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Gorenstein matrices

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Dedicated to Yu.A. Drozd on the occasion of his 60th birthday

Abstract. Let $A = (a_{ij})$ be an integral matrix. We say that A is $(0,1,2)$-matrix if $a_{ij} \in \{0,1,2\}$. There exists the Gorenstein $(0,1,2)$-matrix for any permutation σ on the set $\{1,\ldots,n\}$ without fixed elements. For every positive integer n there exists the Gorenstein cyclic $(0,1,2)$-matrix A_n such that $\mathit{inx}\, A_n = 2$.

If a Latin square $\mathcal{L}_n$ with a first row and first column $(0,1,\ldots n-1)$ is an exponent matrix, then $n = 2^m$ and $\mathcal{L}_n$ is the Cayley table of a direct product of m copies of the cyclic group of order 2. Conversely, the Cayley table $\mathcal{E}_m$ of the elementary abelian group $G_m = (2)\times\ldots\times(2)$ of order 2^m is a Latin square and a Gorenstein symmetric matrix with first row $(0,1,\ldots,2^m-1)$ and

$$\sigma(\mathcal{E}_m) = \begin{pmatrix} 1 & 2 & 3 & \ldots & 2^m-1 & 2^m \\ 2^m & 2^m-1 & 2^m-2 & \ldots & 2 & 1 \end{pmatrix}.$$

1. Introduction

Gorenstein rings appeared in a paper by D. Gorenstein published in 1952 [9]. In [1] H. Bass wrote: "After writing this paper I discovered from Professor Serre that these rings have been encountered by Grothendick the latter having christened in his setting by the fact that a certain module of differentials is locally free of rank one". (See, also [2]).

Let $\mathcal{O}$ be a Dedekind ring with a field of fractions K, and let Λ be an $\mathcal{O}$-order in a finite dimensional separable K-algebra A (see [6]). In this

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case it is natural to consider Λ-lattices, i.e., finitely generated $\mathcal{O}$-torsion free Λ-modules.

Noncommutative Gorenstein $\mathcal{O}$-orders appeared first in [8], (see Definition and Proposition 6.1). An $\mathcal{O}$-order Λ is left Gorenstein if and only if the injective dimension of Λ as a left Λ-module is 1 ($\mathcal{O} \neq K$). Definition and Proposition 6.1 of [8] shows that Λ is left Gorenstein if and only if it is right Gorenstein.

Given a Λ-lattice M, a sublattice N of M is called **pure** if M/N is $\mathcal{O}$-torsion free.

The following theorem is proved in [10]:

An $\mathcal{O}$-order Λ is Gorenstein if and only if each left Λ-lattice is isomorphic to a pure sublattice of a free Λ- lattice.

In the [20] K. Nishida gives an example of a $(0,1)$-order $\Lambda(P_5)$ associated with the finite poset

$P_5 =$ 

such that $inj\,dim\,\Lambda(P_5) = 2$ and $gl.\,dim\Lambda(P_5) = \infty$.

Let Λ be a Gorenstein order. If Λ has the additional property that every $\mathcal{O}$-order containing Λ is also Gorenstein, then Λ is called a Bass order. The following inclusions are easily verified:

$$\text{(maximal orders)} \subseteq \text{(hereditary orders)} \subseteq$$
$$\subseteq \text{(Bass orders)} \subseteq \text{(Gorenstein orders)}$$

(see [6, §37]).

Denote by $\mu_\Lambda(X)$ the minimal number of generators of a finitely generated Λ-module X. The following theorem is proved in [22] (see also [6, Theorem 37.17]).

Let Λ be an $\mathcal{O}$-order such that $\mu_\Lambda(I) \leqslant 2$ for each left ideal I of Λ. Then Λ is a Bass order.

Obviously, the Z-order
$$\begin{pmatrix} \mathbb{Z} & 4\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$
is a Bass order, because for every left ideal J we have $\mu_\Lambda(I) \leqslant 2$, (see also [3]).

In [11] H. Fujita studies an interesting class of algebras which is closely related to tiled orders over discrete valuation rings.

Tiled orders over a discrete valuation rings appeared first in [23] (see also [13, 14]). The Gorenstein condition for exponent matrices of tiled orders is formulated in [15]. Note that the notion of an exponent matrix appeared, first, in the English translation of [24].

A finite directed graph without multiple arrows and multiple loops is called **simply laced**.

Denote by $M_n(B)$ the ring of all $n \times n$ matrices over a ring B.

An integer matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called

- an exponent matrix if $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ and $\alpha_{ii} = 0$ for all i, j, k;
- a reduced exponent matrix if $\alpha_{ij} + \alpha_{ji} > 0$ for all i, j: $i \neq j$.

Recall that a ring A is called a **tiled order** if it is a prime Noetherian semiperfect semidistributive ring with nonzero Jacobson radical (see [4, 5]).

Theorem 1.1. *Each tiled order A is isomorphic to a prime ring of the following form:*

$$A = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \dots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \dots & \pi^{\alpha_{2n}}\mathcal{O} \\ \dots & \dots & \dots & \dots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \dots & \mathcal{O} \end{pmatrix},$$

where $n \geqslant 1$, $\mathcal{O}$ is a discrete valuation ring with a prime element π, and the α_{ij} are integers with $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ for all i, j, k ($\alpha_{ii} = 0$ for all i).

We shall use the following notation: $A = \{\mathcal{O}, \mathcal{E}(A)\}$, where $\mathcal{E}(A) = (\alpha_{ij})$ is the exponent matrix of A, i.e., $A = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$, where the e_{ij} are the matrix units. If a tiled order is reduced, then $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \dots, n$, $i \neq j$, i.e., $\mathcal{E}(A)$ is reduced.

Note that with every reduced tiled order A we associate the following notions (see [4, 5]):

1) the reduced exponent matrix $\mathcal{E}(A)$;
2) the quiver $Q(A)$ which coincides with the quiver $Q(\mathcal{E}(A)$;
3) the width $w(A)$ which coincides with the width $w(\mathcal{E}(A))$ of $\mathcal{E}(A)$;
4) the index of A (*inx* A).

By definition $inx\,\mathcal{E}(A) = inx\,A$.

Let $\mathcal{E} = (\alpha_{ij})$ be a reduced exponent matrix. Set $\mathcal{E}^{(1)} = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ and $\beta_{ii} = 1$ for $i = 1, \ldots, n$, and $\mathcal{E}^{(2)} = (\gamma_{ij})$, where $\gamma_{ij} = \min\limits_{1 \leqslant k \leqslant n} (\beta_{ik} + \beta_{kj})$.

Theorem 1.2. [17]. *The matrix $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is the adjacency matrix of the strongly connected simply laced quiver $Q = Q(\mathcal{E})$.*

A strongly connected simply laced quiver is called admissible if it is the quiver of a reduced exponent matrix.

Theorem 1.3. [18]. *An arbitrary strongly connected simply laced quiver Q with a loop in every vertex is admissible.*

The main concept of this paper is the notion of a **Gorenstein matrix**.

A reduced exponent matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ shall be called **Gorenstein** if there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \ldots, n$.

The permutation σ is denoted by $\sigma(\mathcal{E})$. Notice that $\sigma(\mathcal{E})$ of a Gorenstein matrix $\mathcal{E}$ has no cycles of length 1.

A Gorenstein matrix $\mathcal{E}$ is called **cyclic** if $\sigma(\mathcal{E})$ is a cycle.

A simply laced quiver Q shall be called **Gorenstein** if $Q = Q(\mathcal{E})$ for a Gorenstein matrix $\mathcal{E}$.

2. Examples

Let σ be a permutation of $\{1, 2, \ldots, n\}$. Then $P_\sigma = \sum\limits_{i=1}^{n} e_{i\tau(i)}$ is called a **permutation matrix** (here e_{ij} stand for the matrix units).

In [18, Theorem 4.5] the following theorem was proved:

Theorem 2.1. *The adjacency matrix of the quiver of a cyclic Gorenstein matrix $\mathcal{E}$ with permutation $\sigma = \sigma(\mathcal{E})$ is a sum of some powers of the permutation matrix P_σ.*

We will give examples of Gorenstein matrices.

Examples.

I. The $(n \times n)$-matrix

$$H_n = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}$$

is a Gorenstein cyclic matrix with permutation

$$\sigma = \sigma(H_n) = \begin{pmatrix} 1 & 2 & \dots & n-1 \\ n & 1 & \dots & n \end{pmatrix}.$$

For the adjacency matrix $[Q(H_n)]$ we have that $[Q(H_n)] = P_{\sigma^{n-1}}$.

Remark. The matrices H_n appeared in the theorem by Michler [19], which we state below after giving some notation.

Let $\mathcal{O}$ be a (possibly non-commutative) discrete valuation ring with the division ring of fractions $\mathcal{D}$ and let $\mathcal{M}$ be its unique maximal ideal. Denote by $H_n(\mathcal{O})$ the subring of the matrix ring $M_n(\mathcal{D})$ of the form

$$H_n(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{M} & \mathcal{O} & \dots & \mathcal{O} \\ \dots & \dots & \dots & \dots \\ \mathcal{M} & \mathcal{M} & \dots & \mathcal{O} \end{pmatrix}.$$

Clearly, the ring $H_n(\mathcal{O})$ is hereditary and $\mathcal{E}(H_n(\mathcal{O})) = H_n$.

Theorem 2.2. [19]. *Every semiprime semiperfect Noetherian hereditary ring is Morita equivalent to the finite direct product of division rings and some rings of the form $H_m(\mathcal{O})$.*

II.

The $(2m \times 2m)$-matrix

$$G_{2m} = \left|\begin{array}{c|c} H_m & H_m^{(1)} \\ \hline H_m^{(1)} & H_m \end{array}\right|$$

is Gorenstein with permutation

$$\sigma(G_{2m}) = \begin{pmatrix} 1 & 2 & \dots & m & m+1 & m+2 & \dots & 2m \\ m+1 & m+2 & \dots & 2m & 1 & 2 & \dots & m \end{pmatrix}.$$

If $m = 1$ then

$$[Q(G_2)] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = E + P_\tau,$$

where

$$\tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

In general case, $[Q(G_{2m})] = P_{\tau^{m-1}} + P_{\tau^{2m-1}}$, where

$$\tau = \begin{pmatrix} 1 & 2 & \dots & 2m \\ 2m & 1 & \dots & 2m-1 \end{pmatrix}$$

is a cycle and $inx\, G_{2m} = 2$.

III.

The matrix

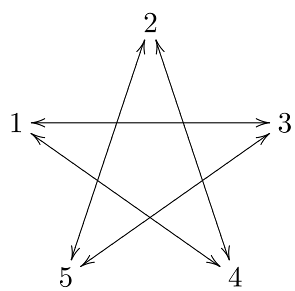
$$\mathcal{E}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

is cyclic Gorenstein with permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$$

and $[Q(\mathcal{E}_5)] = P_{\tau^2} + P_{\tau^3}$.

$Q(\mathcal{E}_5) =$



IV.

The matrix

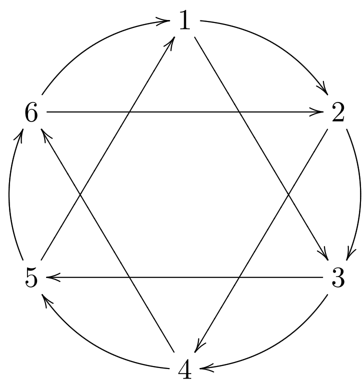
$$\mathcal{E}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 2 & 0 \end{pmatrix}$$

is cyclic Gorenstein with permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

and $[Q(\mathcal{E}_6)] = P_{\tau^4} + P_{\tau^5}$.

$Q(\mathcal{E}_6) =$



V.

The matrix

$$\Gamma_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 4 & 3 & 3 \\ 4 & 0 & 0 & 4 & 2 & 2 \\ 4 & 0 & 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 2 & 0 & 3 \\ 3 & 0 & 1 & 2 & 3 & 0 \end{pmatrix}$$

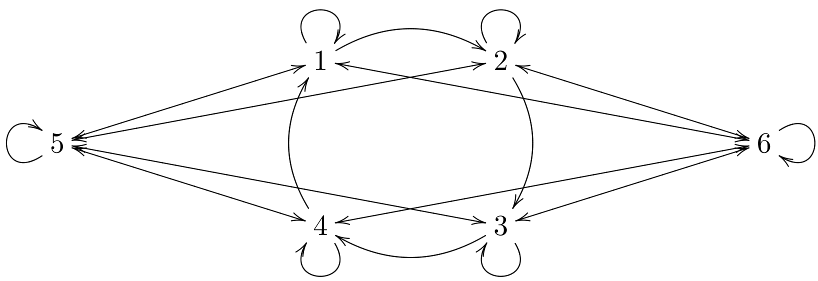
is Gorenstein with permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}.$$

Note that

$$[Q(\Gamma_6)] = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

is not a multiple doubly stochastic matrix. We have that

$Q(\Gamma_6) =$ 

Definition 2.3. [5]. *Two exponents matrices $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ shall be called* **equivalent** *if they can be obtained from each other by transformations of the following two types :*

(1) subtracting an integer from the i-th row with simultaneous adding it to the i-th column;

(2) simultaneous interchanging of two rows and the equally numbered columns.

Proposition 2.4. [5]. *Suppose that $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ are exponent matrices and Θ is obtained from $\mathcal{E}$ by a transformation of type (1). Then $[Q(\mathcal{E})] = [Q(\Theta)]$. If $\mathcal{E}$ is a reduced Gorenstein exponent matrix with permutation $\sigma(\mathcal{E})$, then Θ is also reduced Gorenstein with $\sigma(\Theta) = \sigma(\mathcal{E})$.*

Proposition 2.5. [5]. *Under transformations of the second type the adjacency matrix $[\tilde{Q}]$ of $Q(\Theta)$ changes according to the formula: $[\tilde{Q}] = P_\tau^T[Q]P_\tau$, where $[Q] = [Q(\mathcal{E})]$. If $\mathcal{E}$ is Gorenstein then Θ is also Gorenstein and for the new permutation π we have: $\pi = \tau^{-1}\sigma\tau$, i.e., $\sigma(\Theta) = \tau^{-1}\sigma(\mathcal{E})\tau$.*

Theorem 2.6. *Any Gorenstein $(0,1)$-matrix is equivalent either to H_n or to G_{2m}.*

The proof follows from [16, Theorem 2.1].

Corollary 2.7. *Any cyclic Gorenstein $(0,1)$-matrix is equivalent to a matrix H_n and $\operatorname{inx} H_n = 1$. Conversely, if $\operatorname{inx}\mathcal{E} = 1$, where $\mathcal{E}$ is a reduced exponent matrix, then $\mathcal{E}$ is equivalent to H_n.*

Let σ be a permutation of $\{1, \ldots, n\}$ without fixed elements. There exists a Gorenstein matrix $\mathcal{E}_\sigma$ such that $\sigma(\mathcal{E}_\sigma) = \sigma$ (see [5], Theorem 6.3). The Gorenstein quiver $Q(\mathcal{E}_\sigma)$ shall be called the quiver associated with the permutation σ.

Definition 2.8. *A permutation σ without fixed elements shall be called exceptional if the Gorenstein quiver associated with σ is unique, up to isomorphism.*

Proposition 2.9. *The permutation $\sigma = (12)(345)$ is exceptional.*

Proof. We describe all Gorenstein matrices $\mathcal{E}_\sigma = (\alpha_{ij})$. We can assume that $\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = 0$. So, $\alpha_{12} = \alpha_{22} = \alpha_{32} = \alpha_{42} = \alpha_{52} = 0$.

We have the following system of linear equations for elements of $\mathcal{E}_\sigma$:

$$\begin{cases} \alpha_{12} & = & \alpha_{23} + \alpha_{31} & = & \alpha_{24} + \alpha_{41} & = & \alpha_{25} + \alpha_{51} \\ \alpha_{31} & = & \alpha_{24} & = & \alpha_{34} & = & \alpha_{35} + \alpha_{54} \\ \alpha_{41} & = & \alpha_{25} & = & \alpha_{43} + \alpha_{35} & = & \alpha_{45} \\ \alpha_{51} & = & \alpha_{23} & = & \alpha_{53} & = & \alpha_{54} + \alpha_{43} \end{cases}$$

It is easy to see that

$$\mathcal{E}_\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 4\alpha & 0 & 2\alpha & 2\alpha & 2\alpha \\ 2\alpha & 0 & 0 & 2\alpha & \alpha \\ 2\alpha & 0 & \alpha & 0 & 2\alpha \\ 2\alpha & 0 & 2\alpha & \alpha & 0 \end{pmatrix}.$$

and

$$\mathcal{E}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 4\alpha & 1 & 2\alpha & 2\alpha & 2\alpha \\ 2\alpha & 0 & 1 & 2\alpha & \alpha \\ 2\alpha & 0 & \alpha & 1 & 2\alpha \\ 2\alpha & 0 & 2\alpha & \alpha & 1 \end{pmatrix},$$

$$\mathcal{E}^{(2)} = \begin{pmatrix} 2 & 0 & 1 & 1 & 1 \\ 4\alpha & 2 & 2\alpha+1 & 2\alpha+1 & 2\alpha+1 \\ 2\alpha+1 & 1 & 2 & 2\alpha & \alpha+1 \\ 2\alpha+1 & 1 & \alpha+1 & 2 & 2\alpha \\ 2\alpha+1 & 1 & 2\alpha & \alpha+1 & 2 \end{pmatrix}$$

therefore,

$$[Q(\mathcal{E})] = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

□

3. Gorenstein $(0,1,2)$-matrices

Denote the ring of all square $n\times n$-matrices over the integers $\mathbb{Z}$ by $M_n(\mathbb{Z})$. Let $A\in M_n(\mathbb{Z})$.

Definition 3.1. *A matrix $A=(a_{ij})$ shall be called a* $(0,1,2)$-matrix *if* $a_{ij}\in\{0,1,2\}$.

Theorem 3.2. *For any permutation σ on $\{1,\dots,n\}$ without fixed elements there exists a Gorenstein $(0,1,2)$-matrix.*

Proof. Let $\sigma : i\to\sigma(i)$ be a permutation on $\{1,\dots,n\}$ without fixed elements and $\mathcal{E}_\sigma=(\alpha_{ij})$ be the following $(0,1,2)$-matrix:

- $\alpha_{ii}=0$ and $\alpha_{i\sigma(i)}=2$ for $i=1,\dots,n$;
- $\alpha_{ij}=1$ for $i\neq j$ and $i\neq\sigma(i)$ $(i,j=1,\dots,n)$.

Obviously, $\mathcal{E}_\sigma$ is a Gorenstein matrix with permutation σ.

□

Let σ be an arbitrary permutation on $\{1,\dots,n\}$ without fixed elements and $\mathcal{E}_\sigma$ be a Gorenstein $(0,1,2)$-matrix as in Theorem 3.2. Denote $P_\sigma=\sum\limits_{i=1}^{n}e_{i\sigma(i)}$ the permutation matrix of σ. It is easy to see that $[Q(\mathcal{E}_\sigma)]=U_n-P_\sigma$.

We will show how one can represent the matrix $[Q(\mathcal{E}_\sigma)]$ as a sum of permutation matrices.

Let $\sigma_1,\dots,\sigma_{n-1}$ be the permutations: $\sigma_k(i)=\sigma(i)+k\ (mod\, n)$. Obviously, $\sigma_k(i)\neq\sigma_m(i)$ for $k\neq m$ and $[Q(\mathcal{E}_\sigma)]=\sum\limits_{k=1}^{n-1}P_{\sigma_k}$.

Examples.

I.

Let

$$\mathcal{E}_3=\begin{pmatrix}0&0&0\\2&0&0\\2&2&0\end{pmatrix}$$

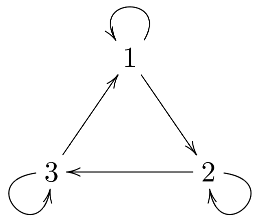
be the Gorenstein matrix with permutation

$$\sigma=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}.$$

Obviously,

$$\mathcal{E}_3^{(2)}=\begin{pmatrix}1&1&0\\2&1&1\\3&2&1\end{pmatrix}\ \text{and}\ [Q(\mathcal{E}_3)]=E+P_{\sigma^2}.$$

Thus, $Q(\mathcal{E}_3)$ has the following form:



II.
Let

$$\mathcal{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \end{pmatrix}$$

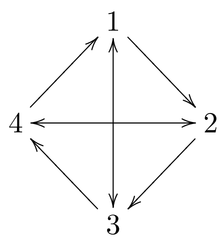
be the Gorenstein matrix with permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

Obviously,

$$\mathcal{E}_4^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 2 & 1 \end{pmatrix} \text{ and } [Q(\mathcal{E}_4)] = P_{\sigma^2} + P_{\sigma^3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Hence, $Q(\mathcal{E}_4)$ has the following form:



III.
Let

$$\mathcal{E}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

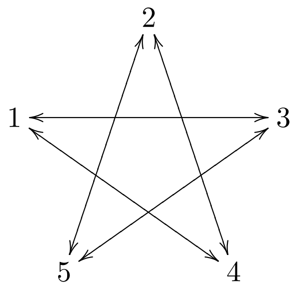
be the Gorenstein matrix with permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

Obviously

$$\mathcal{E}_5^{(2)} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 & 1 \end{pmatrix} \text{ and } [Q(\mathcal{E}_5)] = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

So, $Q(\mathcal{E}_5)$ has the following form



and $[Q(\mathcal{E}_5)] = P_{\sigma^2} + P_{\sigma^3}$.

IV.

Let

$$\mathcal{E}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 2 & 0 \end{pmatrix}$$

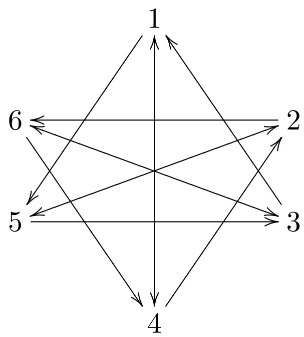
be the Gorenstein matrix with permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Obviously,

$$\mathcal{E}_6^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 & 2 & 1 \end{pmatrix} \quad [Q(\mathcal{E}_6)] = P_{\sigma^2} + P_{\sigma^3} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

We have that $Q(\mathcal{E}_6)$ is of the following form:



In the general case we have that

$$\mathcal{E}_n = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & \dots & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & \dots & 1 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & \dots & 1 & 2 & 0 \end{pmatrix}$$

is a Gorenstein matrix with permutation

$$\sigma(\mathcal{E}_n) = \sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ n & 1 & \dots & n-1 \end{pmatrix}.$$

It is easy to show that $[Q(\mathcal{E}_n)] = P_{\sigma^2} + P_{\sigma^3}$.

Theorem 3.3. *For every positive integer n there exists a Gorenstein cyclic $(0,1,2)$-matrix $\mathcal{E}_n$ such that $\operatorname{inx} \mathcal{E}_n = 2$.*

4. Latin squares and Cayley tables of elementary abelian 2-groups

A Latin square of order n is a square matrix with rows and columns each of which is a permutation of $S = \{s_1, \ldots, s_n\}$.

Every Latin square is a Cayley table of a finite quasigroup. In particular, the Cayley table of a finite group is a Latin square. We take $S = \{0, 1, \ldots, n-1\}$.

Examples.

I.

The Latin square

$$\mathcal{L}_4 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

is a Gorenstein matrix with permutation

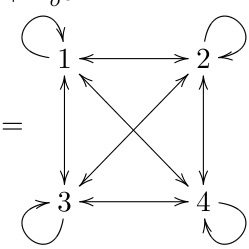
$$\sigma = \sigma(\mathcal{L}_4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

and

$$[Q(\mathcal{L}_4)] = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Obviously, $[Q(\mathcal{L}_4)] = E + P_{\sigma^2} + P_{\sigma^3}$ and

$Q(\mathcal{L}_4) =$



II.

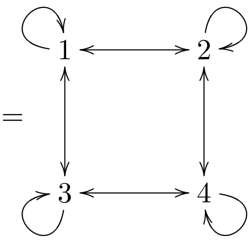
The Latin square

$$\mathcal{E}_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

is the Cayley table of the Klein four-group and is a Gorenstein matrix with permutation $\sigma(\mathcal{E}) = (14)(23)$. By Propositions 2.4 and 2.5 the matrices $\mathcal{E}_2$ and $\mathcal{L}_4$ are non-equivalent.

$$\mathcal{E}_2^{(1)} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 \\ 2 & 3 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{pmatrix}; \quad \mathcal{E}_2^{(2)} = \begin{pmatrix} 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 3 & 3 & 2 & 2 \\ 3 & 3 & 2 & 2 \end{pmatrix}.$$

$Q(\mathcal{E}_2) =$



We introduce the following notation:

$$\mathcal{E}_0 = (0),\ \mathcal{E}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathcal{E}_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix},$$

$$U_n \in M_n(\mathbb{Z}) \text{ and } U_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix},\ X_{k-1} = 2^{k-1} U_{2^{k-1}};$$

$$\mathcal{E}_k = \begin{pmatrix} \mathcal{E}_{k-1} & \mathcal{E}_{k-1} + X_{k-1} \\ \mathcal{E}_{k-1} + X_{k-1} & \mathcal{E}_{k-1} \end{pmatrix} \text{ for } k = 1, 2, \dots .$$

Obviously, $\mathcal{E}_k$ is a Gorenstein matrix with permutation

$$\sigma = \sigma(\mathcal{E}_k) = \begin{pmatrix} 1 & 2 & \dots & k \\ 2^k & 2^k - 1 & \dots & 1 \end{pmatrix}.$$

Main Theorem. *Suppose that a Latin square $\mathcal{L}_n$ with a first row and a first column $(0\,1\,\dots\,n-1)$ is an exponent matrix. Then $n = 2^m$ and $\mathcal{L}_n = \mathcal{E}_m$ is the Cayley table of a direct product of m copies of the cyclic group of order 2.*

Conversely, the Cayley table $\mathcal{E}_m$ of the elementary abelian group $G_m = (2) \times \dots \times (2)$ (m factors) of order 2^m is the Latin square and the Gorenstein symmetric matrix with the first row $(0, 1, \dots, 2^m - 1)$ and

$$\sigma(\mathcal{E}_m) = \begin{pmatrix} 1 & 2 & 3 & \dots & 2^m - 1 & 2^m \\ 2^m & 2^m - 1 & 2^m - 2 & \dots & 2 & 1 \end{pmatrix}.$$

The second part of this theorem was proved in [21, Section 4].

Lemma 4.1. *Let $\mathcal{L}_n = (\alpha_{ij})$ be defined as above. Then*

$$|\, i - j \,| \leqslant \alpha_{ij} \leqslant i + j - 2.$$

Proof. Obviously, $\alpha_{1i} + \alpha_{ij} \geqslant \alpha_{1j}$ and $\alpha_{ij} \geqslant j - 1 - (i-1) = j - i$. Analogously, $\alpha_{ij} + \alpha_{j1} \geqslant \alpha_{i1}$ and $\alpha_{ij} \geqslant i - 1 - (j-1) = i - j$, i.e., $\alpha_{ij} \geqslant |\, i - j \,|$. We have $\alpha_{i1} + \alpha_{1j} \geqslant \alpha_{ij}$ and $\alpha_{ij} \leqslant i + j - 2$. □

Lemma 4.2. *The last row of $\mathcal{L}_n$ is $(n-1, n-2, \ldots, 1)$.*

Proof. We have that $\alpha_{n1} = n - 1$ by the definition of $\mathcal{L}_n$. By Lemma 4.1 we have $\alpha_{ni} \geqslant n - i$. So, $\alpha_{n2} = n - 2$, $\alpha_{n3} = n - 3$ and $\alpha_{nn} = 0$. □

Corollary 4.3. *The last column of $\mathcal{L}_n$ is $(n-1, n-2, \ldots, 1)^T$, where T is the transpose.*

Lemma 4.4. *Let $\mathcal{L}_n = (\alpha_{ij})$ be defined as above. Then*

$$|i + j - (n+1)| \leqslant |n - 1 - \alpha_{ij}|.$$

Proof. By Lemma 4.2 and Corollary 4.3, we have $\alpha_{ij} \leqslant \alpha_{in} + \alpha_{nj} = (n-i) + (n-j) = 2n - (i+j)$. By Lemma 4.1 $\alpha_{ij} \leqslant i + j - 2$. From first inequality we have

$$\alpha_{ij} - (n-1) \leqslant n + 1 - (i+j).$$

From second inequality we have

$$\alpha_{ij} - (n-1) \leqslant (i+j) - (n+1).$$

So

$$|(i+j) - (n+1)| \leqslant |\alpha_{ij} - (n-1)|.$$

□

Corollary 4.5. *An integer n is even.*

Proof. By Lemma 4.4 an integer $n-1$ appears on the (i,j)-th position if $|(n-1) - (n-1)| \geqslant |i + j - (n+1)|$, i.e., $i + j = n + 1$. Hence, the second diagonal has the following form: $(n-1, \ldots, n-1)$. If n is odd then for $i = j = \frac{n+1}{2}$ we have $\alpha_{ii} = n - 1$. This is a contradiction. □

Now we will prove the Main Theorem.

Proof. If $\alpha_{ij} = 1$, then $|i-j| = 1$. We have

$$\mathcal{E} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & * & \cdots & * \\ * & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \cdots & * \\ \cdots & \cdots & \cdots & * \\ * & * & \cdots & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix} = \begin{pmatrix} \mathcal{E}_1 & * & \cdots & * \\ * & \mathcal{E}_1 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & \mathcal{E}_1 \end{pmatrix}.$$

If $\alpha_{ij} = 2$ then $|i-j| \leqslant 2$. It is easy to see that $\alpha_{ij} = 2$ for $|i-j| = 2$, i.e., $\alpha_{4t+1,4t+3} = \alpha_{4t+3,4t+1} = \alpha_{4t+2,4t+4} = \alpha_{4t+4,4t+2} = 2$. In this case $n_1 = 2n_2$ is even.

If $\alpha_{ij} = 3$ then $|i-j| \leqslant 3$.

From $\alpha_{4t+2,4t+3} \leqslant \alpha_{4t+2,4t+1} + \alpha_{4t+1,4t+3} = 1+2 = 3$, $\alpha_{4t+3,4t+2} \leqslant \alpha_{4t+3,4t+1} + \alpha_{4t+1,4t+2} = 2+1 = 3$ and $|i-j| \leqslant 3$ follows $\alpha_{4t+1,4t+4} = \alpha_{4t+2,4t+3} = \alpha_{4t+3,4t+2} = \alpha_{4t+4,4t+1} = 3$.

If n_2 is odd, it is easy to see that $n_2 = 2n_3$ is even.

We obtain the following matrix:

$$\mathcal{E} = \begin{pmatrix} \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{matrix} & * & \cdots & * \\ * & \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{matrix} & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{matrix} \end{pmatrix} =$$

$$= \begin{pmatrix} \mathcal{E}_2 & * & \cdots & * \\ * & \mathcal{E}_2 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & \mathcal{E}_2 \end{pmatrix}.$$

We will prove by induction the following two statements:

- for every k there exists the Latin square $\mathcal{E}_k$ of the order 2^k satisfying Main Theorem;

- for every k the number of the blocks $\mathcal{E}_k$ ($\mathcal{E}_k \neq \mathcal{E}$) is always even.

$$\mathcal{E}_k = \begin{pmatrix} \mathcal{E}_{k-1} & \mathcal{E}_{k-1} + X_{k-1} \\ \mathcal{E}_{k-1} + X_{k-1} & \mathcal{E}_{k-1} \end{pmatrix}.$$

The base of induction was proved.

Assume $\mathcal{E}$ contains n_k blocks $\mathcal{E}_k$ (each of the order 2^k) on the main block diagonal.

$$\mathcal{E}_k = \begin{pmatrix} \mathcal{E}_{k-1} & \mathcal{E}_{k-1} + 2^{k-1}U_k \\ \mathcal{E}_{k-1} + 2^{k-1}U_k & \mathcal{E}_{k-1} \end{pmatrix}.$$

From $\alpha_{ij} = 2^k$ follows that $|i-j| \leqslant 2^k$. It is easy to see $\alpha_{ij} = 2^k$ for $j = 2^k + i$ or $i = 2^k + j$. More precisely

$$\alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+2^k+i} = 2^k$$

$$\alpha_{t\cdot 2^{k+1}+2^k+i,t\cdot 2^{k+1}+i} = 2^k$$

for $t > 0$, $i = 1, 2, \ldots, 2^k$.

It is easy to see that $\alpha_{j,2^k+j} = 2^k$ for $j = 1, 2, \ldots, 2^k$. Analogously, $\alpha_{2^k+i,i} = 2^k$ for $i = 1, 2, \ldots, 2^k$ and

$$\alpha_{t2^{k+1}+i,t2^{k+1}+2^k+i} = 2^k,$$

$$\alpha_{t2^{k+1}+2^k+i,t2^{k+1}+i} = 2^k$$

for $t > 0$, $i = 1, 2, \ldots, 2^k$.

Besides, the blocks $\mathcal{E}_k$ unify pairwise into blocks Y_i, such that in each row and in each column of whose there is an element 2^k. If the number n_k of blocks is odd, then it is impossible to allocate the numbers in the last 2^k rows and columns of the matrix $\mathcal{E}$ in such a way that they appear in each row and each column only once (in the block $\mathcal{E}_{k-1}$ they cannot be allocated). Therefore, the number n_k is even.

From the inequalities

$$2^k \leqslant \alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+2^k+j} \leqslant \alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+j} + \alpha_{t\cdot 2^{k+1}+j,t\cdot 2^{k+1}+2^k+j} =$$
$$= \alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+j} + 2^k < 2^{k+1};$$
$$2^k \leqslant \alpha_{t\cdot 2^{k+1}+2^k+i,t\cdot 2^{k+1}+j} \leqslant \alpha_{t\cdot 2^{k+1}+2^k+i,t\cdot 2^{k+1}+i} + \alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+j} =$$
$$= 2^k + \alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+j} < 2^{k+1}$$

for $j = 1, 2, \ldots, 2^k$ we have

$$Y_i = \begin{pmatrix} \mathcal{E}_k & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_k \end{pmatrix},$$

where $\mathcal{E}_{12}$, $\mathcal{E}_{21}$ are Latin squares.

Since $2^kU_k \leqslant \mathcal{E}_{12} < 2^{k+1}$ then $0 \leqslant \mathcal{E}_{12} - 2^kU_k < 2^k$, and $\mathcal{E}_{12} - 2^kU_k$ is a Latin square on the set $\{1, 2, \ldots, 2^k - 1\}$.

Since $2^kU_k \leqslant \mathcal{E}_{21} < 2^{k+1}$ then $0 \leqslant \mathcal{E}_{21} - 2^kU_k < 2^k$, and $\mathcal{E}_{21} - 2^kU_k$ is a Latin square on the set $\{1, 2, \ldots, 2^k - 1\}$.

Furthermore

$$\begin{aligned}\alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+2^k+1} &\leqslant \\ &\leqslant \alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+1} + \alpha_{t\cdot 2^{k+1}+1,t\cdot 2^{k+1}+2^k+1} = \\ &= (i-1) + 2^k = 2^k + i - 1\end{aligned}$$

$$\begin{aligned}\alpha_{t\cdot 2^{k+1}+2^k+i,t\cdot 2^{k+1}+1} &\leqslant \\ &\leqslant \alpha_{t\cdot 2^{k+1}+2^k+i,t\cdot 2^{k+1}+i} + \alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+1} = \\ &= 2^k + (i-1) = 2^k + i - 1\end{aligned}$$

for all $i = 1, 2, \ldots, 2^k$. For $i = 1, 2, \ldots, 2^k$ we have that

$$\alpha_{t\cdot 2^{k+1}+i,t\cdot 2^{k+1}+2^k+1} = 2^k + i - 1$$
$$\alpha_{t\cdot 2^{k+1}+2^k+i,t\cdot 2^{k+1}+1} = 2^k + i - 1$$

for all $i = 1, 2, \ldots, 2^k$. So the first row (the first column) of the matrices $\mathcal{E}_{12}$ and $\mathcal{E}_{21}$ has the following form $(0, 1, \ldots, 2^k - 1)$ $((0, 1, \ldots, 2^k - 1)^T)$.

By induction we have $\mathcal{E}_{12} - 2^kU_k = \mathcal{E}_{21} - 2^kU_k = \mathcal{E}_k$ and blocks

$$\mathcal{E}_{k+1} = \begin{pmatrix} \mathcal{E}_k & \mathcal{E}_k + X_k \\ \mathcal{E}_k + X_k & \mathcal{E}_k \end{pmatrix}$$

are on the main diagonal.

Thus n_k is divided on two. The following blocks

$$\mathcal{E}_{k+1} = \begin{pmatrix} \mathcal{E}_k & \mathcal{E}_k + 2^kU \\ \mathcal{E}_k + 2^kU & \mathcal{E}_k \end{pmatrix}$$

are on the diagonal of the matrix $\mathcal{E}$. The induction hypothesis is proved.

We have shown that the number n_k of the blocks $\mathcal{E}_k$ on the diagonal $\mathcal{E}$ is $n_{k+1} = \frac{n_k}{2}$ and $n = 2^kn_k$. For some $k = m$ we have $n_m = 1$. Then $n = 2^m$ and $\mathcal{E} = \mathcal{E}_m$.

$$\mathcal{E} = \mathcal{E}_m = \begin{pmatrix} \mathcal{E}_{m-1} & \mathcal{E}_{m-1} + X_{m-1} \\ \mathcal{E}_{m-1} + X_{m-1} & \mathcal{E}_{m-1} \end{pmatrix}.$$

□

Remark. *Example I* shows, that for the Main Theorem the condition for a Latin square to be with the first row and the first column of the form $(01 \ldots n-1)$ is essential.

5. Admissible quivers

Let P be an arbitrary poset. A subset of P is called **a chain** if any two of its elements are related. A subset of P is called **an antichain** if no two distinct elements of the subset are related.

We shall denote a chain of n elements by CH_n and an antichain of n elements by ACH_n.

Theorem 5.1. [7], [12] *Given a poset, the minimal number of disjoint chains which together contain all elements of P is equal to the maximal number of elements in an antichain, if this number is finite.*

Definition 5.2. [12] *The maximal number $w(P)$ of elements in an antichain of P is called the width of P.*

With a reduced exponent $(0,1)$-matrix $\mathcal{E}$ we associate the partially ordered set

$$P_{\mathcal{E}} = \{1, \ldots, n\}$$

with the relation $\leqslant$ defined by the formula: $i \leqslant j \Leftrightarrow \alpha_{ij} = 0$.

Conversely, with any finite poset $P = \{1, \ldots, n\}$ we relate the reduced $(0,1)$-matrix $\mathcal{E}_P = (\lambda_{ij})$ by the following way: $\lambda_{ij} = 0 \Leftrightarrow i \leqslant j$, otherwise, $\lambda_{ij} = 1$.

Let CH_m be a linearly ordered set CH_n. Then $Q(\mathcal{E}_{CH_n})$ is a simple cycle with n vertices. Let

$$ACH_n = \left\{ \begin{matrix} 1 & 2 & \ldots & n-1 & n \\ \bullet & \bullet & \ldots & \bullet & \bullet \end{matrix} \right\}$$

be an antichain of width n. Then $Q(\mathcal{E}_{ACH_n})$ is a complete simply laced quiver with n vertices.

Theorem 5.3. *For every natural m $(1 \leqslant m \leqslant n,\ m \neq n-1)$ there exists an admissible quiver with n vertices and exactly m loops.*

Let $P = ACH_m \cup CH_{n-m}$ $(m \neq n-1)$. Obviously, $Q(\mathcal{E}_P)$ is an admissible quiver with n vertices and exactly m loops.

Remark. It is easy to see that there is no admissible quiver with $n-1$ loops.

Theorem 5.4. *There exists an exponent matrix M_k such that $\operatorname{inx} M_k = k$ for any $1 \leqslant k \leqslant n$.*

Proof. We saw that $inx\, H_n = 1$ and $inx\, \mathcal{E}_{ACH_n} = n$. Let τ be the cyclic permutation:

$$\tau = \begin{pmatrix} 1 & 2 & \dots & n \\ n & 1 & \dots & n-1 \end{pmatrix}.$$

Then the quiver Q_k $(k \geqslant 2)$ with the adjacency matrix: $E + P_\tau + \ldots + P_{\tau^{k-1}}$ is admissible by Theorem 1.3. So, there exists an exponent matrix M_k and $inx\, M_k = k$. □

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