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Miniversal deformations of chains of linear mappings

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Abstract. V.I. Arnold [Russian Math. Surveys, 26 (no. 2), 1971, pp. 29–43] gave a miniversal deformation of matrices of linear operators; that is, a simple canonical form, to which not only a given square matrix A, but also the family of all matrices close to A, can be reduced by similarity transformations smoothly depending on the entries of matrices. We study miniversal deformations of quiver representations and obtain a miniversal deformation of matrices of chains of linear mappings

$$V_1 \text{ —— } V_2 \text{ —— } \cdots \text{ —— } V_t\,,$$

where all V_i are complex or real vector spaces and each line denotes $\longrightarrow$ or $\longleftarrow$.

Introduction

All matrices B that are close to a given square complex matrix A reduce by similarity transformations to their Jordan canonical forms, but these forms and transformations may be discontinuous relative to the entries of B. Arnold [1] (see also [2, § 30]) constructed a normal form, to which not only the matrix A, but all matrices close to it, can be reduced by smooth

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similarity transformations. He called this normal form a *miniversal deformation* of A. A miniversal deformation of real matrices for similarity was given by Galin [5]. A miniversal deformation of pairs of m-by-n matrices with respect to simultaneous equivalence (that is, of matrix pencils) was obtained in the paper [4], which was awarded by the SIAG/LA (SIAM Activity Group on Linear Algebra) Prize in Applied Linear Algebra for the years 1997–2000. The miniversal deformations from [4] and [5] were simplified in [6]. These results are important for applications in which one has matrices that arise from physical measurements, which means that their entries are known only approximately.

The notion of a miniversal deformation was extended to quiver representations in [6]. Recall that a *quiver* is a directed graph, its *representation* $\mathcal{A}$ over a field $\mathbb{F}$ is given by assigning to each vertex p a finite dimensional vector space $\mathcal{A}_p$ over $\mathbb{F}$ and to each arrow $\alpha\colon p \to q$ a linear mapping $\mathcal{A}_\alpha\colon \mathcal{A}_p \to \mathcal{A}_q$. Studying the family of quiver representations whose matrices are close to the matrices of a given representation $\mathcal{A}$, we can independently reduce the matrices of each representation to Belitskii's canonical form [9] losing the smoothness relative to the entries of these matrices. This leads to the problem of constructing a simple normal form to which all representations close to $\mathcal{A}$ can be reduced by smooth changes of bases; that is, to the problem of constructing a miniversal deformation of $\mathcal{A}$.

In Section 1 we recall a theorem from [6] that admits to construct miniversal deformations of quiver representations. In Section 3 we give a direct and constructive proof of this theorem (it was deduced in [6] from some result about miniversal deformations formulated in [3]). In Section 1 we also prove that a miniversal deformation of each quiver representation is easily constructed from miniversal deformations of direct sums of two indecomposable representations. In Section 2 we obtain a miniversal deformation of each quiver $1 - 2 - \cdots - t$ with an arbitrary orientation of its arrows; that is, a miniversal deformation of matrices of chains of linear mappings

$$V_1 \;\text{——}\; V_2 \;\text{——}\; \cdots \;\text{——}\; V_t\,,$$

where all V_i are complex or real vector spaces and each line denotes $\longrightarrow$ or $\longleftarrow$.

1. Miniversal deformations of quiver representations

In this section we consider representations of a quiver Q with vertices $1,\dots,t$. Let $\mathcal{A}$ be any representation of Q over a field $\mathbb{F}$. Choosing bases in the spaces $\mathcal{A}_1,\dots,\mathcal{A}_t$ we may give $\mathcal{A}$ by the matrices of its linear mappings $\mathcal{A}_\alpha\colon \mathcal{A}_p \to \mathcal{A}_q$. This leads to the following definitions. By a *matrix representation* of dimension $\vec{n} = (n_1, \dots, n_t) \in \{0, 1, 2, \dots\}^t$ of Q over $\mathbb{F}$ we mean any set A of matrices $A_\alpha \in \mathbb{F}^{n_q \times n_p}$ assigned to all arrows $\alpha\colon p \to q$. Two matrix representations A and B of dimension $\vec{n}$ are *isomorphic* if there is a sequence $S = (S_1, \dots, S_t)$ of nonsingular $n_1 \times n_1, \dots, n_t \times n_t$ matrices such that

$$B_\alpha = S_q A_\alpha S_p^{-1} \qquad \text{for each arrow } \alpha\colon p \to q.$$

In this case we say that S is an *isomorphism* of the representations A and B and write $S\colon A \xrightarrow{\sim} B$. Clearly, all isomorphic matrix representations define the same (operator) representation with respect to different bases of its spaces. Denote by $\mathcal{R}(\vec{n}, \mathbb{F})$ the vector space of all matrix representations of dimension $\vec{n}$ over $\mathbb{F}$.

From this point on, $\mathbb{F}$ is a field of complex or real numbers, and we consider only matrix representations omitting usually the word "matrix" for abbreviation. A *deformation* of $A \in \mathcal{R}(\vec{n}, \mathbb{F})$ is a matrix representation $\mathcal{A}(\vec{\lambda})$, $\vec{\lambda} = (\lambda_1, \dots, \lambda_k)$, such that the entries of its matrices are convergent in a neighborhood of $\vec{0}$ power series of variables (they are called *parameters*) $\lambda_1, \dots, \lambda_k$ over $\mathbb{F}$ and $\mathcal{A}(\vec{0}) = A$. Two deformations $\mathcal{A}(\vec{\lambda})$ and $\mathcal{B}(\vec{\lambda})$ of $A \in \mathcal{R}(\vec{n}, \mathbb{F})$ are called *equivalent* if the identity isomorphism

$$I_{\vec{n}} = (I_{n_1}, \dots, I_{n_t})\colon A \xrightarrow{\sim} A$$

possesses a deformation $\mathcal{I}(\vec{\lambda})$ (its matrices are convergent in a neighborhood of $\vec{0}$ matrix power series and $\mathcal{I}(\vec{0}) = I_{\vec{n}}$) such that

$$\mathcal{B}_\alpha(\vec{\lambda}) = \mathcal{I}_q(\vec{\lambda})\mathcal{A}_\alpha(\vec{\lambda})\mathcal{I}_p(\vec{\lambda})^{-1} \qquad \text{for each arrow } \alpha\colon p \to q$$

in a neighborhood of $\vec{0}$.

Definition 1. *A deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ of a representation A is called* versal *if every deformation $\mathcal{B}(\mu_1, \dots, \mu_l)$ of A is equivalent to a deformation of the form $\mathcal{A}(\varphi_1(\vec{\mu}), \dots, \varphi_k(\vec{\mu}))$, where $\varphi_i(\vec{\mu})$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_i(\vec{0}) = 0$. A versal deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ of A is called* miniversal *if there is no versal deformation having less than k parameters.*

A miniversal deformation of any representation A of dimension $\vec{n}$ can be constructed as follows. The triples consisting of all arrows $\alpha\colon p \to q$ of Q and indices of the n_q-by-n_p matrices A_α of A form the set

$$\Upsilon_{\vec{n}} := \{(\alpha, i, j) \mid \alpha\colon p \to q, \quad i = 1, \dots, n_q, \quad j = 1, \dots, n_p\}. \qquad (1)$$

For each $(\alpha, i, j) \in \Upsilon_{\vec{n}}$, define the *elementary representation* $E_{\alpha ij}$ whose matrices are zero except for the matrix assigned to α; the $(i,j)^{\text{th}}$ entry of this matrix is 1 and the others are 0.

For each subset $\Gamma \subset \Upsilon_{\vec{n}}$, define the deformation

$$\mathcal{U}_{\Gamma}(\vec{\varepsilon}) := A + \sum_{(\alpha,i,j)\in\Gamma} \varepsilon_{\alpha ij} E_{\alpha ij} \tag{2}$$

of A, in which all $\varepsilon_{\alpha ij}$ are independent parameters. The deformation

$$\mathcal{U}(\vec{\varepsilon}) := \mathcal{U}_{\Upsilon_{\vec{n}}}(\vec{\varepsilon}) \tag{3}$$

is universal in the sense that each deformation $\mathcal{B}(\mu_1, \dots, \mu_l)$ of A has the form $\mathcal{U}(\vec{\varphi}(\mu_1, \dots, \mu_l))$, where $\varphi_{\alpha ij}(\mu_1, \dots, \mu_l)$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_{\alpha ij}(\vec{0}) = 0$. Hence the deformation $\mathcal{B}(\mu_1, \dots, \mu_l)$ in Definition 1 can be replaced by $\mathcal{U}(\vec{\varepsilon})$, which proves the following lemma.

Lemma 2. *The following two conditions are equivalent for any deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ of a representation A:*

(i) *The deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ is versal.*

(ii) *The deformation $\mathcal{U}(\vec{\varepsilon})$ defined in (3) is equivalent to a deformation of the form $\mathcal{A}(\varphi_1(\vec{\varepsilon}), \dots, \varphi_k(\vec{\varepsilon}))$, where $\varphi_i(\vec{\varepsilon})$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_i(\vec{0}) = 0$.*

For a representation A of dimension $\vec{n}$ and each sequence $C_1, \dots, C_t$ of $n_1 \times n_1, \dots, n_t \times n_t$ matrices, we define the representation $[C, A]$ of the same dimension as follows:

$$[C, A]_\alpha = C_q A_\alpha - A_\alpha C_p \qquad \text{for each arrow } \alpha\colon p \to q. \tag{4}$$

Denote by $[\mathbb{F}^{\vec{n}\times\vec{n}}, A]$ the set of such representations.

Due to the next theorem, each representation $A \in \mathcal{R}(\vec{n}, \mathbb{F})$ possesses a miniversal deformation of the form (2), which was called in [6] a *simplest miniversal deformation* of A.

Theorem 3 ([6, Theorem 2.1]). *Let A be a matrix representation of dimension $\vec{n}$ of a quiver Q with vertices $1, \dots, t$ over a field $\mathbb{F}$ of complex or real numbers. For each subset Γ of the set* (1)*, the deformation $\mathcal{U}_\Gamma(\vec{\varepsilon})$ defined in* (2) *is miniversal if and only if the vector space $\mathcal{R}(\vec{n}, \mathbb{F})$ of all representations of dimension $\vec{n}$ decomposes into the direct sum*

$$\mathcal{R}(\vec{n}, \mathbb{F}) = [\mathbb{F}^{\vec{n}\times\vec{n}}, A] \oplus \mathcal{E}_\Gamma, \tag{5}$$

in which $\mathcal{E}_\Gamma$ denotes the subspace spanned by all elementary representations $E_{\alpha ij}$ with $(\alpha, i, j) \in \Gamma$.

In Section 3 we give a direct proof of Theorem 3. A simplest miniversal deformation of $A \in \mathcal{R}(\vec{n},\mathbb{F})$ can be constructed as follows. Let $T_1,\dots,T_r$ be a basis of the space $[\mathbb{F}^{\vec{n}\times\vec{n}},A]$, and let $E_1,\dots,E_l$ be the basis of $\mathcal{R}(\vec{n},\mathbb{F})$ consisting of all elementary representations $E_{\alpha ij}$. Removing from the sequence $T_1,\dots,T_r,E_1,\dots,E_l$ every representation that is a linear combination of the preceding representations, we obtain a new basis $T_1,\dots,T_r,E_{i_1},\dots,E_{i_k}$ of the space $\mathcal{R}(\vec{n},\mathbb{F})$. By Theorem 3, the deformation

$$\mathcal{A}(\varepsilon_1,\dots,\varepsilon_k) = A + \varepsilon_1 E_{i_1} + \dots + \varepsilon_k E_{i_k}$$

is miniversal.

A *direct sum* of two matrix representations A and B is the representation

$$C = A \oplus B, \qquad C_\alpha := A_\alpha \oplus B_\alpha \text{ for all arrows } \alpha.$$

A representation is called *indecomposable* if it is not isomorphic to a direct sum of representations of smaller sizes. It is known that each matrix representation A is isomorphic to a direct sum of indecomposable representations

$$A_1 \oplus A_2 \oplus \dots \oplus A_s$$

determined by A uniquely up to permutation of summands and replacement them by isomorphic representations.

Theorem 4. *Let $A = A_1 \oplus \dots \oplus A_s$ be a matrix representation, and let*

$$\begin{aligned}\mathcal{A}(\vec{\varepsilon}) &= A + B(\vec{\varepsilon})\\ &= \begin{bmatrix} A_1 + B_{11}(\vec{\varepsilon}) & B_{12}(\vec{\varepsilon}) & \dots & B_{1s}(\vec{\varepsilon})\\ B_{21}(\vec{\varepsilon}) & A_2 + B_{22}(\vec{\varepsilon}) & \dots & B_{2s}(\vec{\varepsilon})\\ \vdots & \vdots & \ddots & \vdots\\ B_{s1}(\vec{\varepsilon}) & B_{s2}(\vec{\varepsilon}) & \dots & A_s + B_{ss}(\vec{\varepsilon}) \end{bmatrix}\end{aligned} \tag{6}$$

be its deformation of the form (2)*, whose matrices are partitioned into blocks conformably to the partition of A. Then $\mathcal{A}(\vec{\varepsilon})$ is a simplest miniversal deformation of A if and only if each*

$$\mathcal{A}_{pq}(\vec{\varepsilon}) := \begin{bmatrix} A_p + B_{pp}(\vec{\varepsilon}) & B_{pq}\\ B_{qp}(\vec{\varepsilon}) & A_q + B_{qq}(\vec{\varepsilon}) \end{bmatrix}, \qquad p<q, \tag{7}$$

is a simplest miniversal deformation of $A_p \oplus A_q$.

Proof. For $p,q \in \{1,\dots,s\}$ and for each representation M of dimension $\vec{n}$, whose matrices are partitioned into blocks conformably to the partition of (6), denote by $M^{(p,q)}$ the representation obtained from M as follows: in each of its matrices one replaces by 0 all blocks except for the $(p,q)^{\text{th}}$ block. If $\mathcal{V}$ is a subspace of $\mathcal{R}(\vec{n},\mathbb{F})$, then

$$\mathcal{V}^{(p,q)} := \{M^{(p,q)} \mid M \in \mathcal{V}\}$$

is also a subspace.

Let the deformation (6) be miniversal. Since it has the form (2), the decomposition (5) holds, and so each $M \in \mathcal{R}(\vec{n},\mathbb{F})^{(p,q)}$ is uniquely represented in the form

$$M = P + Q, \qquad P \in [\mathbb{F}^{\vec{n}\times\vec{n}}, A], \quad Q \in \mathcal{E}_{\Gamma}.$$

Then $M = M^{(p,q)} = P^{(p,q)} + Q^{(p,q)}$ and due to the obvious inclusions

$$[\mathbb{F}^{\vec{n}\times\vec{n}}, A]^{(p,q)} \subset [\mathbb{F}^{\vec{n}\times\vec{n}}, A], \qquad \mathcal{E}_{\Gamma}^{(p,q)} \subset \mathcal{E}_{\Gamma}$$

we have

$$P \in [\mathbb{F}^{\vec{n}\times\vec{n}}, A]^{(p,q)}, \qquad Q \in \mathcal{E}_{\Gamma}^{(p,q)},$$

and so

$$\mathcal{R}(\vec{n},\mathbb{F})^{(p,q)} = [\mathbb{F}^{\vec{n}\times\vec{n}}, A]^{(p,q)} \oplus \mathcal{E}_{\Gamma}^{(p,q)} \tag{8}$$

for all $p, q \in \{1, \dots, s\}$. Due to Theorem 3, the deformations (7) are miniversal for all $p < q$.

Conversely, if the deformations (7) are miniversal for all $p < q$, then applying the same reasoning as above to (7) instead of $\mathcal{A}(\vec{\varepsilon})$, we obtain the decompositions (8) for all (p,q). They ensure the decomposition (5), and so the deformation (6) is miniversal by Theorem 3. $\square$

The next lemma helps to construct miniversal deformations and will be used in Section 2.

Lemma 5. *Let A be a representation of Q such that $A_\alpha = I$ for some arrow $\alpha\colon p_1 \to p_2$, $p_1 \neq p_2$. Denote by Q' the quiver obtained from Q by removing the arrow α and replacing p_1 and p_2 by a single vertex p (then each other arrow that connects p_1 and p_2 becomes a loop). Denote by A' the representation of Q' that is obtained from A by removing $A_\alpha = I$. Then each miniversal deformation of A' can be extended to a miniversal deformation of A by assigning the identity matrix to α.*

Proof. Let $\mathcal{A}'(\lambda_1, \dots, \lambda_k)$ be a miniversal deformation of A', and let $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ be the deformation of A obtained from $\mathcal{A}'(\lambda_1, \dots, \lambda_k)$ by assigning the identity matrix to α. We need to prove that $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ satisfies the condition (ii) of Lemma 2. Since $A_\alpha = I$, the deformation

$\mathcal{U}(\vec{\varepsilon})$ of A is equivalent to some deformation $\mathcal{B}(\vec{\varepsilon})$ that is the identity on α. Denote by $\mathcal{B}'(\vec{\varepsilon})$ the deformation of A' obtained from $\mathcal{B}(\vec{\varepsilon})$ by removing $\mathcal{B}(\vec{\varepsilon})_\alpha = I$. Since $\mathcal{A}'(\lambda_1, \dots, \lambda_k)$ is a miniversal deformation of A', by Definition 1 $\mathcal{B}'(\vec{\varepsilon})$ is equivalent to a deformation of the form $\mathcal{A}'(\varphi_1(\vec{\mu}), \dots, \varphi_k(\vec{\mu}))$, where $\varphi_i(\vec{\mu})$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_i(\vec{0}) = 0$. Then $\mathcal{B}(\vec{\varepsilon})$ is equivalent to the deformation $\mathcal{A}(\varphi_1(\vec{\mu}), \dots, \varphi_k(\vec{\mu}))$ and so $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ satisfies the condition (ii) of Lemma 2. □

2. Miniversal deformations of matrices of chains of linear mappings

In this section, we give simplest miniversal deformations of matrices of chains of linear mappings $V_1 - V_2 - \dots - V_t$ over complex or real numbers; that is, of representations of the quiver

$$1 \xrightarrow{\ \alpha_1\ } 2 \xrightarrow{\ \alpha_2\ } \cdots \xrightarrow{\ \alpha_{t-2}\ } (t-1) \xrightarrow{\ \alpha_{t-1}\ } t \tag{9}$$

in which each line denotes $\longrightarrow$ or $\longleftarrow$. Due to Theorem 4, it suffices to give simplest miniversal deformations of those of its representations that are direct sums of two nonindecomposable representations. Each representation A of this quiver is isomorphic to a direct sum, determined uniquely up to permutation of summands, of indecomposable representations of the form

$$L_{ij}: \qquad 1 \xrightarrow{\ 0\ } \cdots \xrightarrow{\ 0\ } i \xrightarrow{\ I_1\ } \cdots \xrightarrow{\ I_1\ } j \xrightarrow{\ 0\ } \cdots \xrightarrow{\ 0\ } t\,, \tag{10}$$

$1 \leqslant i \leqslant j \leqslant t$, having dimension $(0, \dots, 0, 1, \dots, 1, 0, \dots 0)$ (in [10] this direct sum is constructed by A using only unitary transformations). Note that the zero matrices in (10) have sizes 0×0, 0×1, or 1×0; it is agreed that there exists exactly one matrix, denoted by 0_{n0}, of size $n \times 0$ and there exists exactly one matrix, denoted by 0_{0n}, of size $0 \times n$ for every nonnegative integer n; they represent the linear mappings $0 \to \mathbb{F}^n$ and $\mathbb{F}^n \to 0$ and are considered as zero matrices. Then

$$M_{pq} \oplus 0_{m0} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{p0} \\ 0_{mq} & 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} \\ 0_{mq} \end{bmatrix}$$

and

$$M_{pq} \oplus 0_{0n} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \\ 0_{0q} & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} & 0_{pn} \end{bmatrix}$$

for every $p \times q$ matrix M_{pq}.

The next theorem gives simplest miniversal deformations of all direct sums of two indecomposable representations L_{ij} of the quiver (9). Using them and Theorem 4, one can construct a simplest miniversal deformation of any representation decomposed into a direct sum of indecomposable representations.

Theorem 6. *Let L_{pq} and L_{rs} $(p \leqslant r)$ be two nonindecomposable representations of the form* (10) *of the quiver* (9) *over complex or real numbers. Then a miniversal deformation of $A = L_{pq} \oplus L_{rs}$ has at most* 1 *parameter. Moreover, it has no parameters* (*and hence coincides with A*) *in all the cases except for the next cases, in which the representations A and their simplest miniversal deformations $\mathcal{A}(\lambda)$ are the following:*

(i) $L_{pq} \oplus L_{q+1,s}$ $(p \leqslant q < s)$,

$$\cdots \;\text{——}\; q \;\overset{[\lambda]}{\text{———}}\; q+1 \;\text{——}\; \cdots \tag{11}$$

(ii) $L_{pq} \oplus L_{qs}$ $(p < q < s)$,

$$\cdots \;\text{——}\; q-1 \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} q \xrightarrow{\begin{bmatrix}\lambda & 1\end{bmatrix}} q+1 \;\text{——}\; \cdots \tag{12}$$

$$\cdots \;\text{——}\; q-1 \xleftarrow{\begin{bmatrix}1 & 0\end{bmatrix}} q \xleftarrow{\begin{bmatrix}\lambda\\1\end{bmatrix}} q+1 \;\text{——}\; \cdots \tag{13}$$

(iii) $L_{pq} \oplus L_{rs}$ $(p < r \leqslant s < q)$,

$$\cdots \;\text{——}\; r-1 \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} r \;\overset{I_2}{\text{———}}\; \cdots \;\overset{I_2}{\text{———}}\; s \xleftarrow{\begin{bmatrix}1\\\lambda\end{bmatrix}} s+1 \;\text{——}\; \cdots$$

$$\cdots \;\text{——}\; r-1 \xleftarrow{\begin{bmatrix}1 & 0\end{bmatrix}} r \;\overset{I_2}{\text{———}}\; \cdots \;\overset{I_2}{\text{———}}\; s \xrightarrow{\begin{bmatrix}1 & \lambda\end{bmatrix}} s+1 \;\text{——}\; \cdots$$

(*each line denotes $\longrightarrow$ or $\longleftarrow$; all unspecified matrices of $\mathcal{A}(\lambda)$ coincide with the corresponding matrices of A*).

Proof. Let us find a miniversal deformation of $A = L_{pq} \oplus L_{rs}$. We may suppose that the pairs (p, q) and (r, s) are lexicographically ordered; that is, $p \leqslant r$ and if $p = r$ then $q \leqslant s$. Deleting arrows of the quiver (9) that correspond to matrices without rows or columns, we reduce our consideration to the case $p = 1$, $\max(q, s) = t$, and $r \leqslant q + 1$. Due to

Lemma 5, we may suppose that A has no identity matrices. Then the quiver has at most 3 vertices and A is one of the following representations:

$$L_{11} \oplus L_{11}, \quad L_{11} \oplus L_{22}, \quad L_{12} \oplus L_{22}, \quad L_{11} \oplus L_{12}, \quad L_{12} \oplus L_{23}, \quad L_{13} \oplus L_{22},$$

or diagrammatically

$$\begin{bmatrix} \bullet \\ \bullet \end{bmatrix}, \quad \begin{bmatrix} \bullet & \\ & \bullet \end{bmatrix}, \quad \begin{bmatrix} \bullet \!-\! & \bullet \\ & \bullet \end{bmatrix}, \quad \begin{bmatrix} \bullet & \\ \bullet \!-\! & \bullet \end{bmatrix}, \quad \begin{bmatrix} \bullet \!-\! & \bullet & \\ & \bullet \!-\! & \bullet \end{bmatrix}, \quad \begin{bmatrix} \bullet \!-\! & \bullet \!-\! & \bullet \\ & \bullet & \end{bmatrix};$$

here the first row of each matrix represents L_{pq} and the second row represents L_{rs}.

The representation $L_{11} \oplus L_{11}$ has no matrices, and so its deformation has no parameters.

The representation $L_{11} \oplus L_{22}$ consists of the 1-by-1 matrix $[0]$ and has the deformation (11).

The representation $L_{12} \oplus L_{22}$ is

$$1 \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} 2 \qquad \text{or} \qquad 1 \xleftarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} 2$$

depending on the orientation of the arrow. In both the cases, $[\mathbb{F}^{\vec{n}\times\vec{n}}, A] = \mathcal{R}(\vec{n}, \mathbb{F})$ and so by (5) $\mathcal{E}_\Gamma = 0$. Hence each miniversal deformation of A has no parameters. The same holds for $A = L_{11} \oplus L_{12}$.

Depending on the orientation of arrows, $A = L_{12} \oplus L_{23}$ has one of the forms:

$$1 \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} 2 \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} 3 \qquad\qquad 1 \xleftarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} 2 \xleftarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} 3 \tag{14}$$

$$1 \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} 2 \xleftarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} 3 \qquad\qquad 1 \xleftarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} 2 \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} 3 \tag{15}$$

In the first case, the space $[\mathbb{F}^{\vec{n}\times\vec{n}}, A]$ with $\vec{n} = (1, 2, 1)$ consists of the representations

$$[C, A]\colon \qquad 1 \xrightarrow{\begin{bmatrix} b_1 - a \\ b_3 \end{bmatrix}} 2 \xrightarrow{\begin{bmatrix} -b_3 & c - b_4 \end{bmatrix}} 3$$

defined by (4), in which

$$C_1 = [a], \qquad C_2 = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \qquad C_3 = [c].$$

Due to (5), the representation (12) is a miniversal deformation of A. Analogously, (13) is a miniversal deformation of the second representation in (14). If A is of the form (15), then $[\mathbb{F}^{\vec{n}\times\vec{n}}, A]$ coincides with $\mathcal{R}(\vec{n},\mathbb{F})$ and so each miniversal deformation of A has no parameters.

Depending on the orientation of arrows, $A = L_{13} \oplus L_{22}$ has one of the forms:

$$1 \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} 2 \xrightarrow{\begin{bmatrix}1 & 0\end{bmatrix}} 3 \qquad\qquad 1 \xleftarrow{\begin{bmatrix}1 & 0\end{bmatrix}} 2 \xleftarrow{\begin{bmatrix}1\\0\end{bmatrix}} 3 \tag{16}$$

$$1 \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} 2 \xleftarrow{\begin{bmatrix}1\\0\end{bmatrix}} 3 \qquad\qquad 1 \xleftarrow{\begin{bmatrix}1 & 0\end{bmatrix}} 2 \xrightarrow{\begin{bmatrix}1 & 0\end{bmatrix}} 3 \tag{17}$$

All miniversal deformations of (16) have no parameters. The representations (iii) in Theorem 6 are miniversal deformations of (17). □

3. A direct proof of Theorem 3

For each matrix $P = [p_{ij}]$ over a field $\mathbb{F}$ of complex or real numbers, we define its *norm* as follows:

$$\|P\| := \sum |p_{ij}|.$$

By [7, Section 5.6],

$$\|aP + bQ\| \leqslant |a|\,\|P\| + |b|\,\|Q\|, \qquad \|PQ\| \leqslant \|P\|\,\|Q\| \tag{18}$$

for matrices P and Q and $a, b \in \mathbb{F}$.

For each finite set $M = \{M_1, \dots, M_l\}$ of matrices, we put

$$\|M\| := \|M_1\| + \cdots + \|M_l\|.$$

Let Q be a quiver with vertices $1, \dots, t$, let M be its representation of dimension $\vec{n} = (n_1, \dots, n_t)$, and let $S = (S_1, \dots, S_t)$ be a sequence of matrices of sizes $n_1 \times n_1, \dots, n_t \times n_t$ (such sequences will be called *$\vec{n}$-sequences*; they are closed under addition and multiplication). Denote by SM and MS the representations of Q obtained from M by replacing each matrix M_α assigned to $\alpha\colon p \to q$ with $S_q M_\alpha$ and, respectively, $M_\alpha S_p$. Due to (18),

$$\|SM\| \leqslant \sum_{p,\alpha} \|S_p\|\,\|M_\alpha\| = \|S\|\,\|M\|, \qquad \|M\|\,\|S\| \leqslant \|M\|\,\|S\|.$$

If $M_\alpha = [m_{\alpha ij}]$ and $\mathit{\Gamma} \subset \Upsilon_{\vec{n}}$ is a subset of (1), then we put

$$\|M\|_{\mathit{\Gamma}} := \sum_{(\alpha,i,j)\notin \mathit{\Gamma}} |m_{\alpha ij}|;$$

in particular, $\|M\|_{\Upsilon_{\vec{n}}} = \|M\|$.

Lemma 7. *Let A and $\mathit{\Gamma}$ be the representation and the set from Theorem 3 satisfying (5). There exists a natural number m such that for each real numbers ε and δ satisfying*

$$0 < \varepsilon \leqslant \delta < \frac{1}{m}$$

and for each representation M of Q satisfying

$$\|M\|_{\mathit{\Gamma}} < \varepsilon, \quad \|M\| < \delta \tag{19}$$

there exists an $\vec{n}$-sequence

$$S = I_{\vec{n}} + X, \qquad \|X\| < m\varepsilon, \tag{20}$$

in which $I_{\vec{n}} = (I_{n_1}, \dots, I_{n_t})$ and the entries of matrices of X are linear polynomials in entries of M such that

$$S(A+M)S^{-1} = A + M', \qquad \|M'\|_{\mathit{\Gamma}} < m\varepsilon\delta, \quad \|M'\| < \delta + m\varepsilon. \tag{21}$$

Proof. First we construct the $\vec{n}$-sequence (20). By (5), for each elementary representation $E_{\alpha ij}$, $(\alpha, i, j) \in \Upsilon_{\vec{n}}$, (they were introduced after Definition 1), there exists an $\vec{n}$-sequence $X_{\alpha ij}$ such that

$$E_{\alpha ij} + X_{\alpha ij}A - AX_{\alpha ij} \in \mathcal{E}_{\mathit{\Gamma}}.$$

If $M = \sum_{\alpha ij} m_{\alpha ij}E_{\alpha ij}$ (that is, the representation M from Lemma 7 is formed by the matrices $M_\alpha = [m_{\alpha ij}]$), then

$$\sum_{\alpha ij} m_{\alpha ij}E_{\alpha ij} + \sum_{\alpha ij} m_{\alpha ij}X_{\alpha ij}A - \sum_{\alpha ij} m_{\alpha ij}AX_{\alpha ij} \in \mathcal{E}_{\mathit{\Gamma}}$$

and for

$$S = I_{\vec{n}} + X, \qquad X := \sum_{\alpha ij} m_{\alpha ij}X_{\alpha ij},$$

we have

$$M + SA - AS \in \mathcal{E}_{\mathit{\Gamma}}. \tag{22}$$

If $(\alpha, i, j) \in \mathit{\Gamma}$, then $E_{\alpha ij} \in \mathcal{E}_{\mathit{\Gamma}}$ and we can put $X_{\alpha ij} = 0$. If $(\alpha, i, j) \notin \mathit{\Gamma}$, then $|m_{\alpha ij}| < \varepsilon$ by the first inequality in (19). We obtain

$$\|X\| \leqslant \sum_{(\alpha,i,j)\notin\mathit{\Gamma}} |m_{\alpha ij}|\,\|X_{\alpha ij}\| < \sum_{(\alpha,i,j)\notin\mathit{\Gamma}} \varepsilon\|X_{\alpha ij}\| = \varepsilon c, \tag{23}$$

where

$$c := \sum_{(\alpha,i,j)\notin\mathit{\Gamma}} \|X_{\alpha ij}\|.$$

Take $\varepsilon < 1/(2c)$, then

$$\varepsilon c < \frac{1}{2} \tag{24}$$

and so

$$\|X^k\| \leqslant \|X\|^k < (\varepsilon c)^k < 1/2^k \to 0 \qquad \text{if } k \to \infty.$$

Hence,

$$\begin{aligned} S^{-1} = (I_{\vec{n}} + X)^{-1} &= I_{\vec{n}} - X + X^2 - X^3 + \cdots \\ &= I_{\vec{n}} - XS^{-1} = I_{\vec{n}} - X + X^2S^{-1}. \end{aligned} \tag{25}$$

Furthermore,

$$\|S^{-1}\| \leqslant \|I_{\vec{n}}\| + \|X\| + \|X\|^2 + \cdots < n + \varepsilon c + (\varepsilon c)^2 + \cdots,$$

where $n := n_1 + \cdots + n_t$, and by (24)

$$\|S^{-1}\| \leqslant n - 1 + \frac{1}{1 - 1/2} = n + 1. \tag{26}$$

Using (25), we obtain

$$\begin{aligned} S(A+M)S^{-1} = (A + M + XA + XM)S^{-1} = A(I_{\vec{n}} - X + X^2S^{-1}) \\ + (M + XA)(I_{\vec{n}} - XS^{-1}) + XMS^{-1} = A + M', \end{aligned}$$

where

$$\begin{aligned} M' &:= M + XA - AX + N, \\ N &:= AX^2S^{-1} - (M + XA)XS^{-1} + XMS^{-1}. \end{aligned}$$

Then by (23), (26), and (19), and since $\varepsilon \leqslant \delta$, we have

$$\|N\| \leqslant 2\|A\|(\varepsilon c)^2(n+1) + 2\delta(\varepsilon c)(n+1) \leqslant \varepsilon\delta d,$$

where

$$d := 2\|A\|c^2(n+1) + 2c(n+1).$$

By (22), $M + XA - AX \in \mathcal{E}_\Gamma$, and so

$$\|M'\|_\Gamma = \|N\|_\Gamma \leqslant \varepsilon\delta d.$$

Furthermore,

$$\|M'\| \leqslant \|M\| + \|XA - AX\| + \|N\| \leqslant \delta + 2\varepsilon c\|A\| + \varepsilon\delta d = \delta + e\varepsilon,$$

where

$$e := 2c\|A\| + \delta d.$$

Taking any natural number m that is greater than c, d, and e, we obtain (20) and (21). □

Lemma 8. *Let m be any natural number being $\geqslant 3$, and let*

$$\varepsilon_1,\ \delta_1,\ \varepsilon_2,\ \delta_2,\ \varepsilon_3,\ \delta_3,\ \dots$$

be the sequence of numbers defined by induction:

$$\varepsilon_1 = \delta_1 = m^{-7}, \quad \varepsilon_{i+1} = m\varepsilon_i\delta_i, \quad \delta_{i+1} = \delta_i + m\varepsilon_i. \tag{27}$$

Then

$$\varepsilon_i < m^{-4i}, \quad \delta_i < m^{-5} \tag{28}$$

for all i and

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots < 2. \tag{29}$$

Proof. Reasoning by induction, we assume that the inequalities (28) hold for $i = 1, 2, \dots, l$. Then

$$\varepsilon_{l+1} = m\varepsilon_l\delta_l < m^{-4l}mm^{-5} = m^{-4(l+1)}$$

and

$$\begin{aligned}
\delta_{l+1} &= \delta_l + m\varepsilon_l = \delta_{l-1} + m(\varepsilon_{l-1} + \varepsilon_l) = \cdots \\
&= \delta_1 + m(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_l) \\
&< m^{-7} + m(m^{-7} + m^{-4\cdot 2} + m^{-4\cdot 3} + \cdots) \\
&\leqslant m^{-7} + m^{-6}(1 + m^{-1} + m^{-2} + m^{-3} + \cdots) \\
&= m^{-7} + m^{-6}\frac{1}{1 - m^{-1}} \leqslant m^{-6}\left(m^{-1} + \frac{3}{2}\right) < 2m^{-6} < m^{-5}.
\end{aligned}$$

This proves (28) for all i. Then (29) holds too since

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots < m^{-4} + m^{-4\cdot 2} + m^{-4\cdot 3} + \cdots < \frac{1}{1 - m^{-4}} < 2.$$

□

Theorem 3 follows from the next lemma.

Lemma 9. *Let A and Γ satisfy (5), let m be a natural number that is greater than 3 and satisfies Lemma 7, and let M be any representation satisfying $\|M\| < m^{-7}$. Then there exists an $\vec{n}$-sequence $S = I_{\vec{n}} + X$ depending holomorphically on the entries of M in a neighborhood of zero such that*

$$S(A+M)S^{-1} - A \in \mathcal{E}_\Gamma$$

and $S = I_{\vec{n}}$ if $M = 0$.

Proof. We construct a sequence of representations

$$A + M_1,\ A + M_2,\ A + M_3, \dots$$

by induction. Put $M_1 = M$. Let M_i be constructed and let

$$\|M_i\|_\Gamma < \varepsilon_i, \quad \|M_i\| < \delta_i,$$

where ε_i and δ_i are defined in (27). Then by (28) and Lemma 7 there exists

$$S_{i+1} = I_{\vec{n}} + X_{i+1}, \qquad \|X_{i+1}\| < m\varepsilon_{i+1}, \tag{30}$$

such that

$$S_{i+1}(A+M_i)S_{i+1}^{-1} = A + M_{i+1}, \quad \|M_{i+1}\|_\Gamma < \varepsilon_{i+1}, \quad \|M_{i+1}\| < \delta_{i+1}.$$

For each natural number l, put

$$S^{(l)} := S_l \cdots S_3 S_2 S_1 = (I_{\vec{n}} + X_l) \cdots (I_{\vec{n}} + X_2)(I_{\vec{n}} + X_1). \tag{31}$$

Let us prove that the sequence $S^{(1)}, S^{(2)}, S^{(3)}, \dots$ converges. Indeed,

$$S^{(l)} = I_{\vec{n}} + \sum_{l \geqslant i} X_i + \sum_{l \geqslant i > j} X_i X_j + \cdots$$

and so

$$\begin{aligned} \|S^{(l)}\| &\leqslant \|I_{\vec{n}}\| + \sum_{l \geqslant i} \|X_i\| + \sum_{l \geqslant i > j} \|X_i\|\,\|X_j\| + \cdots \\ &\leqslant n - 1 + (1 + \|X_1\|)(1 + \|X_2\|)(1 + \|X_3\|) \cdots, \end{aligned} \tag{32}$$

where $n := n_1 + \cdots + n_t$. By [8, Section III, §4.3], the product (32) converges since the sum

$$\|X_1\| + \|X_2\| + \|X_3\| + \cdots$$

converges due to (30) and (29).

The entries of all matrices forming $S := \lim S^{(l)}$ are holomorphic functions in the entries of M (that satisfies $\|M\| < m^{-7}$) due to Weierstrass' theorem [8, Section III, §4.1] since the sequence

$$I_{\vec{n}} + X_1,\ (I_{\vec{n}} + X_2)(I_{\vec{n}} + X_1),\ (I_{\vec{n}} + X_3)(I_{\vec{n}} + X_2)(I_{\vec{n}} + X_1),\ \dots$$

converges uniformly to (31).

Since $A + M_l \to S(A+M)S^{-1}$ if $l \to \infty$ and $\|M_l\|_\Gamma < \varepsilon_l \to 0$, we have $S(A+M)S^{-1} - A \in \mathcal{E}_\Gamma$. □

References

[1] V.I. Arnold, *On matrices depending on parameters*, Russian Math. Surveys, vol. **26**, no. 2, 1971, pp. 29–43.

[2] V.I. Arnold, *Geometrical methods in the theory of ordinary differential equations.* Springer-Verlag, New York, 1988.

[3] V.I. Arnold, *Dynamical systems VI* (Encyclopaedia Math. Sciences 6), Springer, 1990.

[4] A. Edelman, E. Elmroth, B. Kågström, *A geometric approach to perturbation theory of matrices and matrix pencils. Part I: Versal deformations*, Siam J. Matrix Anal. Appl., vol. **18**, no. 3, 1997, pp. 653–692.

[5] D.M. Galin, *On real matrices depending on parameters*, Uspehi Mat. Nauk, vol. **27**, no. 1, 1972, pp. 241–242.

[6] M.I. Garcia-Planas and V.V. Sergeichuk, *Simplest miniversal deformations of matrices, matrix pencils, and contragredient matrix pencils*, Linear Algebra Appl., vol. **302–303**, 1999, pp. 45–61.

[7] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge U. P., Cambridge, 1985.

[8] A.I. Markushevich, *Theory of Analytic Functions*, State Publishing of Technical and Theoretical Literature, Moscow, 1950 (in Russian).

[9] V.V. Sergeichuk, *Canonical matrices for linear matrix problems*, Linear Algebra Appl., vol. **317**, 2000, pp. 53–102.

[10] V.V. Sergeichuk, *Computation of canonical matrices for chains and cycles of linear mappings*, Linear Algebra Appl., vol. **376**, 2004, pp. 235–263.

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