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Fibrations and cofibrations in a stratified model category

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Abstract. We introduce n-acyclic cofibrations and n-acyclic fibrations in a stratified model category and show that they have the key properties of (acyclic) cofibrations and (acyclic) fibrations in a model category. We analyse their action on sets of homotopy classes and give an application to homotopy colimits and limits.

Introduction

The notion of a model category was introduced by Quillen in [4] and [5] as a way to axiomatize homotopy theory. A model category is an ordinary category with three distinguished classes of maps called weak equivalences, fibrations and cofibrations satisfying certain five axioms which are patterned on the properties of maps with these names in the catogory of topological spaces. It turns out that many algebraic and combinatorial categories also have this structure. Moreover, each model category has a homotopy category, and model category methods can sometimes be used to show that certain geometric categories have homotopy categories equivalent to homotopy categories of algebraic or combinatorial objects.

A stratified model category [6] is a special type of model category. This notion starts with the observation that in many model categories the weak equivalences are *stratified*, in the sense that a map is a weak equivalence if and only if it is an n-equivalence for each nonnegative integer n. This allows one to show that liftings in the fundamental homotopy lifting-extension problem

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$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & Y \end{array}$$

exist (for a stratified model category) in inifinitely many situations other than those specified by the model category axioms.

This paper is a continuation of paper [6]. In Section 2 we introduce the notions of n-acyclic cofibrations and n-acyclic fibrations (Definition 5), which are motivated by Theorems 4.6 and 4.8 in [6]. We show that the main results characterizing (acyclic) cofibrations and (acyclic) fibrations in [4] also hold for n-acyclic cofibrations and n-acyclic fibrations (Theorems 3 and 4 below). This requires (Section 1) adding one further condition in the notion of stratified weak equivalence (namely that n-epimorphisms and n-monomorphisms are closed under retracts) and we show that in all the cases considered this requirement is satisfied. In Section 3 we study the effect of n-acyclic cofibrations and n-acyclic fibrations on sets of homotopy classes. Finally, in Section 4 we give an application of our results to homotopy pushouts and pullbacks. More specifically, we show that certain results about topological spaces which are not hard to show using the cellular approximation theorem in fact hold for an arbitrary stratified model category.

1. Stratified weak equivalences

First, we add one further condition in the notion of a stratified weak equivalence. Then we show that in all the cases considered and most probably in those that may come in the future this requirement is satisfied.

Recall the original definitions.

Definition 1. *A class of morphisms $\mathcal{W}$ in $\mathbf{C}$ is called a class of* **weak equivalences** *if it has the following properties:*

W1 $\mathcal{W}$ contains all isomorphisms and is closed under sequential colimits and retracts.

W2 If f, g are morphisms in $\mathbf{C}$ such that gf is defined, and it two of f, g, gf are in $\mathcal{W}$, then so is the third.

Definition 2. *A class of weak equivalences $\mathcal{W}$ is called* **stratified** *if for each nonnegative integer n there exist a class $\mathcal{W}_n$ of weak equivalences (the n-equivalences), and classes of morhisms $\mathcal{E}_n$ (the n-epimorphisms) and $\mathcal{M}_n$ (the n-monomorphisms) such that*

SW1 $f \in \mathcal{W}$ *if and only if* $f \in \mathcal{W}_n$ *for every nonnegative integer* n.

SW2 $f \in \mathcal{W}_n$ *if and only if* $f \in \mathcal{E}_n$ *and* $f \in \mathcal{M}_n$.

SW3 *If* $h = gf$ *then* $h \in \mathcal{M}_n$ *implies* $f \in \mathcal{M}_n$ *and* $h \in \mathcal{E}_n$ *implies* $g \in \mathcal{E}_n$.

SW4 *The classes* $\mathcal{E}_n$ *and* $\mathcal{M}_n$ *are closed under composition, sequential colimits and arbitrary sums.*

From now on, we also assume that $\mathcal{E}_n$ and $\mathcal{M}_n$ are closed under retracts, i.e. we replace SW4 by

SW4' The classes $\mathcal{E}_n$ and $\mathcal{M}_n$ are closed under composition, retracts, sequential colimits and arbitrary sums.

Example 1. We now show that the stratified weak equivalences of the category of topological spaces satisfy the extra requirement in SW4'. We check that a retract of an n-monomorphism is an n-monomorphism, where n is a fixed nonnegative integer. The proof for the n-epimorphisms is similar. Suppose a morphims $f : X \to Y$ is a retract of a morphism $g : Z \to W$ in **Top**. Hence we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{k} & Z & \xrightarrow{l} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow{u} & W & \xrightarrow{v} & Y \end{array}$$

in which the composites of the horizonal rows are identity maps. Suppose that g is an n-monomorphism. We need to show that f is an n-monomorphism. Choose a basepoint $x_0 \in X$ and let $y_0 = f(x_0)$. Let $z_0 = k(x_0)$ and $w_0 = u(y_0)$. We apply the n-th homotopy group (set if $n = 0$) functor to obtain the diagram

$$\begin{array}{ccccc} \pi_n(X, x_0) & \xrightarrow{k_*} & \pi_n(Z, z_0) & \xrightarrow{l_*} & \pi_n(X, x_0) \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle g_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ \pi_n(Y, y_0) & \xrightarrow{u_*} & \pi_n(W, w_0) & \xrightarrow{v_*} & \pi_n(Y, y_0) \end{array}$$

The diagram obtained shows that f_* is a retract of g_* in the category of groups (sets if $n = 0$). Since the composite g_*k_* is a monomorphism, it follows that f_* is also a monomorphism (by the commutativity of the first square). So f is an n-monomorphism.

Of course, a practically identical argument works for the category **S** of simplicial sets.

The next result shows that the general theorem (5.5 in [6]) about putting a stratified model category structure on a category D related to the category **S** by a family of adjoint functors also leads to stratified weak equivalences satisfying SW4' above. The proof is very similar to the reasoning in the example above and therefore omitted.

Proposition 1. *Let* **D** *be a category,* Λ *an arbitrary index set, and*

$$\Psi_\lambda : \mathbf{S} \leftrightarrows \mathbf{D} : \Phi_\lambda, \qquad \lambda \in \Lambda$$

a family of adjoint functors. Call a map $g : X \to Y$ *in* **D** *an* n*-equivalence (resp.* n*-epimorphism,* n*-monomorphism) if and only if for all* $\lambda \in \Lambda$ *the map* $\Phi_\lambda(g) : \Phi_\lambda X \to \Phi_\lambda Y$ *is an* n*-equivalence (resp.* n*-epimorphism,* n*-monomorpism) in* **S**. *This defines a family of stratified weak equivalences in* **D** *which satisfies SW4'.*

2. The n-acyclic cofibrations and n-acyclic fibrations

Following [2], we recall D.M. Kan's notion of a cofibrantly generated model category.

Definition 3. *A cofibrantly generated model category is a model category* **C** *with arbitrary colimits such that*

- *There exists a set* I *of* **generating cofibrations** *that permits the small object argument with respect to* I *and such that a map is an acyclic fibration if and only if it has the right lifting property with respect to every generating cofibration,*
- *There exists a set* J *of* **generating acyclic cofibrations** *that permits the small object argument with respect to* J *and such that a map is a fibration if and only if it has the right lifting property with respect to every generating acyclic cofibration.*

We now recall from [6] the definition of a stratified model category.

Definition 4. *A* **stratified model category C** *is a cofibrantly generated model category with the following extra structure:*

- *A class* $\mathcal{W}$ *of stratified weak equivalences.*
- *A set* $I = \cup_{n=0}^{\infty} I_n$ *of generating cofibrations such that each* $f \in I_n$ *is an* m*-equivalence for* $m < n-1$ *and an* $(n-1)$*-epimorphism.*

- *A set J of generating acyclic cofibrations. (Hence, in particular, every element in the closure of J is a weak equivalence).*

These are required to satisfy the following axioms:

CC1 If p is a fibration and $n \geq 0$, then p has the RLP with respect to I_n if and only if $p \in \mathcal{E}_n$ and $p \in \mathcal{M}_{n-1}$.

CC2 For all $n \geq -1$, every map in the closure of $J \cup \cup_{m=n+1}^{\infty} I_m$ is an m-equivalence for $m < n$ and an n-epimorphism.

CC3 The set $I \cup J$ permits the small object argument.

The fact that simplicial sets have this structure is contained in [3].

From now on, **C** denotes an arbitrary stratified model category.

Next, recall that $\overline{\mathbb{N}}$ is the ordered set defined as follows:

$$\overline{\mathbb{N}} = \{n \in \mathbb{Z} \mid n \geq -1\} \cup \{\infty\},$$

where $-1 \leq 0 \leq 1 \leq \cdots \leq \infty$.

Now we state the main definition.

Definition 5. *Let $n \in \overline{\mathbb{N}}$. A map $f : X \to Y$ in* **C** *is an n*-**acyclic cofibration** *if and only if f is a cofibration, an m-equivalence for $m < n$ and an n-epimorphism. A map $f : X \to Y$ is and n*-**acyclic fibration** *if and only if f is a fibration, an m-equivalence for $m > n$ and an n-monomorphism.*

Hence a (-1)-acyclic cofibration is just an ordinary cofibration and an ∞-cofibration is an acyclic cofibration in the usual (i.e. Quillen) sense. Similarly, a (-1)-acyclic fibration is an acyclic fibration and an ∞-fibration is an ordinary fibration.

Using these notions we can rewrite Theorems 4.6 and 4.8 in [6] in the following simpler form. One cannot miss the resemblence to the last two axioms of a model category.

Theorem 1. *For every $n \in \overline{\mathbb{N}}$, every map $f : X \to Y$ can be factored in a canonical way as*

$$X \xrightarrow{i} Z \xrightarrow{p} Y$$

where i is an n-acyclic cofibration and p is an n-acyclic fibration.

Theorem 2. *Let $n \in \overline{\mathbb{N}}$. If i is an n-acyclic cofibration and p is an n-acyclic fibration, then a lift exists in every commutative diagram of the form*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y. \end{array}$$

Clearly, if a map is an n-acyclic cofibration, then it is a k-acyclic cofibration for every $k \leq n$. Similarly, if a map is an n-acyclic fibration, then it is a k-acyclic fibration for every $k \geq n$. This observation leads to the following corollary.

Corollary 1. *A lift exists in every commutative diagram of the form*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

where i is an n_1-acyclic cofibration, p an n_2-acyclic fibration and $n_2 \leq n_1$.

Proof. The corollary follows from the fact that under the above assumptions i is an n-acyclic cofibration and p is an n-acyclic fibration for every n such that $n_2 \leq n \leq n_1$. □

Theorem 3. *Let $n \in \overline{\mathbb{N}}$.*

- *The n-acyclic cofibrations are the maps which have the LLP (left lifting property) with respect to the n-acyclic fibrations.*
- *The n-acyclic fibrations in* **C** *are the maps which have the RLP with respect to the n-acyclic cofibrations.*

Proof. We prove the first statement, the proof of the second one is dual. Theorem 2 above states that the n-acyclic cofibrations have the stated property. Conversely, suppose that a map $f : K \to L$ has the LLP with respect to all n-acyclic fibrations. Using Theorem 1 above factor f as a composite

$$K \xrightarrow{i} L' \xrightarrow{p} L$$

where i is an n-acyclic cofibration and p is an n-acyclic fibration.

We now have a comutitative diagram

$$\begin{array}{ccc} K & \xrightarrow{\ i\ } & L' \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ L & \xrightarrow[\text{id}]{} & L \end{array}$$

By Theorem 2 a lifting $g : L \to L'$ exists in the diagram above. The following diagram shows that f is a retract of i:

$$\begin{array}{ccccc} K & \xrightarrow{\text{id}} & K & \xrightarrow{\text{id}} & K \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ L & \xrightarrow[g]{} & L' & \xrightarrow[p]{} & L \end{array}$$

Since in any model category a retract of a cofibration is a cofibration, by SW4' we conclude that f is an n-acyclic cofibration. □

Taking n equal -1 and ∞ in the first statement we obtain Proposition 3.13(1) and 3.13(2), respectively, in [1]. Similarly the second statement corresponds to Proposition 3.13(3) and 3.13(4). These results for $n = -1$ and $n = \infty$ were first established by Quillen in [4].

Corollary 2. *Suppose that* $\mathbf{C}$ *and* $\mathbf{D}$ *are stratified model categories and*

$$F : \mathbf{C} \leftrightarrows \mathbf{D} : G$$

is a pair of adjoint functors.

- *If F preserves n-acyclic cofibrations then G preserves n-acyclic fibrations.*
- *If G preserves n-acyclic fibrations then F preserves n-acyclic cofibrations.*

Proof. We only prove the first statement since the other is dual. Let $p : X \to Y$ be an n-acyclic fibration. We want to show that $G(p)$ is also an n-acyclic fibration. By Theorem 3 it is enough to check that $G(p)$ has the RLP with respect to all n-acyclic cofibrations. Hence we need to show that a lifting exist in the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & G(X) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle G(p)} \\ B & \xrightarrow{\ g\ } & G(Y) \end{array}$$

where $i : A \to B$ is an arbitrary n-acyclic cofibration. By adjointness of F and G, the above diagrams are in bicorrespondence with the diagrams

$$\begin{array}{ccc} F(A) & \xrightarrow{f^\flat} & X \\ \downarrow{\scriptstyle F(i)} & & \downarrow{\scriptstyle p} \\ F(B) & \xrightarrow{g^\flat} & Y \end{array}$$

Since we assume that the functor F preserves n-acyclic cofibrations, by Theorem 2 a lifting exists in the diagram above. By the properties of adjoints a lift exists in the original diagram, and hence we conclude that $G(p)$ is an n-acyclic fibration. □

In any model category the (acyclic) cofibrations are stable under cobase change, and the (acyclic) fibrations are stable under base change (see for example Proposition 3.14 in [1]). It tourns out that these results also hold for n-acyclic cofibrations and n-acyclic fibrations.

Theorem 4. *Let $n \in \overline{\mathbb{N}}$.*

- *The class of n-acyclic cofibrations is stable under cobase change.*
- *The class of n-acyclic fibrations is stable under base change.*

Proof. For the same reasons as before, we only prove the first statement. Assume that $i : K \to L$ is an n-acyclic cofibration. Suppose that j is obtained from i by cobase change, i.e. we have the following pushout diagram

$$\begin{array}{ccc} K & \xrightarrow{f} & K' \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\ L & \xrightarrow{g} & L' \end{array}$$

By the earlier theorem, to show that j is an n-acyclic cofibration it is enough to verify that it has the LLP with respect to the n-acyclic fibrations. So let $p : X \to Y$ be an n-acyclic fibration. We need to verify that a lifting exists in the diagram

$$\begin{array}{ccc} K' & \xrightarrow{f'} & X \\ {\scriptstyle j}\downarrow & & {\scriptstyle p}\downarrow \\ L' & \xrightarrow{g'} & Y \end{array}$$

We can combine the above diagram with the earlier one to obtain

$$\begin{array}{ccccc} K & \xrightarrow{f} & K' & \xrightarrow{f'} & X \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle p}\downarrow \\ L & \xrightarrow{g} & L' & \xrightarrow{g'} & Y \end{array}$$

Since i is an n-acyclic cofibration and p is an n-acyclic fibration, by Theorem 2 there is a lifting $h : L \to X$ in the large rectangle. By the universal property of pushouts the maps h and f' induce a map $k : L' \to X$. This is the desired lifting. We have $kj = f'$ by construction. The maps g' and pk are equal by the universal property of L' (since $g'g = pkg$ and $g'j = pkj$). □

3. Maps induced on sets of homotopy classes by n-acyclic cofibrations and n-acyclic fibrations

First, we briefly recall the notion of left homotopy from [1].

A **cylinder object** for A is an object $A \wedge I$ of $\mathbf{C}$ together with a diagram (the symbol $\sim$ means that the second map is a weak equivalence)

$$A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

which factors the folding map $\mathrm{id}_A + \mathrm{id}_A : A \coprod A \to A$. It is called a **good cylinder object** if the first map is a cofibration. We let

$$i_0 = i \cdot in_0 : A \to A \wedge I \quad \text{and} \qquad i_1 = i \cdot in_1 : A \to A \wedge I.$$

Two maps $f, g : A \to X$ are said to be **left homotopic** if there exists a cylinder object $A \wedge I$ for A and a map $H : A \wedge I \to X$ (called a **left homotopy**) such that $H(i_0 + i_1) = f + g$.

Proposition 2. *Let ϕ be an initial object in $\mathbf{C}$ and $n \in \overline{\mathbb{N}}$. If $\phi \to A$ is an n-acyclic cofibration and $p : Y \to X$ is an n-acyclic fibration, then composition with p induces a bijection*

$$p_* : \pi^l(A, Y) \to \pi^l(A, X), \qquad [f] \mapsto [pf].$$

Proof. First, we check that p_* is well definied. Suppose $f, g : A \to Y$ and $f \overset{l}{\sim} g$. Hence, we have a cylinder object $A \wedge I$ for A and a map $H : A \wedge I \to Y$ such that $H(i_0 + i_1) = f + g : A \coprod A \to Y$. It is easy to check that pH is a left homotopy from pf to pg (it is useful to draw a diagram).

Next, we check that p_* is onto. Choose an $f \in \pi^l(A, X)$. Consider the commutative diagram

$$\begin{array}{ccc} \phi & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow p \\ A & \underset{f}{\longrightarrow} & X \end{array}$$

Since the left vertical arrow is an n-acyclic cofibration and the right vertical arrow is an n-acyclic fibration, by Theorem 2 we have a lift $g : A \to Y$, i.e. a map such that $pg = f$. Hence we have $p_*([g]) = [f]$, and we conclude that p_* is onto.

Finally, we check that p_* is one-to-one. Suppose that $f, g : A \to Y$, and $pf \overset{l}{\sim} pg : A \to X$. By Lemma 3.6 in [1] we can choose a good left homotopy H from pf to pg. Hence we have a commutative diagram

$$\begin{array}{ccc} A \coprod A & \xrightarrow{f+g} & Y \\ i\big\downarrow & & \big\downarrow p \\ A \wedge I & \underset{H}{\longrightarrow} & X \end{array}$$

A lift, call it K, in the above diagram would be the desired left homotopy from f to g. To apply Theorem 2 we need to check that $i : A \coprod A \to A \wedge I$ is an n-acyclic cofibration.

First, we check that $\mathrm{in}_0 : A \to A \coprod A$ is an n-acyclic cofibration. Because in_0 is defined by the pushout diagram

$$\begin{array}{ccc} \phi & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \mathrm{in}_0 \\ A & \underset{\mathrm{in}_1}{\longrightarrow} & A \coprod A \end{array}$$

the fact that in_0 is an n-acyclic cofibration follows from Theorem 4.

Now consider the following compostion of maps, which is equal to the identity map on A

$$A \xrightarrow{\mathrm{in}_0} A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

Since the last map is a weak equivalence, and the composite of the three maps (being the identity map on A) is a weak equivalence, it follows that the composite of the first two maps is a weak equivalence. By the two out of three property for m-equivalences (where $m < n$), we conclude that the map i and an m-equivalence for $m < n$. Moreover, since the composite of the first two maps is a weak equivalence, by SW1 and SW2 it is an n-epimorphism. Hence, by SW3, the map i is an n-epimorphism. We conclude that the map i is an n-acyclic cofibration. Hence the desired lift K giving a left homotopy from f to g exists. This finishes the proof. □

By duality between n-acyclic fibrations and n-acyclic cofibrations the above result implies the following.

Proposition 3. *Let $*$ be a terminal object in $\mathbf{C}$ and $n \in \overline{\mathbb{N}}$. If $X \to *$ is an n-acyclic fibration and $i : A \to B$ is an n-acyclic cofibration, then composition with i induces a bijection*

$$i^* : \pi^r(B, X) \to \pi^r(A, X), \qquad [f] \mapsto [fi].$$

4. An application to homotopy pushouts and pullbacks

Homotopy colimits and limits are now a basic tool in algebraic topology. We give information on some spacial, but very common, examples, namely homotopy pushouts and pullbacks.

It is well known that pushouts and pullbacks do not exist in the homotopy category. To give a simple example, consider the diagram

$$\begin{array}{ccccc} D^2 & \xleftarrow{\text{id}} & S^1 & \xrightarrow{\text{id}} & D^2 \\ \downarrow & & \downarrow{\scriptstyle \text{id}} & & \downarrow \\ * & \longleftarrow & S^1 & \longrightarrow & * \end{array}$$

where D^2 is the 2-dimensional disk and S^1 is the circle. The pushout of the top row is S^2, while the pushout of the bottom row is $*$, the one point space. Although the vertical arrows are weak homotopy equivalences, the map induced on the pushouts is not a weak homotopy equivalence. Experience shows that the pushout of the top row is "the correct one". Section 10 of [1] gives a jutstification for this. We briefly recall the main constructions of that section and then show how the results of Section 2 can be applied to establish some homotopy relations between the input data for a homotopy pushout and the result of the construction. We then state the corresponding results for homotopy pullbacks. These results are well known for the category of topological spaces and the category of simplicial sets by the use of the cellular approximation theorem. Here, we show that they also hold for an arbitrary stratified model category.

Let $\mathbf{D}$ denote the category with three objects and two nonidentity arrows $\mathbf{D} = \{a \leftarrow b \to c\}$. Consider the category $\mathbf{C}^{\mathbf{D}}$ whose objects are functors $X : \mathbf{D} \to \mathbf{C}$. A morphism $f : X \to Y$ in $\mathbf{C}^{\mathbf{D}}$ is a diagram of morphisms in $\mathbf{C}$

$$\begin{array}{ccccc} X(a) & \longleftarrow & X(b) & \longrightarrow & X(c) \\ {\scriptstyle f_a}\downarrow & & {\scriptstyle f_b}\downarrow & & {\scriptstyle f_c}\downarrow \\ Y(a) & \longleftarrow & Y(b) & \longrightarrow & Y(c) \end{array}$$

To define a model category structure on $\mathbf{C}^{\mathbf{D}}$, we first need to define the object $(\delta_a(f), \delta_b(f), \delta_c(f))$ of $\mathbf{C}^{\mathbf{D}}$ determinded by $f : X \to Y$. We let

$$\delta_b(f) = X(b)$$
$$\delta_a(f) = \text{Pushout}\left(Y(b) \xleftarrow{f_b} X(b) \to X(a)\right)$$
$$\delta_c(f) = \text{Pushout}\left(Y(b) \xleftarrow{f_b} X(b) \to X(c)\right)$$

This object fits into the following commutative diagram

$$\begin{array}{ccccc} X(a) & \longleftarrow & X(b) & \longrightarrow & X(c) \\ \downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ \delta_a(f) & \longleftarrow & \delta_b(f) & \longrightarrow & \delta_c(f) \\ {\scriptstyle i_a(f)}\downarrow & & {\scriptstyle i_b(f)}\downarrow & & {\scriptstyle i_c(f)}\downarrow \\ Y(a) & \longleftarrow & Y(b) & \longrightarrow & Y(c) \end{array}$$

There is a model category structure on $\mathbf{C}^{\mathbf{D}}$ in which a map $f : X \to Y$ in $\mathbf{C}^{\mathbf{D}}$ is

- a weak equivalence if f_a, f_b and f_c are weak equivalences in $\mathbf{C}$,
- a fibration if f_a, f_b and f_c are fibrations in $\mathbf{C}$,
- a cofibration if the maps $i_a(f), i_b(f)$ and $i_c(f)$ defined above are cofibrations in $\mathbf{C}$.

We start with a lemma.

Lemma 1. *Suppose that we have a diagram*

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle l} \\ A & \xrightarrow{k} & P \end{array}$$

in which the object P is the pushout of the the diagram consisting of the left vertical arrow and the top horizontal arrow. Suppose f is an n_1-acyclic cofibration and g is an n_2-acyclic cofibration, where $n_1, n_2 \in \overline{\mathbb{N}}$.

Then the map kf *is an* $\min\{n_1, n_2\}$*-acyclic cofibration, where* min *stands for the minimum of two integers.*

Proof. It follows from Theorem 4 that k (as the cobase change of g along f) is an n_2-acyclic cofibration. The composition kf of two cofibrations is again a cofibration. It follows from SW4' that it is $\min\{n_1, n_2\}$-acyclic. □

Theorem 5. *Let* $n_1, n_2 \in \overline{\mathbb{N}}$ *and* $n = \min\{n_1, n_2\}$*. Suppose that we have the diagram*

$$A \xleftarrow{f} B \xrightarrow{g} C$$

in which f *is an* m*-equivalence for* $m < n_1$ *and an* n_1*-epimorphism and* g *is an* m*-equivalence for* $m < n_2$ *and an* n_2*-epimorphism and* B *is cofibrant. Then the canonical map* $B \to \text{hocolim}\left(A \xleftarrow{f} B \xrightarrow{g} C\right)$ *is an* n*-acyclic cofibration.*

Proof. According to Section 10 of [1], in order to find the homotopy pushout we need first to find a cofibrant representative of

$$A \xleftarrow{f} B \xrightarrow{g} C$$

in the model category structure on $\mathbf{C}^{\mathbf{D}}$.

First, we apply Theorem 1 with n_1 to the map $f : B \to A$ to obtain a factorization $f = f''f'$ where $f' : B \to A'$ is an n_1-acyclic cofibration and $f'' : A' \to A$ is an n_1-acyclic fibration. Since, by the assumptions the map f is an m-equivalence for $m < n_1$, by the two out of three property for n-equivalences (i.e. W2), we conclude that the map f'' is an m-equivalence for $m < n_1$. Moreover, by SW3, since f is an n_1-epimorphism, so is f''. We conclude that f'' is an m-equivalence for all m, i.e. that it is a weak equivalence.

Next, we apply Theorem 1 with n_2 to the map $g : B \to C$ to obtain a factorization $g = g''g'$ where $g' : B \to C'$ is an n_2-acyclic cofibration and $g'' : C' \to C$ is an n_2-acyclic fibration. By a reasoning similar to the one earlier we conclude that the map g'' is a weak equivalence.

The above maps fit into the following commutative diagram:

$$\begin{array}{ccccc} A' & \xleftarrow{f'} & B & \xrightarrow{g'} & B' \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle g''} \\ A & \xleftarrow{f} & B & \xrightarrow{g} & C \end{array}$$

Hence the top row is a cofibrant representative of $A \xleftarrow{f} B \xrightarrow{g} C$. It follows from p. 117 of [1] that the desired homotopy direct limit is an ordinary pushout of $A' \leftarrow B \rightarrow C'$. The result now follows from the previous lemma. □

We now state the dual result (the proof is dual to the one above and hence omitted).

Theorem 6. *Let $n_1, n_2 \in \overline{\mathbb{N}}$ and $n = \max\{n_1, n_2\}$. Suppose that we have the diagram*

$$A \xrightarrow{f} B \xleftarrow{g} C$$

in which f is an m-equivalence for $m > n_1$ and an n_1-monomorphism and g is an m-equivalence for $m > n_2$ and an n_2-monomorphism and B is fibrant. Then the canonical map holim $\left(A \xleftarrow{f} B \xrightarrow{g} C\right) \rightarrow B$ is an n-acyclic fibration.

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