frac{1}{2\pi i}\int\limits_{1+b-i\infty}^{1+b+i\infty} \frac{sx^{1-z-s}}{(1-z)(s-1+z)} Z_\varphi(\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}; 1-z)dz + R(z) =$$

$$= e(-\alpha\gamma - \beta\delta)\frac{1}{2\pi i}\int\limits_{1+b-i\infty}^{1+b+i\infty}\frac{sx^{1-z-s}}{(1-z)(s-1+z)}\frac{\Gamma(z)}{\Gamma(1-z)}\left(\frac{\pi}{\sqrt{D}}\right)^{-(-1+2z)}\times$$

$$\times Z_\psi(\left|\begin{array}{cc} -\gamma & -\delta \\ \alpha & \beta \end{array}\right|; z)dz + R(z),$$

where

$$R(z) = res_{w=0,1}\left(\frac{sx^{w-s}}{w(s-w)}Z_\varphi(\left|\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right|; w)\right).$$

The series $Z_\psi(\left|\begin{array}{cc} -\gamma & -\delta \\ \alpha & \beta \end{array}\right|; z)$ is absolutely convergent on the line $Re\, z = 1 + b$. Integration termwise we obtain

$$I = sx^{1-s}\sum_{n=1}^{\infty} b_n \frac{1}{2\pi i}\int\limits_{1+b-i\infty}^{1+b+i\infty}\frac{\pi}{\sqrt{D}}\frac{\Gamma(z)\left(\frac{\pi}{\sqrt{D}}\sqrt{\mu_n x}\right)^{-2z}}{\Gamma(1-z)(1-z)(s-1+z)}dz + R(z). \quad (4)$$

We have the Mellin pair $J_1(x)x^{-1}$ and $2^{z-2}\frac{\Gamma(\frac{1}{2}z)}{\Gamma(2-\frac{1}{2}z)}$ (here $J_1(x)$ is Bessel function).Whence for $v > 0$:

$$J_1(v)v^{-1} = \frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty}\frac{2^{z-2}\Gamma(\frac{1}{2}z)}{\Gamma(2-\frac{1}{2}z)}v^{-z}dz =$$

$$= \frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty}\frac{2^{2w-1}\Gamma(w)}{\Gamma(1-w)(1-w)}v^{-2w}dw.$$

Multiplying this by v^{1-2s} and integrating over the interval $[2\pi\sqrt{\frac{\mu_n x}{D}}, \infty)$ we arrive at the formula

$$\int\limits_{2\pi\sqrt{\frac{\mu_n x}{D}}}^{\infty} J_1(v)v^{-2s}dv =$$

$$= \frac{1}{4}\left(2\pi\sqrt{\frac{\mu_n x}{D}}\right)^{2-2s}\frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty}\frac{\Gamma(w)\left(\frac{4\pi^2\mu_n x}{D}\right)^{-w}}{\Gamma(1-w)(1-w)(s-1+w)}dw. \quad (5)$$

The path of integration we can move to $Re\, w = 1 + b$. Now from (4)-(5) we infer

$$I = sx^{1-s}\sum_{n=1}^{\infty} b_n \frac{\pi}{\sqrt{D}}\left(\frac{4\pi^2\mu_n x}{D}\right)^{s-1}\int\limits_{2\pi\sqrt{\frac{\mu_n x}{D}}}^{\infty} J_1(v)v^{-2s}dv + R(z). \quad (6)$$

Hence, by (2),(6) we obtain

$$Z_\varphi(\left|\begin{matrix}\alpha & \beta\\ \gamma & \delta\end{matrix}\right|; s) - \sum_{\lambda_n\le x}\frac{a_n}{\lambda_n^s} + x^{-s}\sum_{\lambda_n\le x} a_n =$$

$$= 4s\left(\frac{4\pi^2}{D}\right)^{s-1}\sum_{n=1}^{\infty}\frac{\pi}{\sqrt{D}}\frac{b_n}{\mu_n^{1-s}}\int\limits_{2\pi\sqrt{\frac{\mu_n x}{D}}}^{\infty} J_1(v)v^{-2s}dv + R(z). \tag{7}$$

Further,

$$res_{w=0}\left(\frac{sx^{w-s}}{w(s-w)}Z_\varphi(\left|\begin{matrix}\alpha & \beta\\ \gamma & \delta\end{matrix}\right|; w)\right) = -x^{-s}e^{-2\pi i(\alpha\gamma+\beta\delta)},$$

$$res_{w=1}\left(\frac{sx^{w-s}}{w(s-w)}Z_\varphi(\left|\begin{matrix}\alpha & \beta\\ \gamma & \delta\end{matrix}\right|; w)\right) = \epsilon(\alpha,\beta)\frac{sx^{1-s}}{s-1}\frac{\pi}{\sqrt{D}},$$

where $\epsilon(\alpha,\beta) = \begin{cases} 0 & \text{if } (\alpha,\beta)\notin\mathbb{Z}^2,\\ 1 & \text{if } (\alpha,\beta)\in\mathbb{Z}^2.\end{cases}$

Thus from (7) we obtain

$$Z_\varphi(\left|\begin{matrix}\alpha & \beta\\ \gamma & \delta\end{matrix}\right|; s) = \sum_{\lambda_n\le x}\frac{a_n}{\lambda_n^s} + \chi_\varphi(s)\sum_{\mu_n\le x}\frac{b_n}{\mu_n^{1-s}} -$$

$$-x^{-s}\left(\sum_{\lambda_n\le x} a_n - \epsilon(\alpha,\beta)\frac{\pi}{\sqrt{D}}x\right) + \chi_\varphi(s)\sum_{\mu_n\le y}\frac{b_n}{\mu_n^{1-s}}u_n +$$

$$+\sum_{\mu_n>y}\frac{sD}{\pi^2}\left(\frac{\pi^2}{D}\right)^s\int\limits_{2\pi\sqrt{\frac{\mu_n x}{D}}}^{\infty} J_1(v)v^{-2s}dv + \epsilon(\alpha,\beta)\frac{x^{1-s}}{s-1}\frac{\pi}{\sqrt{D}}, \tag{8}$$

where

$$u_n = \chi_\varphi(1-s)\frac{sD}{\pi^2}\left(\frac{\pi^2}{D}\right)^s\int\limits_{2\pi\sqrt{\frac{\mu_n x}{D}}}^{\infty} J_1(v)v^{-2s}dv - 1.$$

From (8) we have

$$Z_\varphi(\left|\begin{matrix}\alpha & \beta\\ \gamma & \delta\end{matrix}\right|; s) = \sum_{\lambda_n\le x}\frac{a_n}{\lambda_n^s} + \chi_\varphi(s)\sum_{\mu_n\le y}\frac{b_n}{\mu_n^{1-s}} + R_\varphi(s,x).$$

In order to calculate the integral

$$I_n(s) = \int\limits_{2\pi\sqrt{\frac{\mu_n x}{D}}}^{\infty} J_1(v)v^{-2s}dv$$

we can apply lemma 1 [11] or lemma III.1.2 [12]. Then after the calculation of $I_n(s)$ (by Jutila's method [11]) we have

$$R_\varphi(s,x) \ll |t|^{1/2} x^{-\sigma} \min(1, \frac{x}{|t|}) \log |t| \log(\frac{|t|\sqrt{D}}{x} + \frac{x}{|t|\sqrt{D}})+$$

$$+x^{1-\sigma}(|t|\sqrt{D})^{-1}(1+\frac{|t|\sqrt{D}}{x}) \min(x^\epsilon + \log|t|,\ y^\epsilon + \log|t|).$$

Forthemore, from (8) we have for $x = y = \frac{t\sqrt{D}}{\pi} = \tau$, $0 \le \sigma \le 1$,

$$\chi_\varphi(1-s) R_\varphi(s,\tau) = -\sqrt{2}\tau^{-\frac{1}{2}} \Delta_\varphi(\tau) + O(t^{-\frac{1}{4}} D^{\frac{1}{8}}), \tag{9}$$

where

$$\Delta_\varphi(x) = \sum_{\substack{u,v\in\mathbb{Z} \\ \varphi(u+\gamma,v+\delta)\le x}} e(\alpha u + \beta v) - \epsilon(\alpha,\beta)\frac{\pi}{\sqrt{D}} x.$$

Remark 1. The estimate of $\Delta_\varphi(x)$ can be obtained by Perron's formula for $Z_\varphi(\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}; s)$. The same reasoning as in the circle problem we easy obtain

$$\Delta_\varphi(x) = -\frac{(Dx)^{\frac{1}{4}}}{\pi} \sum_{\lambda_n \le N} \frac{a_n}{\lambda_n^{\frac{3}{4}}} \cos(2\pi\sqrt{\frac{nx}{D}} + \frac{\pi}{4}) + O(x^\epsilon + \left(\frac{x}{D}\right)^{\frac{1}{2}+\epsilon} N^{-\frac{1}{2}}).$$

Trivially we have

$$\Delta_\varphi(x) \ll x^{\frac{1}{3}+\epsilon} D^{\frac{1}{2}}.$$

Thus from (9) we obtain the estimate for $R_\varphi(s,x)$ in case $x = y = \frac{t\sqrt{D}}{\pi}$

$$R_\varphi(s,x) \ll \tau^{-\frac{1}{6}+\epsilon}.$$

However, the error term in the asymptotic formula in the approximate functional equation, which we obtain, is large for the construction of an asymptotic formula for $\int\limits_{\substack{0 \\ Re\, s=\frac{1}{2}}}^{T} |Z_\varphi(\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}; s)|^2\, dt$. Thus we build a formula for $|Z_\varphi(\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}; s)|^2$ in which the error term is sufficiently small.

We shall use by the idea of D.R. Heath-Brown [6].
Let $\alpha = \beta = \gamma = \delta = 0$. We define

$$f(w) =: \left\{ \left(\frac{\pi}{\sqrt{D}}\right)^{-2w} \Gamma(w+it)\Gamma(w-it) Z_\varphi(w+it) Z_\psi(w-it) \right\}.$$

Since

$$Z_\varphi(\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}; s) =: Z_\varphi(s) = \sum_{\substack{u,v\in\mathbb{Z}\\(u,v)\neq(0,0)}} \frac{1}{\varphi(u,v)^s} = Z_\psi(s) = \sum_{\substack{u,v\in\mathbb{Z}\\(u,v)\neq(0,0)}} \frac{1}{\psi(v,u)^s}$$

we have $f(1-w) = f(w)$, $f(\frac{1}{2}-w) = f(\frac{1}{2}+w)$. Moreover $f(w)$ is meromorphic on the complex plane, the only pole being at $w = \pm it$ and $w = 1 \pm it$. We consider the integral

$$J = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} f(\frac{1}{2}+z) e^{z^2/T} \frac{dz}{z}.$$

If we move the path of integration to $Re\, z = -1$ and set $w = -z$, then we obtain

$$J = -J + res_{z=0}\left(f(\frac{1}{2}+z)e^{z^2/T}\frac{1}{z}\right) + res_{z=\pm\frac{1}{2}\pm it}\left(f(\frac{1}{2}+z)e^{z^2/T}\frac{1}{z}\right)$$

We can show that for $\frac{1}{2}T \le t \le 5T$

$$res_{z=\pm\frac{1}{2}\pm it}\left(f(\frac{1}{2}+z)e^{z^2/T}\frac{1}{z}\right) \ll T^2 e^{-\frac{t^2}{T}-\pi t}.$$

Hence,

$$f(\frac{1}{2}) = 2J + O(T^2 e^{-\frac{t^2}{T}-\pi t}). \tag{10}$$

Now we have

Theorem 2. *Let* $\varphi(u,v) = au^2 + 2buv + cv^2$, $(a,b,c) = 1$ *and* $r_\varphi(n)$ *denote the number of the representations of* n *by form* $\varphi(u,v)$. *Then*

$$\int_0^T |Z_\varphi(\frac{1}{2}+it)|^2\, dt = 2T \sum_{n\le\frac{T\sqrt{D}}{\pi}} \frac{r_\varphi^2(n)}{n} - \frac{2\pi}{\sqrt{D}} \sum_{n\le\frac{T\sqrt{D}}{\pi}} r_\varphi^2(n) +$$

$$+2\sum_{mn\le\frac{T^2D}{\pi^2}} \frac{r_\varphi(m)r_\varphi(n)}{(mn)^{1/2}} \left(\frac{m}{n}\right)^{iT} \left(i\log\frac{m}{n}\right)^{-1} + O((T\sqrt{D})^{1/2+\epsilon}). \tag{11}$$

Proof. We have $\varphi(u,v) = \psi(-v,-u)$. Hence, $r_\varphi(n) = r_\psi(n)$, $Z_\varphi(s) = Z_\psi(s)$.

Now from (10) we obtain uniformly for $T \le t \le 2T$

$$|Z_\varphi(\frac{1}{2}+it)|^2 = \frac{\sqrt{\pi}}{|\Gamma(\frac{1}{2}+it)|^2} f(\frac{1}{2}) = 2\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{\sqrt{\pi}}{|\Gamma(\frac{1}{2}+it)|^2} \pi^{-\frac{1}{2}-z} \times$$

$$\times\Gamma(\frac{1}{2}+z+it)\Gamma(\frac{1}{2}+z-it)Z_\varphi(\frac{1}{2}+z+it)Z_\varphi(\frac{1}{2}+z-it)e^{\frac{z^2}{T}}\frac{dz}{z}+O(T^{-2})=$$

$$=2\sum_{m,n=1}^{\infty}\frac{r_\varphi(m)r_\varphi(n)}{(mn)^{1/2}}\left(\frac{m}{n}\right)^{iT}I(mn,t)+O(T^{-2}), \tag{12}$$

where

$$I(n,t)=:\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\left(\frac{\pi n}{\sqrt{D}}\right)^{-z}G(z,t)e^{\frac{z^2}{T}}\frac{dz}{z},$$

$$G(z,t)=:\frac{\Gamma(\frac{1}{2}+z+it)\Gamma(\frac{1}{2}+z-it)}{|\Gamma(\frac{1}{2}+it)|^2}.$$

Therefore, by Stirling's series for $\log\Gamma(z)$,

$$I(n,t)=\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\left(\frac{t\sqrt{D}}{\pi n}\right)^{z}e^{\frac{z^2}{T}}\frac{dz}{z}+O\left(T^{-\frac{1}{6}}e^{-\frac{T}{8}\log^2\left(\frac{t\sqrt{D}}{\pi n}\right)}\right). \tag{13}$$

Further, we have for $\left|\log\frac{t\sqrt{D}}{\pi n}\right|\gg T^{-\frac{1}{2}}\log T$

$$I(n,t)=\begin{cases}1+O(e^{-\frac{T}{8}\log^2\left(\frac{t\sqrt{D}}{\pi n}\right)}), & \text{if } n<\frac{t\sqrt{D}}{\pi}\\ O(e^{-\frac{T}{8}\log^2\left(\frac{t\sqrt{D}}{\pi n}\right)}), & \text{if } n>\frac{t\sqrt{D}}{\pi}.\end{cases} \tag{14}$$

For $\left|\log\frac{t\sqrt{D}}{\pi n}\right|\ll T^{-\frac{1}{2}}\log T$

$$I(n,t)<<\log T. \tag{15}$$

(In detail, see ([6], lemma 1)).
Now, by (12)-(15) we infer for any T_1, T_2 with $T\le T_1<T_2\le 2T$

$$\int_{T_1}^{T_2}|Z_\varphi(\frac{1}{2}+it)|^2\,dt=2\sum_{n^2\le cT^2D}\frac{r_\varphi^2(n)}{n}\int_{T_1}^{T_2}H(n^2,t)\,dt+2\sum_{\substack{mn\le cT^2D,\\ m\ne n}}\frac{r_\varphi(m)\,r_\varphi(n)}{(mn)^{1/2}}\times$$

$$\times\int_{T_1}^{T_2}H(mn,t)\left(\frac{m}{n}\right)^{iT}dt+O((T\sqrt{D})^{1/2+\epsilon}), \tag{16}$$

where

$$H(n,t)=\begin{cases}1, & \text{if } n<\frac{t\sqrt{D}}{\pi},\\ 0, & \text{if } n>\frac{t\sqrt{D}}{\pi}.\end{cases} \tag{17}$$

Therefore, from (17)

$$\int_{T_1}^{T_2} H(m^2,t)dt = \begin{cases} 2(T_2-T_1), & \text{if } m < \frac{T_1}{\pi}, \\ 2(T_2-\pi m), & \text{if } \frac{T_1}{\pi} \le m \le \frac{T_2}{\pi}, \\ 0, & \text{if } m > \frac{T_2}{\pi}. \end{cases}$$

and for $m \neq n$

$$\int_{T_1}^{T_2} H(mn,t)\left(\frac{m}{n}\right)^{iT} dt = \left(\frac{m}{n}^{it}\right)\left(i\log\frac{m}{n}\right)^{-1} H(mn,t)\Big|_{T_1}^{T_2} +$$

$$+O((T\sqrt{D})^{1/2+\epsilon}).$$

Now we can obtain the following correlation by taking $T_1 = T_0$, $T_2 = 2T_0$, $T_0 = \frac{T}{2^n}$ and summing for $2 \le 2^n \le T$:

$$\int_0^T |Z_\varphi(\frac{1}{2}+it)|^2\, dt = 2T \sum_{n\le\frac{T\sqrt{D}}{\pi}} \frac{r_\varphi^2(n)}{n} - \frac{2\pi}{\sqrt{D}} \sum_{n\le\frac{T\sqrt{D}}{\pi}} r_\varphi^2(n) +$$

$$+2 \sum_{\substack{mn\le\frac{T^2D}{\pi^2}, \\ m\neq n}} \frac{r_\varphi(m)\, r_\varphi(n)}{(mn)^{1/2}} \left(\frac{m}{n}\right)^{iT} \left(i\log\frac{m}{n}\right)^{-1} + O_\epsilon((T\sqrt{D})^{1/2+\epsilon}).$$

□

Remark 2. Since $r_\varphi(n) << d(n)$, we can obtain instead the third sum such estimate

$$T\sqrt{D}\log^3(TD).$$

To this end it suffices to use lemma 4 [3]. Bellow we will obtain more precise result.

3. Proof of theorem 3

In order to prove theorem 3 we shall need several auxiliary assertions.

Lemma 1. *Let the Dirichlet series*

$$\Phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \quad \Psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s}, \quad s = \sigma + it,$$

be absolutely convergent for $Re\, s > 1$, *and assumed that* $\Phi(s)$, $\Psi(s)$ *can be continued analytically over whole s- plane (except at the finite number singular points), moreover the functional equation*

$$A^s\Gamma(ms+v)\Phi(s) = B^{1-s}\Gamma(m(1-s)+v)\Psi(1-s),$$

(A, B *are constants) holds.*
Then, for every $\tau \in \mathbb{C}$, $arg\,\tau = \left(\frac{\pi}{2} - \frac{1}{t}\right) sign\, t$, *and for any fixed strip* $a \le \sigma \le b$ *uniformly for* $|t| \ge t_0$, A, B, τ, *the approximate functional equation*

$$\Phi(s) = \sum a_n\lambda_n^{-s}F(s, \frac{\lambda_n\tau^m}{A}) + \sum_{z\ne s} res\left\{\left(\frac{A}{\tau^m}\right)^{z-s}\frac{\Gamma(mz+v)\Phi(z)}{z-s}\right\}$$

$$+\frac{B^{1-s}\Gamma(m(1-s)+v)}{A^s\Gamma(ms+v)}\sum_{\mu_n\le y\log y} b_n\mu_n^{s-1}F(1-s, \frac{\mu_n\tau^{-m}}{B}) + O(x^{-M}+y^{-M})$$

holds, where $M > 0$ *is any fixed constant,*

$$F(w,X) = \frac{1}{\Gamma(mw+v)}\frac{1}{2\pi i}\int_{(\Delta)}\Gamma(m(w+z)+v)\frac{X^s}{z}dz,$$

Δ *is such that in region* $Re\, s \ge \Delta$ *there are no singularities of the integrating.*
Moreover, we have uniformly for all parameters:

$$F(w,X) = l+$$

$$+O\left(exp\left(-\frac{|X|^{\frac{1}{m}}}{|t|}\right)\left(\frac{|X|}{|t|^m}\right)^{Re\,w+\frac{1}{m}Re\,v}\right)\left(1+\left|m\sqrt{|t|}-\frac{|X^{\frac{1}{m}}|}{\sqrt{|t|}}\right|^{-1}\right),$$

where

$$l = \begin{cases} 1, & if\ \lambda_n \le x, \mu_n \le y, \\ 0, & else, \end{cases}$$

$x = m^m|\tau|^{-1}A|t|^m$, $y = m^m|\tau|B|t|^m$.
This lemma is a special case of Lavrik's theorem ([13]).

Corollary 1. *Let* $\Phi(s) = \sum\limits_{n=1}^{\infty} a_n n^{-s}$, $\quad \Psi(s) = \sum\limits_{n=1}^{\infty} b_n n^{-s}$, *where*

$$a_n = \begin{cases} r_\varphi(n), & if\ n \equiv l(mod\, q), \\ 0, & else, \end{cases} \quad b_n = \frac{1}{q}\sum_{\substack{(u,v)\in\mathbb{Z}^2,\\ \psi(u,v)=n}}\ \sum_{\substack{l_1,l_2\ (mod\, q),\\ \varphi(l_1,l_2)\equiv l(mod\, q)}} e_q(l_1u+l_2v). \tag{18}$$

Then for $s = \frac{1}{2} + it$, $|t| \geq t_0$, m=1, v=0, $A = B = \frac{\sqrt{D}}{\pi}q$, $x = A|t\tau^{-1}|$, $y = B|t\tau|$, $arg\,\tau = arg\,s$, $|\tau| = 1$, *we have*

$$\Phi(s) = \sum_{\substack{n \leq \frac{|s|q^2\sqrt{D}}{\pi}, \\ n \equiv l (mod\, q)}} \frac{r_\varphi(n)}{n^{\frac{1}{2}+it}} + \left(\frac{\pi^2}{D}\right)^{it} \frac{\Gamma(\frac{1}{2} - it)}{\Gamma(\frac{1}{2} + it} \sum_{n \leq \frac{|s|\sqrt{D}}{\pi}} \frac{b_n}{n^{\frac{1}{2}-it}} +$$

$$+O(q^{-1}\log(Mq|t|)) + O((\sqrt{D}|t|)^{-M}), \tag{19}$$

(O- constants can depends on only M, t_0 *).*

The proof of this statement carry out in lemma 5 [15].

Lemma 2. *Let* $l, q \in \mathbb{N}$, $1 \leq l \leq q$. *Then for* $(l, q) = 1$

$$\sum_{\substack{l_1, l_2\, (mod\, q), \\ \varphi(l_1, l_2) \equiv l (mod\, q)}} e_q(l_1 u + l_2 v) \ll q^{\frac{1}{2}}(u, v, q)^{\frac{1}{2}} d(q),$$

(here $d(q)$ *is the number of divisors of n).*

This statement is the well-known Weil's estimate [16] of a trigonometric sum along a curve over a finite field.

Lemma 3. *Let* $\mathcal{B}$ *denotes the set of points* (u, v) *for which* $\varphi(u, v) \equiv l(mod\, q)$ *and* $0 < \varphi(u, v) < 2q$. *Then for* $0 < \epsilon < 1/2$, $T > 1$, *in a rectangle*
$-\epsilon \leq Re\, s \leq 1 + \epsilon$, $1 \leq |Im\, s| \leq T$,

$$|\frac{1}{q^{2s}} \sum_{\substack{l_1, l_2 (mod\, q) \\ \varphi(l_1, l_2) \equiv l (mod\, q)}} Z_\varphi(\left| \begin{matrix} 0 & 0 \\ \frac{l_1}{q} & \frac{l_2}{q} \end{matrix} \right|; s) - \sum_{(u,v) \in \mathcal{B}} \varphi(u, v)^{-s}| =$$

$$= O\left(\left(|t|\sqrt{D}\right)^{\frac{2(1+\epsilon)(1+\epsilon-\sigma)}{1+2\epsilon}} \epsilon^{-2} q^{\frac{\frac{1}{2} - \frac{3}{2}\sigma - \frac{\epsilon}{2}}{1+2\epsilon}}\right),$$

(The O- constant does not depend on t, σ, ϵ, T*).*

This statement is a corollary of lemma 2 and Phragmen-Lindelöf's theorem.

Now we come to the proof of the theorem 3.
If we put $T_0 = \max(t_0, q^\epsilon)$ with t_0 from corollary 1 of lemma 1, then

$$\int\limits_{\substack{0 \\ Re\, s = \frac{1}{2}}}^{T} |\frac{1}{q^{2s}} \sum_{\substack{l_1, l_2 (mod\, q) \\ \varphi(l_1, l_2) \equiv l (mod\, q)}} Z_\varphi(\left| \begin{matrix} 0 & 0 \\ \frac{l_1}{q} & \frac{l_2}{q} \end{matrix} \right|; s) - \sum_{(u,v) \in \mathcal{B}} \varphi(u, v)^{-s}|^2\, dt =$$

$$= \int_{0}^{T_0} + \int_{T_0}^{T} = I_1 + I_2,$$

say.
By lemma 3 it is easily to see that

$$I_1 \ll q^{-1+2\epsilon}\epsilon^{-2}. \tag{20}$$

In order calculate I_2 we applay the corollary 1 from lemma 1, and then obtain

$$I_2 \ll \int_{T_0}^{T} |\sum_{2q\le n\le U} r_\varphi(n) n^{-\frac{1}{2}-it}|^2 dt + \int_{T_0}^{T} |\sum_{n\le V} b_n n^{-\frac{1}{2}+it}|^2 dt +$$

$$+\sqrt{D}Tq^{-1}\log^2(MTq) + (\sqrt{D}T_0)^{-M+1}, \tag{21}$$

(here $U = V = \frac{1}{\pi}|s|\sqrt{D}$.)

The integrals on the right-hand side of (21) can be estimated by the general scheme of the estimation of the mean values of the Dirichlet series (see, for example, [14], Chapt. 6 and 7). Hence we get

$$I_2 \ll (T + N_0) \sum_{2q<n\le U_0} \frac{a_n^2}{n} + (T + V_0) \sum_{n\le V_0} \frac{b_n^2}{n},$$

where $N_0 = \sum_{\substack{2q<n\le cqT\sqrt{D} \\ a_n\neq 0}} 1 \ll T\sqrt{D}$; $U_0 \ll T\sqrt{D}$,$V_0 \ll cT\sqrt{D}$.
Since $r_\varphi(n) \ll d(n)$ we get (using the notations (18)):

$$I_2 \ll \frac{T\sqrt{D}}{q}((TDq)^{2\epsilon} + \log^2(TMq) + (\sqrt{D}T_0)^{-M+1}). \tag{22}$$

The assertion of the theorem follows from (20) and (22) if we put $M = -1 + \frac{1}{\epsilon}$.

4. Proof of Theorem 4

Consider a quadratic form $\varphi_0(u, v) = u^2 + Av^2$, $A \in \mathbb{N}$. Well-known (see, for example, [4]) that there is finite number of the negative discriminants of the quadratic form for which a kind consists out of one class. Let A is such number.

Lemma 4. *Let a kind of the quadratic form* $\varphi_0(u,v) = u^2 + Av^2$, $A > 0$, $A \equiv 1, 2(mod\,4)$, *consists out of one class and let*

$$r_{\varphi_0}(n) = \sum_{\substack{u,v\in\mathbb{Z},\\ \varphi_0(u,v)=n}} 1.$$

Then $\frac{1}{2}r_{\varphi_0}(n)$ *is a multiplicative function if* $A > 1$, *and* $\frac{1}{4}r_{\varphi_0}(n)$ *is a multiplicative function if A=1.*

Proof. Let for some $n \in \mathbb{N}$ we have $n = u_0^2 + Av_0^2$, and let $\varphi_j(u,v)$ be a primitive quadratic form of discriminant $-4A$ also represent of n,$\varphi_j(u_1,v_1) = n$. We shall show that φ_j is equivalent to φ_0 ($\varphi_j \sim \varphi_0$). Indeed , we take into account the connection between the classes of divisors of field $\mathbb{Q}(\sqrt{-\mathbb{A}})$ and the classes of quadratic forms of a discriminant $-4A$ (in a case $A \equiv 1, 2(mod\,4)$). Let a quadratic form $\varphi_j(u,v)$ represent of n(i.e. $n = \varphi_j(u_1,v_1)$), then in a appropriate class of divisors has a divisor $\Re_j$ for which $N(\Re_j) = n$ (norma of $\Re_j$). The quadratic form φ_0 belongs to main kind G_0. Hence the divisor $\Re_0$ belongs to main kind G_0 of divisors, and then by theorem 6 (Ch. III,§ 8) the divisor $\Re_j$ also belongs to G_0. But the kind G_0 consists only one class. Therefore $\Re_0$ and $\Re_1$ belongs the same class and hence $\varphi_0 \sim \varphi_j$. Further, if $A = 1$ we have $\frac{1}{4}r_{\varphi_0}(n) = \sum\limits_{\substack{d|n,\\ d\,is\,odd}} (-1)^{\frac{d-1}{2}}$, and hence $\frac{1}{4}r_{\varphi_0}(n)$ is a multiplicative function.

Let $A > 1$. Then the field $\mathbb{Q}(\sqrt{-A})$ contains only two the roots of 1. We assume that the form φ_0 represent each of numbers n_1 and n_2, $(n_1,n_2) = 1$. Let $\Re_1,\ldots,\Re_{h_1}$ and $\Im_1\ldots\Im_{h_2}$ are all different divisors each of which has a norma n_1 or n_2 respectively. Then the divisors $\Re_i$, $\Im_j$ belongs to the kind G_0. But the product n_1n_2 also can be represented by φ_0. Hence $\Re_i\Im_j \in G_0$, $i = 1,\ldots,h_1$, $j = 1,\ldots,h_2$ (here $h_1 = \frac{1}{2}\,r_{\varphi_0}(n_1)$, $h_2 = \frac{1}{2}\,r_{\varphi_0}(n_2)$). Since $\Re_i\Im_j$ are all different divisors we have $\frac{1}{2}\,r_{\varphi_0}(n_1n_2) \geq \frac{1}{2}\,r_{\varphi_0}(n_1)\frac{1}{2}\,r_{\varphi_0}(n_2)$. On the other hand, any integer divisor $\mathcal{C}$, $N(\mathcal{C}) = n_1n_2$, can be represented in the form of a product of coprime divisors $\Re_i$, $\Im_j$. Hence

$$\frac{1}{2}\,r_{\varphi_0}(n_1n_2) \leq \frac{1}{2}\,r_{\varphi_0}(n_1)\frac{1}{2}\,r_{\varphi_0}(n_2).$$

Therefore

$$\frac{1}{2}\,r_{\varphi_0}(n_1n_2) = \frac{1}{2}\,r_{\varphi_0}(n_1)\frac{1}{2}\,r_{\varphi_0}(n_2).$$

□

Remark 3. Let $\varphi_0(u,v) = u^2 + Av^2$ belongs to the one-class kind G_0, and let p be prime number. For any $k \in \mathbb{N}$

$$r_{\varphi_0}(p^k) = \begin{cases} 2(k+1), & \text{if } \left(\frac{-A}{p}\right) = 1; \\ 1 + (-1)^k, & \text{if } \left(\frac{-A}{p}\right) \neq 1; \\ 2\,, & \text{if } p\,|A. \end{cases}$$

Lemma 5. *Let $\varphi_0(u,v) = u^2 + Av^2$ belongs to the one-class kind G_0. Then*

$$\sum_{n\le x} r^2_{\varphi_0}(n) = c_0 x \log x + c_1 x + O(x^{1/2+\epsilon})$$

with constants, which can depend from A.

Proof. For $Re\, s > 1$ we have

$$\frac{1}{4}\sum_{n=1}^{\infty} \frac{r^2_{\varphi_0}(n)}{n^s} = \prod_{\substack{p,\\ \chi(p)=1}} (1 + \frac{4}{p^s} + O\left(\frac{1}{|p^{2s}|}\right)) \prod_{p\,|D} (1 + \frac{1}{p^s} + O\left(\frac{1}{|p^{2s}|}\right)) \times$$

$$\times g_0(s) = \prod_{\substack{p,\\ \chi(p)=1}} (1+\frac{1}{p^s})^4 \prod_{p\,|D}(1+\frac{1}{p^s}) g_1(s) = \zeta^2(s) \prod_{p\,|D}(1+\frac{1}{p^s})^{-1} g_2(s),$$

where $g_0(s)$, $g_1(s)$, $g_2(s)$ are the regular functions for $Re\, s > \frac{1}{2}$. Now by the Perron's formula we easily get our assertion. □

Lemma 6. *Let $l, q \in \mathbb{N}$, $(l,q) = 1$. Then in the conditions of Lemma we have for any $\epsilon > 0$*

$$\sum_{\substack{n\equiv l (mod\, q),\\ n\le x}} r_{\varphi_0}(n) = \frac{\pi x}{\sqrt{D}} \frac{1}{q^2} J_q(l,A) + O\left(\frac{x^{\frac{1}{2}+\epsilon}}{q^{\frac{1}{4}}}\right),$$

where $J_q(l,A) = \sum\limits_{\substack{l_1, l_2 (mod\, q),\\ l_1 + Al_2 \equiv l (mod\, q)}} 1.$

Proof. For $Re\, s > 1$

$$\sum_{\substack{n=1,\\ n\equiv l (mod\, q)}}^{\infty} \frac{r_{\varphi_0}(n)}{n^s} = \sum_{\substack{l_1, l_2 (mod\, q)\\ l_1^2 + Al_2^2 \equiv l (mod\, q)}} \frac{1}{q^{2s}} Z_{\varphi}(\left| \begin{matrix} 0 & 0 \\ \frac{l_1}{q} & \frac{l_2}{q} \end{matrix} \right|; s).$$

Hence, for $c > 1$, $T > 1$

$$\sum_{\substack{n\equiv l (mod\, q),\\ n\le x}} r_{\varphi_0}(n) =$$

$$= \frac{1}{2\pi i} \int\limits_{c-iT}^{c+iT} \Big(\sum_{\substack{l_1, l_2 (mod\, q) \\ l_1^2 + A l_2^2 \equiv l (mod\, q)}} \frac{1}{q^{2s}} Z_{\varphi_0}\Big(\begin{vmatrix} 0 & 0 \\ \frac{l_1}{q} & \frac{l_2}{q} \end{vmatrix} ; s\Big) - \sum_{(u,v)\in\mathcal{B}} \varphi_0(u,v)^{-s} \Big) \frac{x^s}{s}\, ds +$$

$$+ O\left(\frac{x^c}{Tq(c-1)} \right) + O(x^\epsilon).$$

After shifting the contour of integration to the line $Re\, s = -\epsilon$, applying the functional equation for $Z_{\varphi_0}\Big(\begin{vmatrix} 0 & 0 \\ \frac{l_1}{q} & \frac{l_2}{q} \end{vmatrix} ; s\Big)$ and lemma 3 we obtain

$$\sum_{\substack{n \equiv l (mod\, q), \\ n \le x}} r_{\varphi_0}(n) = \frac{\pi x}{\sqrt{D}} \frac{1}{q^2} \sum_{\substack{l_1, l_2 (mod\, q), \\ l_1 + A l_2 \equiv l (mod\, q)}} 1 + \sum_{(u,v)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{\varphi_0(u,v)^{1+\epsilon}} \times$$

$$\times \sum_{\substack{l_1, l_2 (mod\, q), \\ l_1 + A l_2 \equiv l (mod\, q)}} e^{-2\pi i (\frac{l_1 v + l_2 u}{q})} \cdot \frac{1}{2\pi i} \int\limits_{-\epsilon - iT}^{-\epsilon + iT} \frac{\Gamma(1-s)}{\Gamma(s)} \left(\frac{\pi}{\sqrt{D}} \right)^{-1+2s} \frac{x^s}{s}\, ds +$$

$$+ O\left(\frac{x^c}{Tq(c-1)} \right) + O(x^\epsilon) + O(T^\epsilon q^{\frac{1}{2}+\epsilon}). \qquad (23)$$

Now trivially estimating the integral and applying lemma 2 we get the assertion of lemma if set $T = \frac{x^{\frac{1}{2}}}{q^{\frac{3}{4}}}$. □

Remark 4. A non-trivial estimate the integral in (23) give an estimate of the error term as

$$\ll x^{\frac{1}{3}+\epsilon}.$$

Corollary 2. *Uniformly for* $1 \le h \le x^{\frac{5}{6}-\epsilon}$ *there exist constant* $c_0(h)$ *such that*

$$\sum_{n \le x} r_{\varphi_0}(n)\, r_{\varphi_0}(n+h) = c_0(h) x + O(x^{\frac{5}{6}+\epsilon}),$$

where ϵ *is an arbitrarily small, positive constant. Besides,* $c_0(h) \ll d(h)$.

This statement can be proved similarly the proof of the analogies assertion in [1], [8].

The proof of theorem 4 follows by Heath-Brown's method [2] from theorem 2 with using lemma 5 and corollary from lemma 6.

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