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Nilpotent subsemigroups of a semigroup of order-decreasing transformations of a rooted tree

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Abstract. This paper deals with a semigroup of order-decreasing transformations of a rooted tree. Such are the transformations α of some rooted tree G satisfying following condition: for any x from G $\alpha(x)$ belongs to a simple path from x to the root vertex of G. We describe all subsemigroups of the mentioned semigroup, which are maximal among nilpotent subsemigroups of nilpotency class k in our semigroup. In the event when rooted tree is a ray we prove that all these maximal subsemigroups are pairwise nonisomorphic.

Introduction

Let T be a rooted tree with a natural partial order defined on the set of vertices (i.e. $x < y$ if x belongs to a simple path from y to the root of the tree). Let $\mathcal{T}_T$ be a symmetric semigroup of all transformations of set of vertices of the rooted tree T. We do transformation from left to right, i.e. $(\varphi \cdot \psi)(x) = \psi(\varphi(x))$. A transformation $\alpha \in \mathcal{T}_T$ is called an order-decreasing transformation if for any x from T an inequality $\alpha(x) \leqslant x$ holds. It is easy to see that the set D_T of all order-decreasing transformations from $\mathcal{T}_T$ forms a semigroup. In case of T is a finite chain this semigroup is called D_n. The semigroup D_n has been studied by many algebraists. Being introduced in Pin's monograph([4]) in connection with some problems of formal languages it was later considered by Howie at

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his lectures given in the University of Lisbon on combinatoric and arithmetical problems of the theory of transformation semigroup (some combinatoric results on D_n can be viewed in [3]) and also by Higgings. Umar wrote a series of papers (see, e.g. [5], [6]), investigating ideals, Rees congruences, idempotent rank and Green relations on D_n. More general semigroup of all contraction endomorphisms of arbitrary finite graph was considered by Vernitskii ([7]).

As D_T contains a zero 0, a transformation mapping all the vertices into the root, a question on study of non-trivial nilpotent subsemigroups from D_T naturally arises. For any mapping s from some nilpotent subsemigroup of D_T, we name by domain of s ($doms$) the set of vertices, which s does not map into the root; by the range of s ($rans$) we name the set of non-root vertices from $s(T)$; and by the rank of s we name the number of elements of $rans$. Let $Nil(T,k)$ denote the set of subsemigroups from D_T, which are maximal among nilpotent subsemigroups from D_T of nilpotency class k. The case when the tree is a finite chain was investigated in [8]. In our paper we describe all the semigroups from $Nil(T,k)$, and prove that all these semigroups are pairwise nonisomorphic in case when rooted tree T is a ray. Proving that we used the method of matching of nilpotent subsemigroups of the transformations semigroup to special partially ordered sets, this method first appeared in [2] and is explicitly described in [1].

1. The structure of maximal nilpotent subsemigroups from D_T

Let m be a vertex of T and A be a subset of the set of all vertices of T and $m \notin A$. Then we denote by $Less(m,A)$ the set of all vertices from A less than m; by $Up(m,A)$ we denote the set of all vertices from A greater than m. By $less(m,A)$ and $up(m,A)$ we denote cardinalities of sets $Less(m,A)$ and $Up(m,A)$ correspondingly; by the $Less(m)$ and $Up(m)$ the sets $Less(m, T\setminus\{m\})$ and $Up(m, T\setminus\{m\})$ correspondingly. We fix some natural k less than the number of vertices of T and define $\Lambda(T,k)$ as a set of ordered partitions (i.e. with defined order of blocks (subsets)) of the non-root vertices of T into k nonempty non-overlapping blocks $Q_1,\dots,Q_k$, such that

$$\forall 1 \leqslant i < k, \quad \forall l \in Q_i \quad \exists m \in Q_{i+1} \quad m < l; \tag{1.1}$$

and $\forall 1 < i \leqslant k$, $\forall h \in Q_i$

$$(Q_i \cap Less(h) \neq \varnothing) \Rightarrow (\exists l_1 \in Q_1, \cdots, \exists l_{i-1} \in Q_{i-1} \quad l_1 > \cdots > l_{i-1} > h) \tag{1.2}$$

Let's denote the root of tree T as r. For some partition λ from $\Lambda(T,k)$ with blocks $Q_1,\dots,Q_k$ we consider
$\mathcal{T}_\lambda=\{\varphi\in D_T|\forall m\leqslant k, \forall i\in Q_m \quad \varphi(i)\in(Q_{m+1}\cup\dots\cup Q_k\cup\{r\})\cap Less(i)\}$.

It is easy to verify that $\mathcal{T}_\lambda$ is a subsemigroup from D_T.

Lemma 1. $\mathcal{T}_\lambda\in Nil(T,k)$.

Proof. For any $\varphi_1,\dots,\varphi_k$ from $\mathcal{T}_\lambda$ and for any non-root vertex i from T we have:

$$\varphi_1(i)\in Q_2\cup\dots\cup Q_k\cup\{r\};\quad \varphi_2(\varphi_1(i))\in Q_3\cup\dots\cup Q_k\cup\{r\};\ \dots;$$

$$\varphi_k(\varphi_{k-1}(\dots\varphi_1(i)\dots))\in\{r\}.$$

Therefore $\mathcal{T}_\lambda$ is nilpotent of nilpotency class not greater than k. Simultaneously, one can choose $k-1$ elements from $\mathcal{T}_\lambda$, such that their product is not equal to zero. (e.g., one can select $\varphi_1^*,\dots,\varphi_{k-1}^*$, such that for some l_1 from Q_1 $\varphi_1^*(l_1)=l_2\in Q_2$, $\varphi_2^*(l_2)=l_3\in Q_3,\dots,\varphi_{k-1}^*(l_{k-1})=l_k\in Q_k$, $l_k\neq r$. The existence of $l_2\in Q_2,\dots,l_k\in Q_k$ such that $l_1<l_2<\dots<l_k$, follows from the definition of λ, (1.1). Then $\varphi_1^*\cdot\varphi_1^*\cdot\ldots\cdot\varphi_{k-1}^*\neq 0$).

Hence we have that $\mathcal{T}_\lambda$ is of nilpotency class k. Now we show the maximality of $\mathcal{T}_\lambda$. Indeed, let $\mathcal{T}_\lambda$ be contained in some semigroup $\mathcal{T}$ from $Nil(T,k)$ and $\mathcal{T}\neq\mathcal{T}_\lambda$. We consider ψ from $\mathcal{T}\setminus\mathcal{T}_\lambda$. Then there exist block Q_m and vertex $i\in Q_m$, such that $\xi=\psi(i)$ belongs to $Q_1\cup\dots\cup Q_m$. From (1.1) it follows that
there exists $\varphi_1\in\mathcal{T}_\lambda\setminus\{0\}$ such that $\varphi_1(\xi)\in Q_{m+1}$;
there exists $\varphi_2\in\mathcal{T}_\lambda\setminus\{0\}$ such that $\varphi_2(\varphi_1(\xi))\in Q_{m+2}$;
$\dots$;
there exists $\varphi_{k-m}\in\mathcal{T}_\lambda\setminus\{0\}$ such that $\varphi_{k-m}(\dots\varphi_1(\xi)\dots)\in Q_k$.
Next, if $m=1$ then $\psi\cdot\varphi_1\cdot\ldots\cdot\varphi_{k-m}(i)\in Q_k$, otherwise from $\xi\in Q_1\cup\ldots\cup Q_m$ and (1.1) it follows that $Q_m\cap Less(i)\neq\varnothing$. Then
there exist $\psi_1\in\mathcal{T}_\lambda$, $i_1\in Q_{m-1}$, such that $\psi_1(i_1)=i$;
there exist $\psi_2\in\mathcal{T}_\lambda$, $i_2\in Q_{m-2}$; such that $\psi_2(i_2)=i_1$;
$\dots$;
there exist $\psi_{m-1}\in\mathcal{T}_\lambda$, $i_{m-1}\in Q_1$, such that $\psi_{m-1}(i_{m-1})=i_{m-2}$.
Then $\psi_1\cdot\psi_2\ldots\cdot\psi_{m-1}\cdot\psi\cdot\varphi_1\cdot\ldots\cdot\varphi_{k-m}(i_{m-1})\in Q_k$. So, we have come to contradiction with the condition $\mathcal{T}\in Nil(T,k)$. □

Let $\mathcal{T}$ be a semigroup from $Nil(T,k)$. We define partial order $<^{\mathcal{T}}$ on T as following:

$$i<^{\mathcal{T}} j\Leftrightarrow\exists\varphi\in\mathcal{T}\quad\varphi(j)=i.$$

Since $\mathcal{T}$ is a subsemigroup from D_T then obviously $i<^{\mathcal{T}} j$ implies $i<j$. For some vertex m from T let $Less_{\mathcal{T}}(m)$ stand for the set of all vertices

j from T from T such that $j <^{\mathcal{T}} m$. Next we consider following sets

$$P_1 = \{i \in T \setminus \{r\} \mid Less_{\mathcal{T}}(i) = \{r\}\};$$

$$P_2 = \{i \in T \setminus (\{r\} \cup P_1) \mid Less_{\mathcal{T}}(i) \subset (P_1 \cup \{r\})\};$$

$$P_3 = \{i \in T \setminus (\{r\} \cup P_1 \cup P_2) \mid Less_{\mathcal{T}}(i) \subset (P_1 \cup P_2 \cup \{r\})\};$$

$$\cdots;$$

$$P_p = \{i \in T \setminus (\{r\} \cup P_1 \cup \cdots \cup P_{p-1}) \mid Less_{\mathcal{T}}(i) \subset (P_1 \cup P_2 \cdots \cup P_{p-1} \cup \{r\})\};$$

$$\cdots$$

Obviously, $P_1 \cup \cdots \cup P_p \cup \cdots = T \setminus \{r\}$. Let p_{max} be the greatest among indices p, for which $P_p \neq \varnothing$. From the fact, that $\mathcal{T}$ is a semigroup of nilpotency class k we conclude that $p_{max} = k$. What is more, for any i, j less than k we have that $P_i \cap P_j = \varnothing$.

Sets $Q_1 = P_{p_{max}}, \ldots, Q_{p_{max}} = P_1$ form a partition of set $T \setminus \{1\}$ written in following as $\lambda_{\mathcal{T}}$. (3)

Lemma 2. *For any semigroup $\mathcal{T} \in Nil(T, k)$, partition $\lambda_{\mathcal{T}}$ belongs to $\Lambda(T, k)$.*

Proof. Let i be from Q_l, $l < k$. Then there exist $\varphi \in \mathcal{T}$, $j \in Q_{l+1}$, such that $\varphi(i) = j$; hence $j < i$. Thus $\lambda_{\mathcal{T}}$ meets the requirement (1.1). Next, if for all $w > 1$ there are no vertices m and l from the block Q_w such that $m < l$, then $\lambda_{\mathcal{T}}$ satisfies condition (1.2). Now, let $m, h \in Q_w$ and $h < m$. We take a transformation φ with $dom\varphi = m$, $ran\varphi = h$. It is easy to see that φ belongs to D_T. Let there be no sequence $q_1 \in Q_1, \ldots, q_{w-1} \in Q_{w-1}$ satisfying $q_1 > \ldots > q_{w-1} > m$. Then for any $\phi_1, \phi_2, \ldots, \phi_k$ elements from the semigroup $< \mathcal{T}, \varphi >$ (obtained from adjoining φ to $\mathcal{T}$) it is true that $\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_k = 0$. Since $\mathcal{T}$ is maximal, we have come to the contradiction. Thus (1.2) must be satisfied and $\lambda_{\mathcal{T}}$ belongs to $\Lambda(T, k)$. □

Theorem 1. *There are reciprocal mappings φ and ψ which set up one-to-one correspondence between $\Lambda(T, k)$ and $Nil(T, k)$ defined as follows*

1. $\varphi : \Lambda(T, k) \to Nil(T, k)$, $\forall \lambda \in \Lambda(T, k)$ $\varphi(\lambda) = \mathcal{T}_\lambda$ *(see (1.1)-(1.2))*
2. $\psi : Nil(T, k) \to \Lambda(T, k)$, $\forall \mathcal{T} \in Nil(T, k)$ $\psi(\mathcal{T}) = \lambda_{\mathcal{T}}$ *(see (3)).*

Proof. Let λ be a partition from $\Lambda(T, k)$ with blocks $Q_1, \ldots, Q_k$. We consider $\psi(\varphi(\lambda)) = \psi(\mathcal{T}_\lambda)$. It is a partition from $\Lambda(n, k)$ with blocks $Q_1^\psi, \ldots, Q_k^\psi$. For any j from the block Q_k we have: $Less_{\mathcal{T}_\lambda}(j) = \{r\}$. Therefore $Q_k \subset Q_k^\psi$. Now we take an arbitrary i from $T \setminus Q_k$. From the definition of $\mathcal{T}_\lambda$ it follows that there exists β from $\mathcal{T}_\lambda$, such that $i \in dom\beta$. Therefore $Less_{\mathcal{T}_\lambda}(i) \neq \{r\}$ and $Q_k = Q_k^\psi$. Next, for any j from Q_{k-1} we have: $Less_{\mathcal{T}_\lambda}(j) \subset Q_k \cup \{r\} = Q_k^\psi \cup \{r\}$. Therefore $Q_{k-1} \subset Q_{k-1}^\psi$. Now we take an arbitrary i from $T \setminus (Q_k \cup Q_{k-1})$. From the definition of $\mathcal{T}_\lambda$ it follows that there exists β from $\mathcal{T}_\lambda$, such that $i \in dom\beta$ and $\beta(i) \in Q_{k-1}$. Therefore $i \notin Q_{k-1}^\psi$ and $Q_{k-1} = Q_{k-1}^\psi$.

Further we move by induction. Let an equality $Q_{k-l+1} = Q_{k-l+1}^\psi$ be held. For any i from Q_{k-l} an inclusion $Less_{\mathcal{T}_\lambda}(i) \subset Q_k \cup \cdots \cup Q_{k-l+1} \cup \{r\}$ holds. Simultaneously, for any i from $T \setminus (Q_k \cup \ldots \cup Q_{k-l})$ there exists β from $\mathcal{T}_\lambda$, such that $i \in dom\beta$ and $\beta(i) \in Q_{k-l}$. Therefore $Q_{k-l} = Q_{k-l}^\psi$. Hence $\psi(\varphi(\lambda)) = \lambda$. Now let us take some $\mathcal{T}$ from $Nil(T,k)$. We consider $\varphi(\psi(\mathcal{T})) = \varphi(\lambda_\mathcal{T})$. It is a semigroup from $Nil(T,k)$. $\lambda_\mathcal{T}$ is a partition from $\Lambda(T,k)$ with blocks $Q_1, \ldots, Q_k$. Let us take some α from $\mathcal{T}$. For any element j from the block Q_k we have that $\alpha(j) = r$ (for the definition of $\lambda_\mathcal{T}$). Let now $j \in Q_{k-1}$. Then $\alpha(j)$ belongs to the set $Q_k \cup \{r\}$ (for the definition of $\lambda_\mathcal{T}$). Next, for the definition of $\lambda_\mathcal{T}$ for any element i from the block Q_{k-l} we have that $\alpha(i)$ does not belong to any of the sets $Q_1, Q_2, \ldots, Q_{k-l}$. Hence we get that $\mathcal{T}$ belongs to $\varphi(\lambda_\mathcal{T})$. As $\mathcal{T}$ is maximal among the nilpotent subsemigroups from D_n of nilpotency class k, then $\mathcal{T} = \varphi(\psi(\mathcal{T})) = \varphi(\mathcal{T}_\lambda)$. □

2. Equivalence relations $\sim^R$ and $\sim^L$ and their properties

Here and in the following we consider the case when T is a ray. Surely, one can number vertices in such a way that T is isomorphic to the set of natural numbers $\mathbb{N}$. Let's define relations $\sim^R$ and $\sim^L$ on some $\mathcal{T} \in Nil(\mathbb{N},k)$ as follows: for any elements x, y from $\mathcal{T} \in Nil(\mathbb{N},k)$

1. $x \sim^R y \Leftrightarrow$ for all $t \in \mathcal{T}$ $tx = ty$;
2. $x \sim^L y \Leftrightarrow$ for all $t \in \mathcal{T}$ $xt = yt$.

It is easy to prove that $\sim^R$ and $\sim^L$ are equivalence relations.

Proposition 1. *1.* $x \sim^R y \Leftrightarrow \forall m \in \mathbb{N} \setminus Q_1 \quad (x(m) = y(m))$*;*

2. $x \sim^L y \Leftrightarrow \forall m \in \mathbb{N}$ *if* $x(m) \in \bigcup_{i=1}^{k-1} Q_i$ *then* $x(m) = y(m)$*,*
if $x(m) \in Q_k$ *then* $y(m) \in Q_k$*.*

Proof. Let x and y be from the semigroup $\mathcal{T}$ and $\forall t \in \mathcal{T} : tx = ty$. For any m from $\mathbb{N} \setminus Q_1$ let s be from $\mathcal{T}$ such that $rans = m$. Then $sx = sy$ implies $x(m) = y(m)$.

Simultaneously, let x and y be from $\mathcal{T}$ and for any m from $\mathbb{N} \setminus Q_1$ $x(m) = y(m)$. Let's take an arbitrary element s from $\mathcal{T}$. Then $domsx = domsy$ and $domsx \in \mathbb{N} \setminus Q_1$, so $sx = sy$ and $x \sim^R y$. The second part of the proposition can be proved analogously. □

Corollary 1. *Let $\mathcal{T}$ be a semigroup from $Nil(\mathbb{N}, k)$ with the correspondent partition λ from $\Lambda(\mathbb{N}, k)$. Then all the blocks of the partition λ except Q_1 are finite if and only if the number of equivalency classes generated by the equivalency relation $\sim^R$ on the semigroup $\mathcal{T}$ is finite.*

Proof. It is obvious that if at least one of the blocks $Q_2, \ldots, Q_k$ is infinite, then $\mathcal{T}$ has infinite number of equivalence classes for the relation $\sim^R$. Simultaneously, if all the blocks Q_2, Q_3,…,Q_k are finite then the number of equivalence classes is also finite and equals

$$\prod_{m \in Q_i,\ 2 \leqslant i \leqslant k} (less(m, Q_{i+1} \cup \cdots \cup Q_k) + 1).$$

□

3. Non-isomorphism theorem

Theorem 2. *Let $k > 2$. Then all semigroups from $Nil(\mathbb{N}, k)$ are pairwise non-isomorphic.*

Proof. We show that it is possible to restore the correspondent partition λ from $\Lambda(\mathbb{N}, k)$ from the properties of an arbitrary semigroup from $Nil(\mathbb{N}, k)$ as an abstract semigroup. To do this, we use induction for nilpotency class k. First we consider the case of $k = 3$. Let $\mathcal{T}$ be a semigroup from $Nil(\mathbb{N}, 3)$ with a correspondent partition $\lambda \in \Lambda(\mathbb{N}, 3)$, which consists of the blocks

$$Q_1 = \{\ldots, a_i, \ldots, a_2, a_1\} \quad (a_1 < a_2 < \ldots < a_i < \ldots),$$

$$Q_2 = \{\ldots, b_i, \ldots, b_2, b_1\} \quad (b_1 < b_2 < \ldots < b_i < \ldots),$$

$$Q_3 = \{\ldots, c_i, \ldots, c_2, c_1\} \quad (c_1 < c_2 < \ldots < c_i < \ldots).$$

Let's show that

$$\min_{s \in \mathcal{T},\ |s\mathcal{T}| \neq 1,\ |s\mathcal{T}| < \infty} |s\mathcal{T}| = less(b_1, Q_3) + 1.$$

Indeed, if $|s\mathcal{T}| \neq 1$, then there exist a and b ($a \in Q_1$, $b \in Q_2$) such that $s(a) = b$. Then for any $c \in Q_3 \cap Less(b, Q_3)$ an ideal $s\mathcal{T}$ contains an element of rank 1 that maps a into c. Therefore $|s\mathcal{T}| \geqslant less(b, Q_3) + 1 \geqslant$

$less(b_1, Q_3) + 1$.
On the other hand, $\mathcal{T}$ contains a mapping of range b_1. At the same time, $|s_0\mathcal{T}| = less(b_1, Q_3) + 1$. Therefore we have:

$$\min_{s\in\mathcal{T},\ |s\mathcal{T}|\neq 1,\ |s\mathcal{T}|<\infty} |s\mathcal{T}| = less(b_1, Q_3) + 1.$$

For each subset A of Q_1 there is a right ideal $s\mathcal{T}$, satisfying $|s\mathcal{T}| = less(b_1, Q_3) + 1$:

$$s\mathcal{T} = \big\{\varphi \in \mathcal{T},\ |ran\varphi| = 1, dom\varphi = A, ran\varphi \in Less(b_1, Q_3)\big\},$$

and there is no other ideal $s\mathcal{T}$ of cardinality $less(b_1, Q_3) + 1$. We denote the set of such ideals by Θ_1. By B_1 we stand for the set of all numbers b of Q_2, for which $less(b, Q_3) = less(b_1, Q_3)$. Next, let
$W = \{s \in \mathcal{T} |\text{if for some } t_1 \in \mathcal{T}\ st_2 \neq 0 \text{ and } t_1t_2 \neq 0 \text{ then for all } t_3 \in \mathcal{T}, st_3 \neq 0 \Rightarrow t_1t_3 \neq 0\}$.
It is easy to verify that for all s from W $|rans \cap Q_2| = 1$.

For any set X from Θ_1 we consider the number of equivalence classes of $\sim^L$ on the set of elements s from W such that $s\mathcal{T} = X$. If $s\mathcal{T} \in \Theta_1$ for some s of W, then there exists only one b of Q_2, which belongs to $rans$, and $b \in B_1 \cap Less\Big(\min\limits_{a\in s^{-1}(b)} a, Q_2\Big)$, as $B_1 \cap \Bigg(\bigcap\limits_{a\in s^{-1}(b)} Less(a, Q_2)\Bigg) = B_1 \cap Less\Big(\min\limits_{a\in s^{-1}(b)} a, Q_2\Big)$ holds. Using proposition 1 we can conclude that among the numbers $\Big|\{s \in W, sT = X\}/_{\sim^L}\Big|$ of equivalency classes for the relation $\sim^L$ on the set $\{s \in W |\ s\mathcal{T} = X\}$ where $X \in \Theta_1$ one can find only the numbers $|B_1|$ and $|less(a, Q_2)|$, where $|less(a, Q_2)| < |B_1|$, $a \in Q_1$. Hence we can say whether the set B_1 is finite or not. Next, from the abstract properties of $\mathcal{T}$ we can get numbers
$less(b_1, Q_3) = \cdots = less\big(b_{|B_1|}, Q_3\big)$, $|B_1|$ and the set of numbers
$\Omega_1 = \{\alpha_1, \ldots, \alpha_{i_1}\} = \{less(a, Q_2)|a \in Less(b_{|B_1|}, Q_1)\}$.
Now we consider

$$\Theta_2 = \Bigg\{s\mathcal{T}\ :\ \exists X \in \Theta_1,\ s\mathcal{T} = \bigcap_{\tau\in\mathcal{T},\ \tau\mathcal{T}\cap\Theta_1 = X,\ X\neq\tau\mathcal{T}} \tau\mathcal{T}\Bigg\}.$$

If Θ_2 is an empty set, then $B_1 = Q_2$. If $\Theta_2 \neq \varnothing$, then $Q_2 \setminus B_1 \neq \varnothing$ and for every X from Θ_2 equality $|X| = less\big(b_{|B_1|+1}, Q_3\big)$ holds. Indeed, if $X = s\mathcal{T}$ belongs to Θ_2, then there exists only one element b from the second block of the partition λ, which belongs to $rans$, because otherwise X would have two different ideals from Θ_1. It is easy to see that $b \in Q_2 \setminus B_1$. In such a case $|X| = less\big(b, Q_3\big) \geqslant less\Big(b_{|B_1|+1}, Q_3\Big)$. Let s_0 be an element of rank 1 and $rans_0 = b_{|B_1|+1}$. Then $|s_0\mathcal{T}| = less(b_{|B_1|+1}, Q_3)$ and so $|X| = less\Big(b_{|B_1|+1}, Q_3\Big)$.

We define set B_2 as following:

$$B_2 = \Big\{b \in Q_2, less(b, Q_3) = less\big(b_{|B_1|+1}, Q_3\big)\Big\}.$$

For any set X of Θ_2 we consider the number of equivalency classes for the relation $\sim^L$ on the set of elements s from $\mathcal{T}$ such that $s\mathcal{T} = X$. If $s\mathcal{T} \in \Theta_2$ for s of $\mathcal{T}$, then there exists only one b of Q_2, which belongs to $rans$, and as $B_2 \cap \left(\bigcap\limits_{a\in s^{-1}(b)} Less(a, Q_2)\right) = B_2 \cap Less\left(\min\limits_{a\in s^{-1}(b)} a, Q_2\right)$, then $b \in B_2 \cap \left(\bigcap\limits_{a\in s^{-1}(b)} Less(a, Q_2)\right)$. Hence we conclude that among the numbers $\Big|\{s \in \mathcal{T}|sT = X\}/_{\sim^L}\Big|$ of equivalence classes for the relation $\sim^L$ on the set $\{s \in \mathcal{T}|\ s\mathcal{T} = X\}$ for all $X \in \Theta_2$ there are numbers $|B_2|$ and $|less(a, Q_2)| - |B_1|$ $\big($where $a \in \big(Less(b_{|B_2|+|B_1|}, Q_1) \setminus Less(b_{|B_1|}, Q_1)\big)\big)$ only. Hence for the general properties of semigroup $\mathcal{T}$ we can say whether the set B_2 is finite or not. Next, we get numbers $less(b_{|B_1|+1}, Q_3)= \dots$ $=less\big(b_{|B_1|+|B_2|}, Q_3\big)$, $|B_2|$ and the set of numbers $\Omega_2 = \big\{\alpha_1^2, ..., \alpha_{i_2}^2\} =$

$$= \{less(a, Q_2)|a \in(Less(b_{|B_2|+|B_1|}, Q_1) \setminus Less(b_{|B_1|+1}, Q_1))\}$$

$$(\alpha_1^2 < \alpha_2^2 < \ldots < \alpha_i^2 < \alpha_{i+1}^2 < \ldots)$$

Then we define sets

$$\Theta_j = \{s\mathcal{T} \mid \exists X \in \Theta_{j-1},\ s\mathcal{T} = \bigcap_{\tau\in\mathcal{T},\ \tau\mathcal{T}\cap(\bigcup\limits_{m=1}^{j-1}\Theta_m)=X,\ \tau\mathcal{T}\neq X} \tau\mathcal{T})\};$$

$$B_j = \Big\{b \in Q_2,\ less(b, Q_3) = less\big(b_{|B_1|+\ldots+|B_{j-1}|+1}, Q_3\big)\Big\}.$$

If Θ_j is an empty set, then $\bigcup\limits_{i=1}^{j-1} B_i = Q_2$ and the process of considering Θ_j is finished. If not, then $Q_2 \setminus \bigcup\limits_{i=1}^{j-1} B_i \neq \varnothing$ and it is easy to prove by induction that for each set X of Θ_j $|X|=less(b_{|B_1|+|B_2|+\ldots+|B_{j-1}|+1}, Q_3)$. Indeed, if $X = s\mathcal{T}$ belongs to Θ_j, then there exists only one element b from the second block of the partition λ, which belongs to $rans$, because otherwise X has two different ideals from Θ_1 for some $i < j$. It is clear that $b \in Q_2 \setminus \bigcup\limits_{i=1}^{j-1} B_i$. In such a case $|X| = less(b, Q_3) \geqslant less(b_{|B_1|+|B_2|+\ldots+|B_{j-1}|+1}, Q_3)$. Let s_0 be an element of rank 1 and $rans_0 =$

$b_{|B_1|+|B_2|+\ldots+|B_{j-1}|+1}$. Then

$$|s_0\mathcal{T}| = less(b_{|B_1|+|B_2|+\ldots+|B_{j-1}|+1}, Q_3)$$

and so $|X| = less(b_{|B_1|+|B_2|+\ldots+|B_{j-1}|+1}, Q_3)$.
For each set X of Θ_j we consider number of equivalency classes for the relation $\sim^L$ on the set $\{s \in \mathcal{T}|\ s\mathcal{T} = X\}$. If $s\mathcal{T} \in \Theta_j$ for some s from $\mathcal{T}$, then there exists only one b of Q_2 which belongs to $doms$, and at that $b \in B_j \cap \left(\bigcap_{a\in s^{-1}(b)} Less(a, Q_2)\right) = B_j \cap Less\left(\min_{a\in s^{-1}(b)} a, Q_2\right)$.
Hence we have that among the numbers of equivalency classes for the relation $\sim^L$ on the set $\{s \in Ann_L\mathcal{T}|\ s\mathcal{T} = X\}$, where X is an element of Θ_j, there are numbers $|B_j|$ and $|less(a, Q_2)| - \sum_{i=1}^{j-1} |B_i|$ (where $a \in (Less(b_{|B_1|+\cdots+|B_j|}, Q_1) \setminus Less(b_{|B_1|+\cdots+|B_{j-1}|}, Q_1)))$ only. Hence we can say whether the set B_j is finite or not. So, for the general properties of $\mathcal{T}$ we can obtain numbers $less\left(b_{|B_1|+\cdots+|B_{j-1}|+1}, Q_3\right) = \cdots = less(b_{|B_1|+\cdots+|B_j|}, Q_3)$, $|B_j|$ and the set of numbers $\Omega_j = \{\alpha^j_1, \ldots, \alpha^j_{i_j}\} =$

$$=\left\{less(a, Q_2) - \sum_{i=1}^{j-1} |B_i| : a \in Less(b_{|B_1|+\cdots+|B_j|}, Q_1) \setminus Less(b_{|B_1|+\cdots+|B_{j-1}|}, Q_1)\right\}$$

At last we have next sets of numbers:

1. $\{less(b, Q_3), b \in Q_2\}$
2. $\{|B_1|, \ldots, |B_i|, \ldots\}$
3. $\Omega_j =$
$\left\{less(a, Q_2) - \sum_{i=1}^{j-1} |B_i| : a \in Less\left(b_{\left(\sum_{i=1}^{j}|B_i|\right)}, Q_1\right) \setminus Less\left(b_{\left(\sum_{i=1}^{j-1}|B_i|\right)}, Q_1\right)\right\}$.

It is clear that $\forall a \in Q_1$ either there exists α^l_m from some Ω_m such that $less(a, Q_2) = \alpha^l_m + |B_1| + \cdots + |B_{m-1}|$ or there exists such j that $less(a, Q_2) = |B_j|$.

Let's now consider such ideals X from Θ_1, for which

1. number of equivalence classes for the relation $\sim^L$ on the set $\{s \in \mathcal{T},\ s\mathcal{T} = X\}$ equals α^1_2 — the next to the least number of Ω_1;
2. the set $\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}$ is finite.

To each such ideal we conform a number $|\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}|$. Considering all sets X satisfying 1-2, we get some set of numbers $|\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}|$ with repetition, which we denote by $\Psi_{\alpha_1^1}$ (under a set with a repetition we mean a set where each number has it's repetition factor). If number of equivalence classes for the relation $\sim^L$ on the set $\{s \in \mathcal{T},\ s\mathcal{T} = X\}$ is equal to α_2^1, then for $A^1_{\alpha_1} = \{a \in Q_1 : less(a, Q_2) = \alpha_1^1\}$, $\big((s \in Ann_L\mathcal{T}) \wedge (s\mathcal{T} = X)\big)$ implies $s(A^1_{\alpha_1}) \subset Q_3$. Also for some a from Q_1 $less(a, Q_2) < |B_1|$ implies $Less(a, Q_3) = Less(b_{k_2}, Q_3)$ and therefore $less(a, Q_3) = less(b_{k_2}, Q_3)$. Thus the least element of $\Psi_{\alpha_1^1}$ is the number $\alpha_2^1\Big(less(b_{k_2}, Q_3)+1\Big)^{\left|A^1_{\alpha_1^1}\right|}$. Hence we can get $|A^1_{\alpha_1^1}|$ from the general properties of semigroup $\mathcal{T}$. Now let's consider ideals X from Θ_1 such that the number of equivalence classes for the relation $\sim^L$on the set $\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}$ equals α_3^1 — the number from Ω_1, next to α_2^1, and the set $\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}$ is finite. To each such ideal we conform the number $|\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}|$, considering all such X, we get some set of natural numbers $|\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}|$ with repetition, which we denote by $\Psi_{\alpha_2^1}$. Let $A^1_{\alpha_2^1} = \{a \in Q_1 : less(a, Q_2) = \alpha_2^1\}$. The least element of $\Psi_{\alpha_2^1}$ is the number $\alpha_3(less(b_{k_2}, Q_3)+1)^{|A^1_{\alpha_1}|+|A^1_{\alpha_2}|}$. Hence we get the number $|A^1_{\alpha_2}|$ from the abstract properties of $\mathcal{T}$. Now let $A^1_{\alpha_i} = \Big\{a \in Q_1 : less(a, Q_2) = \alpha_i^1\Big\}$ for every α_i^1 of Ω_1. Next we consider ideals X from Θ_1 such that the number of equivalence classes for the relation $\sim^L$ on the set $\{s \in \mathcal{T},\ s\mathcal{T} = X\}$ is equal to an element α_i^1 from Ω_1, and the set $\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}$ is finite. To each such ideal we conform a number $|\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}|$; taking all such X, we get some set of natural numbers $|\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}|$ with repetition, which we denote by $\Psi_{\alpha_i^1}$. The least element of $\Psi_{\alpha_i^1}$ is $\alpha_3\Big(less(b_{k_2}, Q_3)+1\Big)^{\sum_{l=1}^{i-1}|A^1_{\alpha_l}|}$. Therefore we can get $|A^1_{\alpha_{i-1}}|$ from the general properties of our semigroup. Let $A_{|B_1|}$ denote the set of all a from Q_1 such that $less(a, Q_2) = |B_1|$. Now let's assume that the set B_1 is finite. We investigate ideals X of Θ_1 for which equivalence classes for the relation $\sim^L$ on the set $\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}$ equals $|B_1|$, and the set $\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}$ is finite. To each such ideal we conform the number $|\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}|$. We get the set of numbers $|\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}|$ with repetition, let's denote it by Γ. Clearly, each mapping s from the left annulator $\mathcal{T}$, for which $s\mathcal{T} = X$, maps some nonempty subset from $Q_1 \setminus \bigcup_{\alpha_i^1 \in \Omega_1} A_{\alpha_i^1}$ into an element from B_1, and all the other elements from Q_1 — into elements from $Q_3 \cup \{1\}$. At that s maps elements $a \in \bigcup_{\alpha \in \Omega_1} A_\alpha$ into $Q_3 \cup \{1\}$, and at least one element of $A_{|B_1|}$ must be mapped into Q_2. Therefore Γ contains numbers of type $|B_1|\Big(\prod_{a \in A} \big(less(a, Q_3)+1\big)\Big)$, where $A \subset Q_1$, $A \cap A_{|B_1|} \neq A_{|B_1|}$, and $\bigcup_{\alpha_i^1 \in \Omega_1} A_{\alpha_i^1} \subset A$. The least element among all elements of Γ is $|B_1|\Bigg(less(b_{k_2}, Q_3) + 1\Bigg)^{\sum_{\alpha_i^1 \in \Omega_1} |A_{\alpha_i^1}|}$. Hence we get $|A_{\alpha_{i_1}}|$ (α_{i_1} is the greatest number of Ω_1). Let's denote the least element of Γ by ξ, and $\bigcup_{\alpha_i^1 \in \Omega_1} A_{\alpha_i^1}$ by A_{Ω_1}. Now we consider ideals X from Θ_2 such that number of equivalence classes for the relation $\sim^L$ on the set $\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}$ is equal to the least element α_1^2 of Ω_2; and the set $\{s \in Ann_L\mathcal{T} : s\mathcal{T} = X\}$ is finite. To each such ideal we conform the number $|\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}|$. We get some set of numbers with repetition, which we denote by $\Psi_{\alpha_1^2}$. If Θ_2 is empty, then $\Psi_{\alpha_1^2}$ is also empty and thus Q_2 is finite and for all $a \in Q_1$ $less(a, Q_3) = less(b_1, Q_3)$. If Θ_2 is not empty, then $\Psi_{\alpha_1^2}$ is not empty too, and the least element of $\Psi_{\alpha_1^2}$ is the number $\alpha_1^2\Bigg(less(b_1, Q_3) + 1\Bigg)^{\sum_{\alpha \in \Omega_1} |A_\alpha^1|}\Bigg(\prod_{a \in Q_1,\ less(s, Q_2) = |B_1|} (less(a, Q_3) + 1)\Bigg)$, in case of the set $A_{|B_1|} = \big\{a \in Q_1 :\ less(a, Q_2) = |B_1|\big\}$ is nonempty, and $\alpha_1^2(less(b_1, Q_3)+1)^{\sum_{\alpha \in \Omega_1} |A_\alpha^1|}$ otherwise. Hence we can say whether the set $A_{|B_1|}$ is empty, and if not we have the number

$$\prod_{a \in A_{|B_1|}} (less(a, Q_3) + 1).$$

Let

$$\eta = \begin{cases} \prod_{a \in A_{|B_1|}} \Big(less(a, Q_3) + 1\Big), & A_{|B_1|} = \varnothing; \\ 1, & A_{|B_1|} <> \varnothing. \end{cases}$$

Let's remove one number ξ from Γ. Now the least element of Γ and the one next to it are $\xi \cdot \big(less(a_{|A_{\Omega_1}|+1}, Q_3)+1\big)$ and $\xi \cdot \big(less(a_{|A_{\Omega_1}|+2}, Q_3)+1\big)$. So, we get numbers $less\big(a_{|A_{\Omega_1}|+1}, Q_3\big)$ and $less\big(a_{|A_{\Omega_1}|+2}, Q_3\big)$.
We remove the number $\xi \cdot \big(less(a_{|A_{\Omega_1}|+2}, Q_3) + 1\big)$ from Γ;
if $\eta \neq \big(less(a_{|A_{\Omega_1}|+1}, Q_3) + 2\big)$ then we take away a number $\xi \cdot less\big(a_{|A_{\Omega_1}|+1}, Q_3\big)$ from Γ.
Next, if $\eta \neq \big(less(a_{|A_{\Omega_1}|+2}, Q_3) + 1\big)\big(less(a_{|A_{\Omega_1}+1|}, Q_3) + 1\big)$ and $\eta \neq \big(less(a_{|A_{\Omega_1}|}, Q_3) + 1\big)$, then we take away the number

$\xi \cdot \big(less(a_{|A_{\Omega_1}|+1}, Q_3) + 1\big)\big(less(a_{|A_{\Omega_1}|+2}, Q_3) + 1\big)$ from Γ.
Now the least element of Γ is $\xi\big(less(a_{|A_{\Omega_1}|+3}, Q_3) + 1\big)$. So we get $less\big(a_{|A_{\Omega_1}|+3}, Q_3\big)$. We remove next numbers from Γ

- $\xi\big(less(a_{|A_{\Omega_1}|+3}, Q_3) + 1\big)$;
- $\xi\big(less(a_{|A_{\Omega_1}|+3}, Q_3) + 1\big)\big(less(a_{|A_{\Omega_1}|+1}, Q_3) + 1\big)$;
- $\xi\big(less(a_{|A_{\Omega_1}|+3}, Q_3) + 1\big)\big(less(a_{|A_{\Omega_1}|+1}, Q_3) + 1\big)$, if $\big(less(a_{|A_{\Omega_1}|+1}, Q_3) + 1\big) \neq \eta)$;
- $\big(less(a_{|A_{\Omega_1}|+3}, Q_3)+1\big)\big(less(a_{|A_{\Omega_1}|+1}, Q_3)+1\big)\big(less(a_{|A_{\Omega_1}|+2}, Q_3)+1\big)$, if $\big(less(a_{|A_{\Omega_1}|+1}, Q_3) + 1\big)\big(less(a_{|A_{\Omega_1}|+2}, Q_3) + 1\big) \neq \eta$ and $\big(less(a_{|A_{\Omega_1}|+3}, Q_3)+1\big)\big(less(a_{|A_{\Omega_1}|+1}, Q_3)+1\big)\big(less(a_{|A_{\Omega_1}|+2}, Q_3)+1\big)\neq\eta$.

Now the least element of Γ is $|B_1|\Big(less\big(a_{|A_{\Omega_1}|+4}, Q_3\big) + 1\Big)$. We remove each time the least element and it's products with already removed numbers from Γ. Gradually we obtain numbers $less(a, Q_3)$ for all numbers a of the first block of the partition.
Now let B_1 be an infinite set. Obviously, in such a case Q_3 must be finite and thus for every element a from Q_1 an equality $less(a, Q_3) = less(b_1, Q_3) + 1$ holds. Next, Ω_1 also is an infinite set and $Q_1 = \bigcup_{\alpha\in\Omega_1} A_\alpha$ (implies from the definition of the set $\Lambda(\mathbb{N}, k)$); considering minimal elements of described above sets $\Psi_{\alpha_i^1}(\alpha_i^1 \in \Omega_1)$ we can get cardinalities of sets $|A_{\alpha_i^1}|$.

For any natural n we denote by A_n the set

$$\{a \in A : less(a, Q_2) = n\}.$$

To each of sets Ω_j we add the number $\alpha_m^j = |B_j|$, $m = \max_{\alpha_i^j\in\Omega_j} i + 1$. Now we have some set Ω'_j. We consider ideals X of Θ_j such that the number of equivalence classes for the relation $\sim^L$ on the set $\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}$ equals some $\alpha_l^j \in \Omega'_j$, and the set $\{s \in Ann_L\mathcal{T} :\ s\mathcal{T} = X\}$ is finite. To every such ideal we conform the number $|\{s \in Ann_L\mathcal{T},\ s\mathcal{T} = X\}|$. Hence we get some set of numbers with repetition $\Phi_{\alpha_l^j}$ with the least element

$$\alpha_l^j\Bigg(\prod_{a\in Q_1:\ less(a,Q_2)\leqslant|B_{j-1}|}(less(a, Q_3)+1)\Bigg)\Bigg(less\Big(b_{|B_1|+\cdots+|B_{j-1}|+1}, Q_3\Big)+1\Bigg)^{\sum_{q=1}^{l-1}|A^j_{\alpha_q}|},$$

if α_l^j is not the least element of Ω_j'; and

$$\alpha_l^j \cdot \Bigg(\prod_{a\in Q_1\colon less(a,Q_2)\leqslant |B_{j-1}|} \big(less(a,Q_3)+1\big)\Bigg),$$

if α_l^j is the least element of Ω_j'. Hence we gradually get numbers $|A_{\alpha_l^j}|$ for all α_i^j from Ω_j. Now we divide the least element of the set $\Phi_{\alpha_m^{j-1}}$ by the least element of the set $\Phi_{\alpha_1^j}$ ($\alpha_m^{j-1} = \max\limits_{\alpha\in\Omega_{j-1}'} \alpha$). If the obtained number equals 1, then the set $A_{|B_j|}$ is empty; otherwise we get the number $\prod\limits_{a\in A_{|B_j|}} (less(a,Q_3)+1)$. As we already know numbers $A_{|\alpha|}$ where $\alpha \in \Omega_1' \cup \Omega_2$ and $less(a,Q_3)$ for each $a \in Q_1$, then we can find i such that the obtained number $\prod\limits_{a\in A_{|B_j|}} (less(a,Q_3)+1)$ is equal to the number

$$\prod_{\sum\limits_{\alpha\in\Omega_1'\cup\Omega_2} |A_\alpha|+1 \leqslant l \leqslant \sum\limits_{\alpha\in\Omega_1'\cup\Omega_2} |A_\alpha|+1+i} less(a_l,Q_3).$$

So, we get $|A_{|B_j|}| = i$. Analogously we get numbers $|A_{|B_j|}|$, $j > 2$.
It is necessary to note that if at some step B_j is an infinite set, then it means that the block Q_3 is finite and $Q_1 = \bigcup\limits_{\alpha\in \bigcup\limits_{1\leqslant i\leqslant j} \Omega_i} A_\alpha$. As for any a from Q_1 $less(a,Q_3)$ belongs to $\bigcup\limits_{i=1}^{\infty} \Omega_i'$, then for any a from Q_1 we have the number $less(a,Q_2)$. So, we get such numbers :

- $less(a,Q_2)\ \forall a \in Q_1$;
- $less(a,Q_3)\ \forall a \in Q_1$;
- $less(b,Q_3)\ \forall b \in Q_2$.

Now we show that one can obtain the elements of the blocks Q_1, Q_2, Q_3 from these numbers. Really, we can get all the numbers of the first block. Indeed, for some $a_j \in Q_1$ we have:

$$a_j = less(a_j,Q_3) + less(a_j,Q_2) + 1 + j.$$

Next, for $b_j \in Q_2$ we have that $less(b_j,Q_3) = |\{a \in Q_1 : less(a,Q_2) < j\}|$; and for $c_j \in Q_3$ it is true that $less(c_j,Q_2) = |\{b \in Q_2 : less(b,Q_3) < j\}|$ and $less(c_j,Q_1) = |\{b \in Q_1 : less(b,Q_3) < j\}|$.

Hence we get elements of blocks Q_2 and Q_3:

$$b_j = less(b_j, Q_3) + less(b_j, Q_1) + j + 1;$$

$$c_j = less(c_j, Q_1) + less(c_j, Q_2) + j + 1.$$

So, for abstract properties of semigroup $\mathcal{T}$ it is possible to restore the corresponding partition from $\Lambda(\mathbb{N}, 3)$; then it means that non-isomorphic semigroups correspond to different partitions, so the theorem is proved for $k = 3$.

Now suppose the statement of the theorem holds for all $k \leqslant k_0$. Let $\mathcal{T}$ be a semigroup from $Nil(\mathbb{N}, k_0 + 1)$, and partition λ is the respective partition from $\Lambda(\mathbb{N}, k_0+1)$ with blocks $Q_1, Q_2, \dots, Q_{k_0}, Q_{k_0+1}$. Now let's consider the set

$$S_1 = \{s \in \mathcal{T} : \forall a_1, \dots, a_{k_0-1} \in \mathcal{T} \quad s \cdot a_1 \cdot \ldots \cdot a_{k_0-1} = 0\}.$$

It is easy to see that a transformation of $\mathcal{T}$ belongs to S_1 if and only if its range has empty intersection with the second block of the partition λ. It is also obvious that S is a subsemigroup of $\mathcal{T}$. More, S_1 belongs to $Nil(\mathcal{T}, k_0)$. Really, there is a correspondent partition from $\Lambda(\mathcal{T}, k_0)$ with blocks $Q_1 \cup Q_2, \dots, Q_{k_0+1}$. Then for the induction assumption one can obtain the numbers of blocks $Q_1 \cup Q_2,\ Q_3, \dots, Q_{k_0+1}$. Next, let's consider the set

$$S_2 = \{s \in \mathcal{T} : \forall a_1, \dots, a_{k_0-1} \in \mathcal{T} \quad a_1 \cdot \ldots \cdot a_{k_0-1} \cdot s = 0\}.$$

Analogously, a transformation of $\mathcal{T}$ belongs to S_2 if and only if its domain has empty intersection with the next to the last block of the partition λ; and S_2 is a maximal nilpotent subsemigroup of nilpotency class k_0 with a corresponding partition having blocks $Q_1, Q_2, \dots, Q_{k_0} \cup Q_{k_0+1}$. For the induction assumption one can obtain the numbers of the first block Q_1. So, we have the numbers of all the blocks of the partition λ. So, for the properties $\mathcal{T}$ as abstract semigroup we get elements of the blocks of the correspondent partition, so the theorem is proved. □

Corollary 2. *Let $Nil(n, k)$ denote the set of all maximal nilpotent subsemigroups of the semigroup of oreder-decreasing transformations of the set $\{1, \dots, n\}$. Then all the semigroups from $\bigcup_{\substack{n \geqslant 4 \\ 3 \leqslant k \leqslant n-1}} Nil(n, k)$ are pairewise non-isomorphic.*

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