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RESEARCH ARTICLE

Pseudodiscrete balleans

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Abstract. A ballean $\mathcal{B}$ is a set X endowed with some family of subsets of X which are called the balls. The properties of the balls are postulated in such a way that a ballean can be considered as an asymptotic counterpart of a uniform topological space. A ballean is called pseudodiscrete if "almost all" balls of every pregiven radius are singletons. We give a filter characterization of pseudodiscrete balleans and their classification up to quasi-asymorphisms. It is proved that a ballean is pseudodiscrete if and only if every real function defined on its support is slowly oscillating. We show that the class of irresolvable balleans are tightly connected with the class of pseudodiscrete balleans.

1. Ball structures and balleans

A *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radiuses.*

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^*(x,\alpha) = \{y \in X : x \in B(y,\alpha)\}, \quad B(A,\alpha) = \bigcup_{a \in A} B(a,\alpha).$$

A ball structure $\mathbb{B} = (X, P, B)$ is called

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- *lower symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B^*(x, \alpha') \subseteq B(x, \alpha), \quad B(x, \beta') \subseteq B^*(x, \beta);$$

- *upper symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \quad B^*(x, \beta) \subseteq B(x, \beta');$$

- *lower multiplicative* if, for any α, $\beta \in P$ there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta);$$

- *upper multiplicative* if, for any $\alpha, \beta \in P$ there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

Let $\mathcal{B} = (X, P, B)$ be a lower symmetric, lower multiplicative ball structure. Then the family

$$\{\bigcup_{x \in X} B(x, \alpha) \times B(x, \alpha) : \alpha \in P\}$$

is a base of entourages for some (uniquely determined) uniformity on X. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x, U) = \{y \in X : (x, y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

A ball structure is said to be a *ballean* if it is upper symmetric and upper multiplicative. The balleans arouse independently in asymptotic topology [1,4] under the name coarse structure and in combinatorics [5]. For good motivation to study the ballean related to metric space see the survey [1].

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \longrightarrow X_2$ is called a $\prec$*-mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f\big(B_1(x, \alpha)\big) \subseteq B_2(f(x), \beta).$$

By the definition, $\prec$-mappings can be considered as the asymptotic counterparts of the uniformly continuous mappings between the uniform topological space.

If $f : X_1 \longrightarrow X_2$ is a bijection such that f and f^{-1} are the $\prec$-mappings, we say that the balleans $\mathcal{B}_1$ and $\mathcal{B}_2$ are *asymorphic*. If $X_1 = X_2$ and the identity mapping $id : X_1 \longrightarrow X_2$ is a $\prec$-mapping, we write $\mathcal{B}_1 \preceq \mathcal{B}_2$. If $\mathcal{B}_1 \preceq \mathcal{B}_2$ and $\mathcal{B}_2 \preceq \mathcal{B}_1$, we write $\mathcal{B}_1 = \mathcal{B}_2$. Given an arbitrary ballean $\mathcal{B} = (X, P, B)$, we can replace every ball $B(x, \alpha)$ to $B(x, \alpha) \bigcup B^*(x, \alpha)$ and get the same ballean on X,so in what follows we assume that $B(x, \alpha) = B^*(x, \alpha)$ for all $x \in X, \ \alpha \in P$.

To determine the subject of this paper we need some more definitions. Let $\mathcal{B} = (X, P, B)$ be a ballean.

A subset A of X is called *bounded* if there exists $x \in X$ and $\alpha \in P$ such that $A \subseteq B(x, \alpha)$. A ballean is called bounded if its support is bounded.

Given any elements $x, y \in X$, we say that x, y are *connected* if there exists $\alpha \in P$ such that $y \in B(x, \alpha)$. The connectedness is an equivalence on X, so X disintegrates into connected components. A ballean $\mathcal{B}$ is called connected if any two elements from X are connected.

A ballean $\mathcal{B}$ is called *proper* if $\mathcal{B}$ is connected and unbounded.

We use also the preodering $\leq$ on P defined by the rule: $\alpha \leq \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$, there exists $\beta \in P'$ such that $\beta \geq \alpha$. The minimal cardinality $cf\ \mathcal{B}$ of cofinal subsets is called *cofinality* of $\mathcal{B}$.

Given a ballean $\mathcal{B} = (X, P, B)$, a subset $Y \subseteq X$ is called *large* if there exists $\alpha \in P$ such that $X = B(Y, \alpha)$. The large subsets of a ballean are the asymptotic counterparts of the dense subsets of a uniform space.

2. Pseudodiscrete balleans

A ballean $\mathcal{B} = (X, P, B)$ is called *discrete* if $B(x, \alpha) = \{x\}$ for all $x \in X, \ \alpha \in P$. Following [7], we say that $\mathcal{B}$ is *pseudodiscrete* if, for every $\alpha \in P$, there exists a bounded subset V of X such that $B(x, \alpha) = \{x\}$ for every $x \in X \setminus V$. Clearly, every discrete ballean is pseudodiscrete and every bounded ballean is pseudodiscrete.

Let X be a set and let φ be a filter on X. Given any $x \in X$ and $F \in \varphi$, we put

$$B_\varphi(x, F) = \left\{ \begin{array}{r} x, \ if \ x \in F; \\ X \setminus F, \ if \ x \in X \setminus F; \end{array} \right.$$

and denote by $\mathcal{B}(X, \varphi)$ the ballean (X, φ, B_φ). Clearly, a subset V of X is bounded in $\mathcal{B}(X, \varphi)$ if and only if either V is a singleton or $X \setminus V \in \varphi$. It follows that $\mathcal{B}(X, \varphi)$ is pseudodiscrete and $\mathcal{B}(X, \varphi)$ is bounded if and only if X is a singleton. The list of connected components of $\mathcal{B}(X, \varphi)$ is

$X \setminus \bigcap \varphi$ and $\{x\}$, $x \in \bigcap \varphi$. Hence, $\mathcal{B}(X, \varphi)$ is connected if and only if either $\bigcap \varphi = \emptyset$ or $|X| = 1$, $\mathcal{B}(X, \varphi)$ is proper if and only if $\bigcap \varphi = \emptyset$.

Every metric space (M, d) determines the *metric* ballean $\mathcal{B}(M, d) = (M, \mathbb{R}^+, B_d)$, where $\mathbb{R}^+ = \{r \in \mathbb{R} : r \geq 0\}$, $B_d(x, r) = \{y \in M : d(x, y) \leq r\}$. A ballean $\mathcal{B}$ is called *metrizable* if $\mathcal{B}$ is asymorphic to some metric ballean. By [6], $\mathcal{B}$ is metrizable if and only if $\mathcal{B}$ is connected and $cf\ \mathcal{B} \leq \aleph_0$. Hence, a ballean $\mathcal{B}(X, \varphi)$ is metrizable if and only if either $|X| = 1$ of $\bigcap \varphi = \emptyset$ and φ has a countable base.

Theorem 1. *Let $\mathcal{B} = (X, P, B)$ be an unbounded pseudodiscrete ballean. Then there exists a filter φ on X such that $\mathcal{B} = \mathcal{B}(X, \varphi)$.*

Proof. First we show that at most one connected component of $\mathcal{B}$ is not a singleton. Assume the contrary and choose two connected components Y, Z such that $|Y| > 1$, $|Z| > 1$. Then we pick $y, y' \in Y$, $y \neq y'$ and $z, z' \in Z$, $z \neq z'$. Choose $\alpha \in P$ such that $y' \in B(y, \alpha)$, $z' \in B(z, \alpha)$. Since $\mathcal{B}$ is pseudobounded, there exists a bounded subset V of X such that $|B(x, \alpha)| = 1$ for every $x \in X \setminus V$. Since every bounded subset of an arbitrary ballean is contained in some connected component, then either $V \bigcap Y = \emptyset$ or $V \bigcap Z = \emptyset$. If $V \bigcap Y = \emptyset$, then $y \in X \setminus V$ and $|B(y, \alpha)| > 1$. If $V \bigcap Z = \emptyset$, then $z \in X \setminus V$ and $|B(z, \alpha)| > 1$. In both cases we get a contradiction to the choice of V.

Let C be a union of all one-element connected components of X, $A = X \setminus C$. If A is bounded, then $\mathcal{B}$ is determined by the filter $\varphi = \{Y \subseteq X : C \subseteq Y\}$. Suppose that the connected component A is unbounded. Put $\varphi = \{X \setminus V : V \textit{ is a bounded subset of } A\}$ and note that the union of any finite family of bounded subsets of fixed connected component is bounded, so φ is a filter on X. We show that $\mathcal{B} = \mathcal{B}(X, \varphi)$. To prove that $\mathcal{B} \preceq \mathcal{B}(X, \varphi)$, we take an arbitrary $\alpha \in P$ and choose a bounded subset V such that $|B(x, \alpha)| = 1$ for every $x \in X \setminus V$. If $V \subseteq A$, we put $U = B(V, \alpha)$ and note that U is a bounded subset of A, so $X \setminus U \in \varphi$ and $B(x, \alpha) \subseteq B_\varphi(x, X \setminus U)$. If $V \bigcap A = \emptyset$, then V is contained in the connected component which is a singleton. Hence, $|B(x, \alpha)| = 1$ for all $x \in X$ and $B(x, \alpha) \subseteq B_\varphi(x, F)$ for an arbitrary $F \in \varphi$. To check that $\mathcal{B}(x, \varphi) \preceq \mathcal{B}$, we fix an arbitrary $F \in \varphi$. By the choice of φ, the subset $V = X \setminus F$ is bounded in $\mathcal{B}$. Choose $\alpha \in P$ such that $V \subseteq B(x, \alpha)$ for every $x \in V$. Since $B_\varphi(x, F) = \{x\}$ for every $x \in X \setminus V$, $B_\varphi(x, F) \subseteq \mathcal{B}(x, \alpha)$ for every $x \in X$. $\square$

Let $\mathcal{B} = (X, P, B)$ be an arbitrary proper ballean. The family $\varphi = \{X \setminus V : V \textit{ is a bounded subsets of } X\}$ is a filter on X, so $\mathcal{B}$ has the *pseudodiscrete companion* $\mathcal{B}(X, \varphi)$. By the definition, $\mathcal{B}(X, \varphi)$ is the smallest (with respect to $\preceq$) ballean on X such that every bounded subset in $\mathcal{B}$ is bounded in $\mathcal{B}(X, \varphi)$.

3. Subbaleans and factor-balleans

Let $\mathcal{B} = (X, P, B)$ be a ballean, Y be a nonempty subset of X. The ballean $\mathcal{B}_Y = (Y, P, B_Y)$, where $B_Y(y, \alpha) \bigcap X$, is called a *subballean* of X. Clearly, every subballean of pseudodiscrete ballean is pseudodiscrete.

A family $\mathcal{F}$ of subset of X is called *uniformly bounded* in $\mathcal{B}$ if there exists $\alpha \in P$ such that $F \subseteq B(x, \alpha)$ for every $x \in F$. Let $\mathcal{F}$ be a uniformly bounded partition of X. Given any $F \in \mathcal{F}$ and $\alpha \in P$, we put $B_{\mathcal{F}}(F, \alpha) = \{F' \in \mathcal{F} : F' \subseteq B(F, \alpha)\}$. It is easy to check that the ball structure $\mathcal{B}/\mathcal{F} = (\mathcal{F}, P, B_{\mathcal{F}})$ is a ballean which is called a *factor-ballean* of $\mathcal{B}$. We note also that $\mathcal{B}/\mathcal{F}$ is a smallest (by $\preceq$) ballean on $\mathcal{F}$ such that the projection $pr : X \to F$, where $pr(x) = F$ if and only if $x \in F$, is a $\prec$-mapping.

Let X be a set, φ be a filter on X. A family $\mathcal{F}$ of subset of X is uniformly bounded in the ballean $\mathcal{B}(X, \varphi)$ if and only if there exists $A \in \varphi$ such that, for every $F \in \mathcal{F}$, either $F \subseteq X \setminus A$ or F is a singleton. In view of Theorem 1, it follows that a factor-ballean of every pseudodiscrete ballean is pseudodiscrete.

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans, $f : X_1 \to X_2$ be a $\prec$-mapping. We consider the partition $ker\ f$ of X_1 determined by the equivalence: $x \sim y$ if and only if $f(x) = f(y)$. If the partition $ker\ f$ is uniformly bounded in $\mathcal{B}_1$, we get the canonical decomposition $f = i_f \circ pr_f$, where $pr_f : X_1 \to ker\ f$, $i_f : ker\ f \to X_2$. In this case, pr_f is a surjective $\prec$-mapping of $\mathcal{B}_1$ onto $\mathcal{B}_1/_{ker\ f}$, i_f is a surjective $\prec$-mapping of $\mathcal{B}_1/_{ker\ f}$ into $\mathcal{B}_2$.

4. Quasi-asymorphisms

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called an *asymorphic embedding* of $\mathcal{B}_1$ into $\mathcal{B}_2$ if f is an asymorphism between $\mathcal{B}_1$ and the subballean of $\mathcal{B}_2$ determined by the subset $f(X_1)$ of X_2.

A $\prec$-mapping $f : X_1 \to X_2$ is called a *quasi-asymorphic embedding* of $\mathcal{B}_1$ into $\mathcal{B}_2$ if, for every $\beta \in P_2$, there exists $\alpha \in P_1$ such that, for all $x_1, x_2 \in X_2$, $f(x_1) \in B_2(f(x_2), \beta)$ implies $x_1 \in B_1(x_2, \alpha)$. Equivalently, $f : X_1 \to X_2$ is a *qiasi-asymorphic embedding* if, for every uniformly bounded family $\mathcal{F}_1$ of subsets of X_1, the family $f(\mathcal{F}_1) = \{f(F) : F \in \mathcal{F}_1\}$ is uniformly bounded in $\mathcal{B}_2$ and, for every uniformly bounded family $\mathcal{F}_2$ of subsets of X_2, the family $f^{-1}(\mathcal{F}_2) = \{f^{-1}(F) : F \in \mathcal{F}_2\}$ is uniformly bounded in $\mathcal{B}_1$. We note also that a quasi-asymorphic embedding $f : X_1 \to X_2$ is an asymorphic embadding if and only if f is injective. For the case of metric ballean the notion of quasi-asymorphic embedding was introduced by Gromov [2] under the name uniform embedding.

Let $f : X_1 \to X_2$ is a quasi-asymorphic embedding of $\mathcal{B}_1$ into $\mathcal{B}_2$. Then the partition *ker f* is uniformly bounded in $\mathcal{B}_1$ and the mapping $i_f :\ ker\ f \to X_2$ from the canonical decomposition $f = i_f \circ pr_f$ is an asymorphic embedding of $\mathcal{B}_1/_{ker\ f}$ into $\mathcal{B}_2$. On the other hand, if some factor-ballean of $\mathcal{B}_1$ admits an asymorphic embedding into $\mathcal{B}_2$, then $\mathcal{B}_1$ admits a quasi-asymorphic embedding into $\mathcal{B}_2$.

The next definition generalizes the notion of quasi-isometry between metric spaces [3].

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans, $f_1 : X_1 \to X_2$ and $f_2 : X_2 \to X_1$ be $\prec$-mappings. We say that the pair (f_1, f_2) is a *quasi-asymorphism* between $\mathcal{B}_1$ and $\mathcal{B}_2$ if there exist $\alpha \in P_1$, $\beta \in P_2$ such that, for all $x \in X_1$, $y \in X_2$,

$$f_2 f_1(x) \in B(x, \alpha),\ f_1 f_2(y) \in B(y, \beta).$$

Automatically, in this case f_1 and f_2 are quasi-asymorphic embeddings and the subsets $f_1(X_1)$ and $f_2(X_2)$ are large in $\mathcal{B}_2$ and $\mathcal{B}_1$ respectively.

Now we connect quasi-asymorphisms with quasi-asymorphic embeddings. Let $f_1 : X_1 \to X_2$ be a quasi-asymirphic embedding of $\mathcal{B}_1$ into $\mathcal{B}_2$ such that the subset $f_1(X_1)$ of X_2 is large in $\mathcal{B}_2$. We construct a mapping $f_2 : X_2 \to X_1$ such that the pair (f_1, f_2) is a quasi-asymorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$. For every $y \in f(X_1)$, we choose some element $g(y) \in f^{-1}(y)$, so we have the mapping $g : f(X_1) \to X_1$. Since $f(X_1)$ is large in $\mathcal{B}_2$, there exists $\beta \in P_2$ such that $B_2(f(X_1), \beta) = X_2$. To define the mapping $f_2 : X_2 \to X_1$, we take an arbitrary $z \in X_2$, choose $y \in f(x_1)$ such that $z \in B(y, \alpha)$ and put $f_2(z) = g(y)$.

Clearly, every ballean, asymorphic to a pseudodiscrete ballean, is pseudodiscrete, but a ballean, quasi-asymorphic to pseudodiscrete ballean, needs not to be pseudodiscrete.

Theorem 2. *For every ballean $\mathcal{B} = (X, P, B)$, the following statements are equivalent:*

(i) $\mathcal{B}$ is quasi-asymorphically embeddable to some pseudodiscrete ballean;

(ii) $\mathcal{B}$ is quasi-asymorphic with some pseudodiscrete ballean;

(iii) there exists a uniformly bounded partition $\mathcal{F}$ of X such that the factor-ballean $\mathcal{B}/_{\mathcal{F}}$ is pseudodiscrete.

Proof. (i)⇒(ii). If $\mathcal{B}$ is quasi-asymorphically embeddable to a pseudodiscrete ballean $\mathcal{B}'$, the $\mathcal{B}$ is quasi-asymorphic with some subballean of $\mathcal{B}'$ and it suffices to note that every subballean of pseudodiscrete ballean is pseudodiscrete.

(ii)⇒(iii). Let (f,g) be a quasi-asymorphism between $\mathcal{B}$ and some pseudodiscrete ballean $\mathcal{B}'$ with the support Y. We consider the canonical decomposition $f = i_f \circ pr_f$ and note that i_f is an asymorphism between $\mathcal{B}/_{ker\ f}$ and the subballean of $\mathcal{B}'$ defined by the subset $f(X)$ of Y, so we can put $\mathcal{F} = ker\ f$.

(iii)⇒ (i). Since $\mathcal{F}$ is uniformly bounded, the projection $pr : X \to \mathcal{F}$ is a quasi-asymorphic embedding of $\mathcal{B}$ onto $\mathcal{B}/_{\mathcal{F}}$. □

Using Theorem 2 we can easily construct, for every unbounded pseudodiscrete ballean $\mathcal{B}$, quasi-asymorphic ballean $\mathcal{B}'$ which is not pseudodiscrete. By Theorem 1, we may suppose that $\mathcal{B} = \mathcal{B}(X,\varphi)$ where φ is a filter on the support X of $\mathcal{B}$. Let us take a disjoint family $\{Y_x : x \in X\}$ of sets such that $|Y_x| > 1$ for every $x \in X$. Put $Y = \bigcup_{x\in X} Y_x$ and consider the ballean $\mathcal{B}' = (Y, \varphi, B')$, where $B'(y,F)$ is defined by the rule: if $y \in Y_x$ and $x \in F$, then $B'(y,F) = Y_x$, otherwise $B'(y,F) = \bigcup_{x\in X\setminus F} Y_x$. Clearly, $\mathcal{B}'$ is not pseudodiscrete, the family $\mathcal{F} = \{Y_x : x \in X\}$ is uniformly bounded in $\mathcal{B}'$ and $\mathcal{B}'/_{\mathcal{F}}$ is asymorphic to $\mathcal{B}$, so $\mathcal{B}'$ is quasi-asymorphic to $\mathcal{B}$.

Let X, Y be sets, φ and ψ be filters on X and Y. We are going to answer the question: when the pseudodiscrete balleans $\mathcal{B}(X,\varphi)$ and $\mathcal{B}(Y,\psi)$ are quasi-asymorphic? We say that φ and ψ are *equivalent* if there exist the subsets $\Phi \in \varphi$, $\Psi \in \psi$ and a bijection $h : \Phi \to \Psi$ such that, for a subset $F \subseteq \Phi$, we have $F \in \varphi$ if and only if $h(F) \in \Psi$. If the balleans $\mathcal{B}(X,\varphi)$ and $\mathcal{B}(Y,\psi)$ are asymorphic then φ and ψ are equivalent with $\Phi = X$, $\Psi = Y$.

Theorem 3. *Let X, Y be sets, φ, ψ be filters on X and Y such that $\bigcap\varphi = \bigcap\psi = \emptyset$, then the balleans $\mathcal{B}(X,\varphi)$ and $\mathcal{B}(Y,\psi)$ are quasi-asymorphic if and only if φ and ψ are equivalent.*

Proof. Let $\mathcal{B}(X,\varphi)$ and $\mathcal{B}(Y,\psi)$ are quasi-asymorphic. We fix a quasi-asymorphic embedding $f : X \to Y$ such that $f(X)$ is large in Y. Since the partition $ker\ f$ of X is uniformly bounded in $\mathcal{B}(X,\varphi)$, there exists $\Phi \in \varphi$ such that every element $x \in \Phi$ defines the element $\{x\}$ of $ker\ f$. It follows that the restriction f' of f to Φ is injective. Since $X\setminus\Phi$ is bounded in $\mathcal{B}(X,\varphi)$ and $\mathcal{B}(X,\varphi)$ is connected, Φ is large in $\mathcal{B}(X,\varphi)$, hence $f(\Phi)$ is large in $f(X)$. Since $f(X)$ is large in $\mathcal{B}(Y,\psi)$, $f(\Phi)$ is large in $\mathcal{B}(Y,\psi)$. It follows that $f(\Phi) \in \psi$. Put $\Psi = f(\Phi)$. Then $h : \Phi \to \Psi$ is an asymorphism between the subballeans of $B(X,\varphi)$ and $B(X,\psi)$ determined by the subsets Φ and Ψ. Hence, φ and ψ are equivalent.

Assume that φ and ψ are equivalent and $\Phi \in \varphi$, $\Psi \in \psi$, $h : \Phi \to \Psi$ are corresponding sets and bijection. We take an arbitrary extension $f : X \to \Psi$ and note that f is a quasi-asymorphic embedding of $\mathcal{B}(X,\varphi)$ to $\mathcal{B}(Y,\psi)$. □

Let $\mathcal{B}(X,\varphi)$ be a connected pseudodiscrete ballean. We take a symbol ∞ and put $\dot{X} = X\bigcup\{\infty\}$. Then we endow $\dot{X}$ with the topology τ_φ in the following way: every point $x \in X$ is isolated in τ_φ and the family $\{F\bigcup\{\infty\} : F \in \varphi\}$ is a filter of neighborhoods of ∞ in τ_φ. On the other hand, let Y be a topological space with only one non-isolated point y. Let ψ be a filter of neighborhoods of y. We put $X = Y\backslash\{y\}$, $\varphi = \{F\backslash\{y\} : F \in \psi\}$. Then the ballean $\mathcal{B}(X,\varphi)$ is connected and $(\dot{X},\tau_\varphi)$ is homeomorphic to Y. Thus, we have defined the correspondence between the class of connected pseudodiscrete balleans and the class of topological spaces with only one non-isolated points.

Let X, Y be sets, φ, ψ be filters on X and Y such that $\bigcap\varphi = \bigcap\psi = \emptyset$. Then the spaces $(\dot{X},\tau_\varphi)$ and $(\dot{Y},\tau_\psi)$ are homeomorphic if and only if $\mathcal{B}(X,\varphi)$ and $\mathcal{B}(Y,\psi)$ are asymorphic. By Theorem 3, $\mathcal{B}(X,\varphi)$ and $\mathcal{B}(Y,\psi)$ are quasi-asymorphic if and only if the non-isolated points of $(\dot{X},\tau_\varphi)$ and $(\dot{Y},\tau_\psi)$ have homeomorphic neighborhoods.

5. Slowly oscillating functions

Let $\mathcal{B} = (X,P,B)$ be a ballean. A function $f : X \to \mathbb{R}$ is called *slowly oscillating* if , for $\alpha \in P$ and every $\varepsilon > 0$, there exists a bounded subset V of X such that

$$diam\ f(B(x,\alpha)) < \varepsilon$$

for every $x \in X\backslash V$, where $diam A = sup\{|a-b| : a, b \in A\}$.

Theorem 4. *For every ballean $\mathcal{B} = (X,P,B)$, the following statements are equivalent:*

(i) $\mathcal{B}$ is pseudodiscrete;

(ii) every function $f : X \to \mathbb{R}$ is slowly oscillating;

(iii) every function $f : X \to \{0,1\}$ is slowly oscillating.

Proof. (i)⇒(ii). Given $\alpha \in P$ and $\varepsilon > 0$, we take a bounded subset $V \subseteq X$ such that $|B(x,\alpha)| = 1$ for every $x \in X\backslash V$. Then $diam f(B(x,\alpha)) = 0$ for every $x \in X\backslash V$, so $diam f(B(x,\alpha)) < \varepsilon$.

The implication (ii)⇒(iii) is trivial.

(iii)⇒(i). First we show that at most one connected component of X is not a singleton. Suppose not and choose two connected components X_0, X_1 of such that $|X_0| > 1$, $|X_1| > 1$. Let $x_0, x_0' \in X_0$, $x_0 \neq x_0'$ and $x_1, x_1' \in X_1$, $x_1 \neq x_1'$. Then we define a function $f : X \to \{0,1\}$ by the rule $f(x_0) = f(x_1) = 0$ and $f(x) = 1$ for every $x \in X\backslash\{x_0, x_1\}$. Pick $\alpha \in P$ such that $x_0' \in B(x_0,\alpha)$, $x_1' \in B(x_1,\alpha)$. If V is a bounded subset of X, then there exists $i \in \{0,1\}$ such that $V\bigcap X_i = \emptyset$. Then $diam f(B(x_i,\alpha)) = 1$, so f is not slowly oscillating. Hence, to prove that $\mathcal{B}$ is pseudodiscrete, we may suppose that $\mathcal{B}$ is connected.

Fix an arbitrary $\alpha \in P$ and choose a subset Y of X such that $\{B(y,\alpha) : y \in Y\}$ is a maximal disjoint family. We put $Y_0 = \{y \in Y : |B(y,\alpha)| = 1\}$, $Y_1 = Y \backslash Y_0$. It suffices to show that $X \backslash Y_0$ is bounded.

Assume that Y_1 is unbounded and consider the function $f : X \to \{0,1\}$, defined by the rule: $f|_{Y_1} \equiv 1$, $f|_{X\backslash Y_1} \equiv 0$. Given an arbitrary bounded subset V of X, we take $y \in Y_1$ such that $y \notin V$. Then $f(B(y,\alpha)) = \{0,1\}$, so f is not slowly oscillating. Hence, Y_1 is bounded.

Assume that $X \backslash Y_0$ is bounded and define a function $f : X \to \{0,1\}$ by the rule: $f|_{Y_0} \equiv 1$, $f|_{X\ Y_0} \equiv 0$. Pick $\beta \in P$ such that $B(B(x,\alpha),\alpha) \subseteq B(x,\beta)$ for every $x \in X$. By the assumption, f is slowly oscillating, so there exists a bounded subset U of X such that $diam f(B(x,\beta)) < \frac{1}{2}$ for every $x \in X \backslash U$. Since $\mathcal{B}$ is connected and Y_1, U are bounded, the subset $Y_1 \bigcup U$ is bounded. We put $V = B(Y_1 \bigcup U, \beta)$. Since $X \backslash Y_0$ is unbounded and V is bounded, we can choose some $z \in (X \backslash Y_0) \backslash V$. Then $diam f(B(z,\beta)) < \frac{1}{2}$. Since $z \notin B(y_1,\beta)$, by the choice of Y, there exists $y \in Y_0$ such that $B(z,\alpha) \bigcap B(y,\alpha) = \emptyset$, so $y \in B(z,\beta)$ and $diam f(B(z,\beta)) = 1$, a contradiction. $\square$

Under additional (but omited in formulation) assumption of connectedness of $\mathcal{B}$ the above theorem was proved in [7, Proposition 3.2].

Following [7], we say that a ballean $\mathcal{B} = (X, P, B)$ is *pseudobounded* if, for every slowly oscillating function $f : X \to \mathbb{R}$, there exists a bounded subset V of X such that f is bounded on $X \backslash V$. To characterize the pseudodiscrete pseudobounded balleans we use the Stone-Čech compactifications.

Let X be a discrete space, βX be the Stone-Čech compactification of X. We take the points of βX to be the ultrafilters on X with the points of X identified with the principal ultrafilters. For every subset $A \subseteq X$, we put $\bar{A} = \{q \in \beta X : A \in q\}$. The topology of βX can be defined by stating that the family $\{\bar{A} : A \subseteq X\}$ is a base for the open sets. For every filter φ on X, the subset $\bar{\varphi} = \bigcap\{\bar{A} : A \in \varphi\}$ is closed in βX, and, for every nonempty closed subset $K \subseteq \beta G$, there exists a filter φ on X such that $K = \bar{\varphi}$.

A filter φ on a set X is called *countably complete* if φ is closed under countable intersections.

Theorem 5. *Let X be a set and let φ be a filter on X. The ballean $\mathcal{B}(X,\varphi)$ is pseudobounded if and only if the set $\bar{\varphi}$ is finite and every ultrafilter $p \in \bar{\varphi}$ is countably complete.*

Proof. Let $\mathcal{B}(X,\varphi)$ be pseudobounded, but $\bar{\varphi}$ is infinite. Since the space $\bar{\varphi}$ is Hausdorf, we can choose a sequence $(p_n)_{n\in\omega}$ of elements of $\bar{\varphi}$ and a sequence $(P_n)_{n\in\omega}$ of subsets of X such that, for every $n \in \omega$, $P_n \in p_n$ and the family $\{P_n : n \in \omega\}$ is disjoint. We define a function $f : X \to \mathbb{R}$ by the rule: $f|_{P_n} \equiv n$, $n \in \omega$ and $f(x) = 0$ for every

$x \in X \setminus \bigcup_{n\in\omega} P_n$. By Theorem 4, f is slowly oscillating. Let V be a bounded subset of X. Then either $X \setminus V \in \varphi$ or V is singleton. In both cases $f|_{X\ V}$ is unbounded, so $\mathcal{B}(X, \varphi)$ is not pseudodiscrete. Hence, $\bar{\varphi}$ is finite.

We show that every ultrafilter $p \in \bar{\varphi}$ is countable complete. Otherwise, we fix $q \in \bar{\varphi}$ such that q is not countably complete and partition $X = \bigcup_{n\in\omega} X_n$ so that $X_n \neq q$ for every $n \in \omega$. Define a function $f : X \to \mathbb{R}$ by the rule $f|_{X_n} \equiv n$, $n \in \omega$. Let V be an arbitrary bounded subset of X. If there exists $m \in \omega$ such that $X_n \subseteq V$ for every $n \geq \omega$, then $X_i \in q$ for some $i < n$, contradicting to the choice of $\{X_n : n \in \omega\}$. Hence, $X_n \setminus V = \emptyset$ for infinitely many $n \in \omega$ and $f|_{X_n \setminus V}$ is unbounded and $\mathcal{B}$ is not pseudobounded.

Suppose that $\bar{\varphi}$ is finite and every ultrafilter $p \in \bar{\varphi}$ is countably complete. We fix an arbitrary function $f : X \to \mathbb{R}$ and put $X_n = \{x \in X : n \leq |f(x)| < n+1\}$, $n \in \omega$. For every $p \in \bar{\varphi}$, we pick $n(p) \in \omega$ such that $X_{n(p)} \in p$ and put $Y = \bigcup_{p\in\bar{\varphi}} X_{n(p)}$. Then $Y \in \varphi$ and $f|_Y$ is bounded, so $\mathcal{B}(X, \varphi)$ is pseudobounded. $\square$

In view of Theorem 5, there exists a proper pseudodiscrete pseudobounded ballean on a set X if and only if there exists a countably complete free ultrafilter on X, i.e. the cardinal $|X|$ is Ulam-measurable. It is well-known that the first Ulam-measurable cardinal is measurable. Hence, the existence of proper pseudodiscrete pseudobounded ballean is equivalent to the set-theoretical axiom MC.

6. Pseudodiscretness and resolvability

A ballean $\mathcal{B} = (X, P, B)$ is called *irresolvable* if X can not be partitioned to two large subsets. For resolvability of balleans see [8]. Clearly, a ballean $\mathcal{B}$ is irresolvable if and only if at least one connected component of $\mathcal{B}$ is irresolvable, a bounded ballean is irresolvable if and only if it is a singleton, so the irresolvability problem is reduced to the class of proper balleans.

We show that irresolvability is tightly connected with pseudodiscretness.

Theorem 6. *For a proper ballean $\mathcal{B} = (X, P, B)$, the following statements are equivalent:*

(i) $\mathcal{B}$ is irresolvable;

(ii) for every $\alpha \in P$, the subset $X_\alpha = \{x \in X : |B(x, \alpha)| = 1\}$ is unbounded;

(iii) there exists a filter φ on X such that $\bigcap \varphi = \emptyset$ and $\mathcal{B} \preceq \mathcal{B}(X, \varphi)$.

Proof. (i)$\Rightarrow$(ii). Assume the contrary: the subset X_α is bounded for some $\alpha \in P$. We take a subset Y of X such that $|B(y, \alpha)| > 1$, $y \in Y$

and the family $\{B(y,\alpha) : y \in Y\}$ is maximal disjoint. For every $y \in Y$, we pick $y' \in B(y,\alpha)$, $y' \neq y$ and put $Y' = \{y' : y \in Y\}$. Then Y, Y_1 are disjoint large subsets of X and we get a contradiction to irresolvability of $\mathcal{B}$.

(ii)$\Rightarrow$(iii). If $\alpha, \alpha' \in P$ and $\beta \geq \alpha, \beta \geq \alpha'$, then $X_\beta \subseteq X_\alpha \bigcap X_{\alpha'}$. It follows that the family $\{X_\alpha : \alpha \in P\}$ is a base for some filter φ on X. Since $\mathcal{B}$ is connected, $\bigcap \varphi = \emptyset$. For any $x \in X$, $\alpha \in P$, we have $B(x,\alpha) \subseteq B_\varphi(x, X\ X_\alpha)$, so $\mathcal{B} \preceq \mathcal{B}(X,\varphi)$.

(iii)$\Rightarrow$(i). Let A be a subset of X. If A is large in $\mathcal{B}$, then A is large in $\mathcal{B}(X,\varphi)$. If A is large in $\mathcal{B}(X,\varphi)$, then $A \in \varphi$. It follows that any two large in $\mathcal{B}$ subsets of X meets, so $\mathcal{B}$ is irresolvable. ☐

Let $\mathcal{B} = (X,P,B)$ be a ballean, $Y \subseteq X$. We say that subbalean $\mathcal{B}_Y$ of $\mathcal{B}$ is *almost isolated* (almost invariant in terminology of [9]) if, for every $\alpha \in P$, the subset $B(Y,\alpha)\ Y$ is bounded in $\mathcal{B}$.

Lemma 1. *Let $\mathcal{B} = (X,P,B)$ be a proper ballean, $Y \subseteq X$ and subballean $\mathcal{B}_Y$ is unbounded and almost isolated in $\mathcal{B}$. If $\mathcal{B}_Y$ is irresolvable, then $\mathcal{B}$ is irresolvable.*

Proof. Given an arbitrary $\alpha \in P$, it suffices to fined $y \in Y$ such that $B(y,\alpha) \subseteq Y$ and $|B(y,\alpha)| = 1$. Assume that it does not hold. Put $Y_0 = \{y \in Y : B(y,\alpha) \nsubseteq Y\}$, $Y_1 = \{y \in Y : B(y,\alpha) \subseteq Y\}$. Since $\mathcal{B}_Y$ is almost isolated in $\mathcal{B}$, Y_0 is bounded. By the assumption, $|B(y,\alpha)| > 1$ for every $y \in Y_1$. We choose a subset $Z \subseteq Y_1$ such that the family $\{B(z,\alpha) : z \in Z\}$ is maximal disjoint. For every $z \in Z$, we take an arbitrary $z' \in B(z,\alpha)$, $z \neq z'$. Put $Z' = \{z' : z \in Z\}$. If $x \in Y_1$, then $B(x,\alpha) \bigcap B(z,\alpha) \neq \emptyset$ for some $z \in Z$. It follows that Z and Z' are large in Y_1. Since Y_0 is bounded, $Y = Y_0 \bigcup Y_1$ and $\mathcal{B}_Y$ is connected, then Z and Z' are disjoint large subsets in $\mathcal{B}_Y$, so we get a contradiction to irresolvability of $\mathcal{B}_Y$. ☐

Following [7], we say that a ballean $\mathcal{B} = (X,P,B)$ is *ordinal* if there exist a cofinal well-ordered (with respect to $\leq$) subset of P. Note that every metrizable ballean is ordinal.

Lemma 2. *Let $\mathcal{B} = (X,P,B)$ be a proper irresolvable ordinal ballean. Then there exists an unbounded subset Y of X such that the subballean $\mathcal{B}_Y$ is pseudodiscrete and almost isolated in $\mathcal{B}$.*

Proof. We may suppose that P is well-ordered. Let $|P| = \gamma$. We identify P with the set of ordinals $\{\alpha : \alpha < \gamma\}$. Replacing P by some its cofinal subset, we may assume that γ is a regular cardinal. For every $\alpha \in P$, we put $Y_\alpha = \{y \in X : |B(y,\alpha)| = 1\}$. By Theorem 6, Y_α is unbounded. Since γ is regular, every subset of X of cardinality $< \gamma$ is bounded. This observation allows us to construct an injective transfinite sequence $(y_\alpha)_{\alpha<\gamma}$ of elements of X such that, for every $\alpha < \gamma$, we have

$$y_\alpha \in Y_\alpha, \ B(y_\alpha, \alpha) \bigcap (\bigcup_{\lambda<\alpha} B(y_\lambda, \lambda)) = \emptyset.$$

Put $Y = \{y_\alpha : \alpha < \gamma\}$. Then the subset $\{y_\lambda : \lambda < \alpha\}$ is bounded in $\mathcal{B}$ and $B(y_\beta, \alpha) \subseteq Y$, $|B(y_\beta, \alpha)| = 1$ for every $\beta \geq 1$. Hence, $\mathcal{B}_Y$ is pseudodiscrete and almost isolated in $\mathcal{B}$. □

We do not know whether Lemma 2 is true without the ordinalily assumption on $\mathcal{B}$.

Theorem 7. *Let $\mathcal{B} = (X, P, B)$ be a proper ordinal ballean. Then $\mathcal{B}$ is irresolvable if and only if there exists a subset Y of X such that the subbalean $\mathcal{B}_Y$ is pseudodiscrete and almost invariant in $\mathcal{B}$.*

Proof. Apply Lemmas 1 and 2. □

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