Algebra and Discrete Mathematics
Number 4. **(2006).** pp. 12 – 39

RESEARCH ARTICLE

Geometry monoid of the left distributivity and the left idempotency

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Communicated by L. Marki

Abstract. We construct here the geometry monoids of LDI (left distributive idempotent) and of LDLI (left distributive left idempotent) identities. We study their properties and construct a monoid with solvable word problem based on relations of the geometry monoid of LDLI.

Introduction

Geometry monoids and geometry groups are structures that describe the action of identities on terms. The most known ones are the Thompson group F as the geometry group of the associativity and the Thompson group V as the geometry group of the associativity and the commutativity [8]. Actually, every identity has its geometry monoid and its geometry group but the monoid can be too complicated and the group can be too far away from the monoid to tell us something about the investigated identity.

Nevertheless, in some cases, like of the left distributivity [5], studying the geometry monoid brought a solution of the world problem. And the method can be used also for some other identities, like $x(yz) = (xy)(yz)$ [6]. Hence a question arose whether this approach can or cannot solve the world problem of the left distributivity and the idempotency (LDI), that means of identities $x(yz) = (xy)(xz)$ and $x = xx$. Regrettably, only little progress has been achieved so far.

When studying the problem, the author focused on the identity of the left idempotency $xy = (xx)y$ which appears naturally in some left distributive structures (e.g. left distributive left quasigroups). It seems that the combination of the left distributivity and the left idempotency

(LDLI) is very close to LDI [9] and that their word problems could be of the same difficulty to solve.

It is likely that any attempt to attack the word problem of LDI considering its monoid of geometry is hopeless. However, the geometry monoid of LDLI looks more friendly and we manage here to construct a monoid with solvable word problem based on the geometry monoid. This monoid can serve as a corner stone for further constructions involving the geometric monoid of LDLI.

We use in the paper the same approach as Dehornoy used for proving properties of the geometry monoid of the left distributivity [4]. In Section 1 we introduce the geometry monoids for LDI and LDLI. In Section 2 we study some relations in the geometry monoids and write them down in a presentation. Monoids with this presentations are called syntactical. In Section 3 we study the syntactical monoids and establish some more complex relations. In Section 4 we prove that for any pair of elements there exists a common right multiple. We use the result in Section 5 where we prove by the word reversing method that the syntactical monoid of LDLI is left cancellative, has solvable word problem and its left divisibility order forms a lattice.

1. The geometry monoids

The construction of geometry monoids is already standard. We give thus only a brief description of the monoids properties and leave some propositions unproved. A more detailed description is given in [10].

We fix an infinite set of variables $\{x_1, x_2, x_3, \ldots\}$ and we denote by T the set of all terms with these variables. We shall consider three types of identities

left distributivity (LD)	$x(yz) = (xy)(xz)$
idempotency (I)	$x = xx$
left idempotency (LI)	$xy = (xx)y$

To avoid excessive uses of parenthesis we write xyz instead of $x(yz)$. We shall consider two families of identities: left distributivity + idempotency (LDI) and left distributivity + left idempotency (LDLI). We denote by $\overset{\text{LDI}}{=}$ and $\overset{\text{LDLI}}{=}$ the congruences on T generated by these families respectively.

Applying an identity means replacing a subterm of a specific form by a term of another specific form. We shall formalise this situation introducing addresses of subterms. Each subterm is represented by a sequence from $\{0,1\}^*$ where 0 means left and 1 means right. The empty sequence is denoted by $\varnothing$. The set of all addresses is denoted $\mathbf{A}$.

Definition. For each address α, we define D_α as the partial function from T to T that replaces the subterm $t_1t_2t_3$ on the address α by the term $t_1t_2 \cdot t_1t_3$. We also define I_α as the partial function from T to T that replaces the subterm t_1 on the address α by the term t_1t_1. We write $t \bullet \mathrm{D}_\alpha$, respectively $t \bullet \mathrm{I}_\alpha$ to be the images of a term t under these mappings.

Example 1.1. Consider $t = x_1x_2x_3x_4$ (see Figure 1). The term belongs to the domain of the operators $\mathrm{D}_\varnothing$, D_1, $\mathrm{I}_\varnothing$, I_0, I_1, I_{10}, I_{11}, I_{110} and I_{111}.

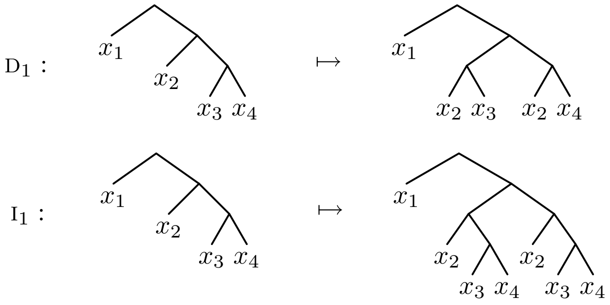

Figure 1: The operators D_1 and I_1 and their actions on the term $x_1 \cdot x_2 \cdot x_3 \cdot x_4$.

We have not defined what is an LI-operator. In fact, we do not need it if we notice that replacing a subterm t_1t_2 on an address α by $t_1t_1 \cdot t_2$ is the same as doubling the subterm on the address $\alpha 0$, *i.e.,* applying the operator $\mathrm{I}_{\alpha 0}$.

We also need a formal notation for an operator that can mean either an D_α or an I_α. Since we will work in different contexts (LD, LDI or LDLI), we should be careful and give an exact definition of what such an operator DI_α means.

Definition. Let $\mathbf{A}_{\mathrm{LD}}$ and $\mathbf{A}_{\mathrm{I}}$ be two disjoint copies of the address set $\mathbf{A}$. We denote by $\mathbf{A}_{\mathrm{LDI}}$ the set $\mathbf{A}_{\mathrm{LD}} \cup \mathbf{A}_{\mathrm{I}}$ and, for each α in $\mathbf{A}_{\mathrm{LDI}}$, we define DI_α either as D_α if α belongs to $\mathbf{A}_{\mathrm{LD}}$, or as I_α if α belongs to $\mathbf{A}_{\mathrm{I}}$. We denote also by $\mathbf{A}_{\mathrm{LI}}$ the subset of $\mathbf{A}_{\mathrm{I}}$ defined as $\{\alpha \in \mathbf{A}_{\mathrm{I}};\ \exists\gamma : \alpha = \gamma 0\}$ and we denote by $\mathbf{A}_{\mathrm{LDLI}}$ the set $\mathbf{A}_{\mathrm{LD}} \cup \mathbf{A}_{\mathrm{LI}}$.

The following lemma results immediately from the definition of the operators:

Lemma 1.2. *For each address* α *in* $\mathbf{A}_{\text{LDI}}$*, the operator* DI_α *is a partial injective mapping to* T*; its inverse is the operator* DI_α^{-1} *defined as* DI_α *but exchanging the roles of* $t_1t_2t_3$ *and* $t_1t_2 \cdot t_1t_3$*, respectively* t_1 *and* t_1t_1*.*

All the following material in the section is written for the family LDI but we can formulate all the things the same way for the family LDLI.

Definition. The *geometry monoid* of LDI is the monoid $\mathcal{G}_{\text{LDI}}$ generated by the operators $\text{DI}_\alpha^{\pm 1}$ with α in $\mathbf{A}_{\text{LDI}}$ using the composition. Analogically, the *positive geometry monoid* is the monoid $\mathcal{G}_{\text{LDI}}^{+}$ generated by the operators DI_α with α in $\mathbf{A}_{\text{LDI}}$

The monoid $\mathcal{G}_{\text{LDI}}$ is not a group because the mapping $\text{DI}_\alpha^{-1} \circ \text{DI}_\alpha$ is not generally the identical mapping on T, it is only the identity on the domain of DI_α.

By definition, the elements of $\mathcal{G}_{\text{LDI}}^{+}$ are finite products of operators DI_α hence they are of the form

$$\text{DI}_{\alpha_p} \circ \cdots \circ \text{DI}_{\alpha_2} \circ \text{DI}_{\alpha_1} .$$

Such elements can be expressed as finite sequences of addresses in $\mathbf{A}_{\text{LDI}}$, *i.e.,* by words on $\mathbf{A}_{\text{LDI}}$. We denote by $\mathbf{A}_{\text{LDI}}^{*}$ the set of such words. The product of two words u and v is the concatenation of the words, denoted by uv or $u \cdot v$. We write the ε symbol for the empty word.

Remark 1.3. We should not confuse $\varnothing$ and ε. The word ε is a word of length 0. The word $\varnothing$ is a word of length 1 that consists of the address $\varnothing$ (the empty address).

We consider that the operators DI_α act on the right, *i.e.,* that we compose the operators from the left to the right. Therefore, in order not to confuse the composition to the right with the composition $\circ$, we use the symbol $\bullet$.

Definition. For $\alpha_1 \cdot \cdots \cdot \alpha_p$ in $\mathbf{A}_{\text{LDI}}^{*}$, the operator DI_u is defined as $\text{DI}_{\alpha_1} \bullet \cdots \bullet \text{DI}_{\alpha_p}$, *i.e.,* as $\text{DI}_{\alpha_p} \circ \cdots \circ \text{DI}_{\alpha_1}$. The operator DI_ε is defined as the identity.

Consider now the monoid $\mathcal{G}_{\text{LDI}}$. Its elements are of the form

$$\text{DI}_{\alpha_1}^{e1} \bullet \text{DI}_{\alpha_2}^{e_2} \bullet \cdots \bullet \text{DI}_{\alpha_p}^{e_p}$$

with α_i in $\mathbf{A}_{\text{LDI}}$ and $e_i = \pm 1$, for i between 1 and p. To specify such a product, it is natural to introduce the formal inverse α^{-1} of an address α in $\mathbf{A}_{\text{LDI}}$. We write $\mathbf{A}_{\text{LDI}}^{-1}$ for the set of formal inverses of addresses in $\mathbf{A}_{\text{LDI}}$, and $\mathbf{A}_{\text{LDI}}^{\pm 1}$ for the set $\mathbf{A}_{\text{LDI}} \cup \mathbf{A}_{\text{LDI}}^{-1}$.

Definition. For $w = \alpha_1^{e_1} \cdot \cdots \cdot \alpha_p^{e_p}$ in $(\mathbf{A}_{\text{LDI}}^{\pm 1})^*$, the operator DI_w is defined as the product $\mathrm{DI}_{\alpha_1}^{e_1} \bullet \cdots \bullet \mathrm{DI}_{\alpha_p}^{e_p}$.

The elements in $\mathbf{A}_{\text{LDI}}^*$ are called *positive words* and the ones in $(\mathbf{A}_{\text{LDI}}^{\pm 1})^*$ are called simply words.

Definition. For a term t and a word w in $(\mathbf{A}_{\text{LDI}}^{\pm 1})^*$, the term $t \bullet w$ is defined as the image of t under the mapping DI_w, if it exists.

By construction, one has the relation $t \bullet w \overset{\text{LDI}}{=} t$ for each word w in $(\mathbf{A}_{\text{LDI}}^{\pm 1})^*$, if t belongs to the domain of DI_w. In fact, if we consider the definition of an equivalence on terms, we get immediately an equivalence:

Proposition 1.4. *Let t and t' be two terms in T. The terms t and t' are LDI-equivalent if and only if an operator in $\mathcal{G}_{\text{LDI}}$ sends t to t', i.e., if we have $t' = t \bullet w$ for a word w on $\mathbf{A}_{\text{LDI}}^{\pm 1}$.*

In order to be able to study the operators more deeply, we have to describe their domains. This can be done by a rather standard technique and hence there is no need to prove the results here properly. The reader can look into Dehornoy's book [4], where the results were proved properly for the left distributivity, and check that it works exactly the same way for LDI (one needs only to notice that a term belongs to the domain of a positive operator DI_α if and only if it is large enough). We give here thus the results of the investigation only.

Definition. A *substitution* h is a homomorphism $h : T \to T$. A *substitute* of a term t is a term $h(t)$ for a substitution h.

Proposition 1.5. [10] *For each positive word u on $\mathbf{A}_{\text{LDI}}$ there exists a unique (up to renaming of variables) pair of terms (t_u^L, t_u^R) such that the operator DI_u sends a term t on a term t' if and only if there exists a substitution h such that $(t_u^L)^h = t$ and $(t_u^R)^h = t'$.*

Example 1.6. Take, for instance, $\mathrm{D}_u = \mathrm{D}_0 \bullet \mathrm{I}_{10}$. Every term in the domain of D_u has to contain the addresses 010 and 10. The most general such a term is $t_u^L = x_1 x_2 x_3 \cdot x_4 x_5$ and one has $t_u^R = t_u^L \bullet u = (x_1 x_2 \cdot x_1 x_3) \cdot x_4 x_4 \cdot x_5$.

Proposition 1.7. [10] *Let $u_1, \ldots, u_m$ be two words on $\mathbf{A}_{\text{LDI}}$. Then the intersection of the domains of the operators $\mathrm{DI}_{u_1}, \ldots, \mathrm{DI}_{u_m}$ is the set of all the substitutes of a unique term $t_{u_1,\ldots,u_m}^L$.*

Example 1.8. Consider $\mathrm{DI}_{u_1} = \mathrm{D}_0 \bullet \mathrm{I}_{10}$ and $\mathrm{DI}_{u_2} = \mathrm{D}_\varnothing \bullet \mathrm{D}_1 \bullet \mathrm{I}_{00}$. Any term in the domain of both the operators have to contain the addresses 010 and 10 (from u_1) as well as the addresses 10, 110 and 0 (from u_2). The most general such a term is $t_{u_1,u_2}^L = x_1 x_2 x_3 \cdot x_4 x_5 x_6$.

2. Relations in the positive geometry monoids

In this section we find some relations true in the positive geometry monoids. We consider the monoid $\mathcal{G}^{+}_{\text{LDLI}}$ as a submonoid of the monoid $\mathcal{G}^{+}_{\text{LDI}}$ and hence it suffices to state all the found results for $\mathcal{G}^{+}_{\text{LDI}}$.

Definition. Let t be a term. An address α is called *internal* if there exists a subterm of t at the address α. The set of all internal addresses is called the *skeleton* of t and denoted by $\mathrm{Skel}(t)$. The *outline* of t, denoted by $\mathrm{Out}(t)$ is the set of all addresses of all leaves of t. An addresses α is called a *prefix* of an address β if we have $\alpha\gamma = \beta$ for some address γ. We write then $\alpha \sqsubseteq \beta$. If neither α is a prefix of β nor β is a prefix of α then we say that they are *orthogonal* and we write $\alpha \perp \beta$.

We first notice that if two operators act on orthogonal addresses then they act independently on independent subterms. Therefore we can commute them and obtain, for all α, β, γ,

$$\textsc{di}_{\alpha 0\gamma} \bullet \textsc{di}_{\alpha 1\beta} = \textsc{di}_{\alpha 1\beta} \bullet \textsc{di}_{\alpha 0\gamma}\,.$$

Hence we can consider only pairs of addresses where one is a prefix of the other. For facilitating the notation we introduce the shifting of addresses.

Definition. For w a word on $\mathbf{A}^{\pm 1}_{\text{LDI}}$ and γ an address in $\mathbf{A}$, we denote by $\mathrm{sh}_{\gamma}(w)$, or simply by γw, the *γ-shift* of w defined as the word obtained from w replacing each address $\alpha^{\pm 1}$ by the address $\gamma\alpha^{\pm 1}$.

Example 2.1. For $\textsc{di}_{u} = \textsc{d}_{01} \bullet \textsc{i}_{110} \bullet \textsc{i}_{\varnothing}$ one has $\textsc{di}_{1u} = \textsc{d}_{101} \bullet \textsc{i}_{1110} \bullet \textsc{i}_{1}$. Analogously, $\textsc{di}_{1\varepsilon} = \textsc{di}_{\varepsilon} = \mathrm{id}_{T}$ since ε is the empty address.

We immediately see

Lemma 2.2. *For each α, $\textsc{di}_{w} = \textsc{di}_{w'}$ implies $\textsc{di}_{\alpha w} = \textsc{di}_{\alpha w'}$.*

Hence we can suppose from now on that one of the operators is $\textsc{di}_{\varnothing}$. Let us start with $\textsc{i}_{\varnothing}$. If an operator $\textsc{di}_{\alpha}$ acts on a term t and we then double the term, it is the same as to apply $\textsc{di}_{0\alpha} \bullet \textsc{di}_{1\alpha}$ on the term $t \cdot t$. Therefore we get

$$\textsc{di}_{\alpha} \bullet \textsc{i}_{\varnothing} = \textsc{i}_{\varnothing} \bullet \textsc{di}_{0\alpha} \bullet \textsc{di}_{1\alpha}\,.$$

Now consider the operator $\textsc{d}_{\varnothing}$. If an operator acts in the left subterm of a term t, for instance $\textsc{di}_{0\alpha}$, its image is copied by $\textsc{d}_{\varnothing}$ to two addresses, namely 00α and 10α. Therefore we obtain

$$\textsc{di}_{0\alpha} \bullet \textsc{d}_{\varnothing} = \textsc{d}_{\varnothing} \bullet \textsc{di}_{00\alpha} \bullet \textsc{di}_{10\alpha}\,.$$

Analogically, the subterm 10 is sent to 01 and the subterm 11 is left untouched. Therefore we have

$$\mathrm{DI}_{10\alpha} \bullet \mathrm{D}_{\varnothing} = \mathrm{D}_{\varnothing} \bullet \mathrm{DI}_{01\alpha},$$
$$\mathrm{DI}_{11\alpha} \bullet \mathrm{D}_{\varnothing} = \mathrm{D}_{\varnothing} \bullet \mathrm{DI}_{11\alpha}\,.$$

However, not all relations are that simple. We present here a "relation" where we have to be careful about the domain of the operators.

Lemma 2.3. *For each term t which is not a variable, we have*

$$t \bullet \mathrm{I}_1 \bullet \mathrm{D}_{\varnothing} = t \bullet \mathrm{I}_{\varnothing}\,.$$

Proof. Let $t = t_1 \cdot t_2$. Left side makes $t \mapsto t_1 \cdot t_2 \cdot t_2 \mapsto (t_1 \cdot t_2) \cdot (t_1 \cdot t_2)$ which is clearly the image of t under $\mathrm{I}_{\varnothing}$. ☐

Although $\mathrm{I}_1 \bullet \mathrm{D}_{\varnothing} = \mathrm{I}_{\varnothing}$ is not true (no variable is in the domain of the left hand side operator), once you multiply the "equality" on the left by anything, you obtain a real equality. Analogously, if you multiply the "equality" by something (not $\mathrm{I}_{\varnothing}$ nor I_0 nor I_1) on the right, you demand that each term in the domain of both operators is more complex than a variable and hence you obtain an equality.

Now there are three more relations that are not so evident:

Lemma 2.4. *One has*

$$\mathrm{D}_1 \bullet \mathrm{D}_{\varnothing} \bullet \mathrm{D}_1 \bullet \mathrm{D}_0 = \mathrm{D}_{\varnothing} \bullet \mathrm{D}_1 \bullet \mathrm{D}_{\varnothing},$$
$$\mathrm{I}_{10} \bullet \mathrm{D}_{\varnothing} \bullet \mathrm{D}_0 = \mathrm{D}_{\varnothing} \bullet \mathrm{I}_0,$$
$$\mathrm{I}_1 \bullet \mathrm{D}_{\varnothing} \bullet \mathrm{D}_1 \bullet \mathrm{D}_0 = \mathrm{D}_{\varnothing} \bullet \mathrm{I}_{\varnothing}\,.$$

Proof. (i) The left side treats a term $t_1 \cdot t_2 \cdot t_3 \cdot t_4$ followingly: $t_1 \cdot t_2 \cdot t_3 \cdot t_4 \mapsto t_1 \cdot (t_2 \cdot t_3) \cdot t_2 \cdot t_4 \mapsto (t_1 \cdot t_2 \cdot t_3) \cdot t_1 \cdot t_2 \cdot t_4 \mapsto (t_1 \cdot t_2 \cdot t_3) \cdot (t_1 \cdot t_2) \cdot t_1 \cdot t_4 \mapsto ((t_1 \cdot t_2) \cdot t_1 \cdot t_3) \cdot (t_1 \cdot t_2) \cdot t_1 \cdot t_4$. The right side makes: $t_1 \cdot t_2 \cdot t_3 \cdot t_4 \mapsto (t_1 \cdot t_2) \cdot t_1 \cdot t_3 \cdot t_4 \mapsto (t_1 \cdot t_2) \cdot (t_1 \cdot t_3) \cdot t_1 \cdot t_4 \mapsto ((t_1 \cdot t_2) \cdot t_1 \cdot t_3) \cdot (t_1 \cdot t_2) \cdot t_1 \cdot t_4$.

(ii) The left side sends a term $t_1 \cdot t_2 \cdot t_3$ to $t_1 \cdot (t_2 \cdot t_2) \cdot t_3 \mapsto (t_1 \cdot t_2 \cdot t_2) \cdot t_1 \cdot t_3 \mapsto ((t_1 \cdot t_2) \cdot t_1 \cdot t_2) \cdot t_1 \cdot t_3$ which is clearly the same as the image of $t_1 \cdot t_2 \cdot t_3$ under the right side. The third relation is $\mathrm{D}_{\varnothing} \bullet \mathrm{I}_{\varnothing} = \mathrm{I}_{\varnothing} \bullet \mathrm{D}_1 \bullet \mathrm{D}_0$ combined with Lemma 2.3. ☐

We finish at this point the study of relations in the geometry monoid since it is not too comfortable: we have to check every time equality of mappings and for more complicated relations it becomes too technical. In order to avoid this computation with operators, we are going to present formal monoids that could be isomorphic to geometry monoids; we simply give the presentations of the formal monoids. These formal monoids are called *syntactical* monoids. Since the monoids can differ from the geometry ones, we have to use formally different elements.

Definition. The set $\boldsymbol{\mathcal{A}}_{\mathrm{LDI}}$ is defined as the set of symbols $\mathtt{d}_\alpha$ and $\mathtt{i}_\alpha$, with α in $\mathbf{A}_{\mathrm{LDI}}$. An *LDI-relation* is a pair of words in $\boldsymbol{\mathcal{A}}_{\mathrm{LDI}}$ among the following relations:

$$\begin{array}{rlcl}
(\mathtt{d}_{\gamma 0\alpha}\cdot\mathtt{d}_{\gamma 1\beta} & ,\ \mathtt{d}_{\gamma 1\beta}\cdot\mathtt{d}_{\gamma 0\alpha}) & & \text{type } \perp\\
(\mathtt{i}_{\gamma 0\alpha}\cdot\mathtt{d}_{\gamma 1\beta} & ,\ \mathtt{d}_{\gamma 1\beta}\cdot\mathtt{i}_{\gamma 0\alpha}) & & \text{type } \perp\\
(\mathtt{d}_{\gamma 0\alpha}\cdot\mathtt{i}_{\gamma 1\beta} & ,\ \mathtt{i}_{\gamma 1\beta}\cdot\mathtt{d}_{\gamma 0\alpha}) & & \text{type } \perp\\
(\mathtt{i}_{\gamma 0\alpha}\cdot\mathtt{i}_{\gamma 1\beta} & ,\ \mathtt{i}_{\gamma 1\beta}\cdot\mathtt{i}_{\gamma 0\alpha}) & & \text{type } \perp\\
(\mathtt{d}_{\gamma 0\alpha}\cdot\mathtt{d}_{\gamma} & ,\ \mathtt{d}_{\gamma}\cdot\mathtt{d}_{\gamma 00\alpha}\cdot\mathtt{d}_{\gamma 10\alpha}) & & \text{type D0}\\
(\mathtt{d}_{\gamma 10\alpha}\cdot\mathtt{d}_{\gamma} & ,\ \mathtt{d}_{\gamma}\cdot\mathtt{d}_{\gamma 01\alpha}) & & \text{type D10}\\
(\mathtt{d}_{\gamma 11\alpha}\cdot\mathtt{d}_{\gamma} & ,\ \mathtt{d}_{\gamma}\cdot\mathtt{d}_{\gamma 11\alpha}) & & \text{type D11}\\
(\mathtt{d}_{\gamma 1}\cdot\mathtt{d}_{\gamma}\cdot\mathtt{d}_{\gamma 1}\cdot\mathtt{d}_{\gamma 0} & ,\ \mathtt{d}_{\gamma}\cdot\mathtt{d}_{\gamma 1}\cdot\mathtt{d}_{\gamma}) & & \text{type D1}\\
(\mathtt{i}_{\gamma\alpha}\cdot\mathtt{i}_{\gamma} & ,\ \mathtt{i}_{\gamma}\cdot\mathtt{i}_{\gamma 0\alpha}\cdot\mathtt{i}_{\gamma 1\alpha}) & & \text{type I}\\
(\mathtt{d}_{\gamma\alpha}\cdot\mathtt{i}_{\gamma} & ,\ \mathtt{i}_{\gamma}\cdot\mathtt{d}_{\gamma 0\alpha}\cdot\mathtt{d}_{\gamma 1\alpha}) & & \text{type DI}\\
(\mathtt{i}_{\gamma 0\alpha}\cdot\mathtt{d}_{\gamma} & ,\ \mathtt{d}_{\gamma}\cdot\mathtt{i}_{\gamma 00\alpha}\cdot\mathtt{i}_{\gamma 10\alpha}) & & \text{type ID0}\\
(\mathtt{i}_{\gamma 10\alpha}\cdot\mathtt{d}_{\gamma} & ,\ \mathtt{d}_{\gamma}\cdot\mathtt{i}_{\gamma 01\alpha}) & & \text{type ID10}\\
(\mathtt{i}_{\gamma 10}\cdot\mathtt{d}_{\gamma}\cdot\mathtt{d}_{\gamma 0} & ,\ \mathtt{d}_{\gamma}\cdot\mathtt{i}_{\gamma 0}) & & \text{type ID10+}\\
(\mathtt{i}_{\gamma 11\alpha}\cdot\mathtt{d}_{\gamma} & ,\ \mathtt{d}_{\gamma}\cdot\mathtt{i}_{\gamma 11\alpha}) & & \text{type ID11}\\
(\mathtt{i}_{\gamma 1}\cdot\mathtt{d}_{\gamma}\cdot\mathtt{d}_{\gamma 1}\cdot\mathtt{d}_{\gamma 0} & ,\ \mathtt{d}_{\gamma}\cdot\mathtt{i}_{\gamma}) & & \text{type ID1}\\
(\mathtt{d}_{\gamma\alpha}\cdot\mathtt{i}_{\gamma 1}\cdot\mathtt{d}_{\gamma} & ,\ \mathtt{d}_{\gamma\alpha}\cdot\mathtt{i}_{\gamma}) & & \text{type C}\\
(\mathtt{i}_{\gamma\alpha}\cdot\mathtt{i}_{\gamma 1}\cdot\mathtt{d}_{\gamma} & ,\ \mathtt{i}_{\gamma\alpha}\cdot\mathtt{i}_{\gamma}) & & \text{type C}\\
(\mathtt{d}_{\gamma}\cdot\mathtt{i}_{\gamma 11}\cdot\mathtt{d}_{\gamma 1} & ,\ \mathtt{d}_{\gamma}\cdot\mathtt{i}_{\gamma 1}) & & \text{type C}\\
(\mathtt{i}_{\gamma 1}\cdot\mathtt{d}_{\gamma}\cdot\mathtt{i}_{\gamma\alpha} & ,\ \mathtt{i}_{\gamma}\cdot\mathtt{i}_{\gamma\alpha}) & \mathrm{lg}(\alpha)\geq 2 & \text{type C}\\
(\mathtt{i}_{\gamma 1}\cdot\mathtt{d}_{\gamma}\cdot\mathtt{d}_{\gamma\alpha} & ,\ \mathtt{i}_{\gamma}\cdot\mathtt{d}_{\gamma\alpha}) & & \text{type C}
\end{array}$$

The set $\boldsymbol{\mathcal{A}}_{\mathrm{LDLI}}$ is defined as the set of symbols $\mathtt{d}_\alpha$ and $\mathtt{i}_\alpha$, with α in $\mathbf{A}_{\mathrm{LDLI}}$. An *LDLI-relation* is an LDI-relation (u, v) such that u and v belong to $\boldsymbol{\mathcal{A}}_{\mathrm{LDLI}}$. The relation $\equiv^{+}_{\mathrm{LDI}}$ is defined as the congruence of the monoid $\boldsymbol{\mathcal{A}}^{*}_{\mathrm{LDI}}$ generated by the LDI-relations. The relation $\equiv^{+}_{\mathrm{LDLI}}$ is defined analogously.

If we want to prove that syntactical monoids are isomorphic to geometry monoids, we have to prove $\mathtt{di}_u \equiv^{+}_{\mathrm{LDI}} \mathtt{di}_v$ if and only if $\mathrm{DI}_u = \mathrm{DI}_v$, for all pairs u, v. One implication is immediate:

Proposition 2.5. *For u, v in $\mathbf{A}^{*}_{\mathrm{LDI}}$, the relation $\mathtt{di}_u \equiv^{+}_{\mathrm{LDI}} \mathtt{di}_v$ gives $\mathrm{DI}_u = \mathrm{DI}_v$.*

Proof. If $(\mathtt{di}_u, \mathtt{di}_v)$ is an LDI-relation, the equality $\mathrm{DI}_u = \mathrm{DI}_v$ falls to one of the types discussed before the definition (possibly up to a shift, which does not make a difference due to Lemma 2.2). The rest is evident. □

We will not prove here the other direction, its validity is unknown for LDI and known to be false for LDLI, as we will see in the last section. Nevertheless, the monoids given by these presentations are rich enough to have all the properties we need in our further study. And, of course, since the geometry monoids are factors of the syntactical monoids, all relations, which are proved in the syntactical monoids, hold also in the geometry monoids.

3. Syntactical relations

In this section we establish some more complex relations. We will work with both the relations $\equiv^+_{\text{LDI}}$ and $\equiv^+_{\text{LDLI}}$ at once. To facilitate the expression, we will write only the symbol $\equiv^+$: the expression "$u \equiv^+ v$, for all words u and v" shall mean $u \equiv^+_{\text{LDI}} v$, for all words u and v in $\mathbf{A}^*_{\text{LDI}}$, as well as $u \equiv^+_{\text{LDLI}} v$, for all words u and v in $\mathbf{A}^*_{\text{LDLI}}$. If some relation involves $\equiv^+_{\text{LDI}}$ only, we write $\equiv^+_{\text{LDI}}$ explicitly.

Proposition 3.1. *For u, u' two words and α in $\mathbf{A}$, the relation $\mathtt{di}_u \equiv^+ \mathtt{di}_{u'}$ gives $\mathtt{di}_{\alpha u} \equiv^+ \mathtt{di}_{\alpha u'}$.* □

Lemma 3.2. *Let u_1 and u_2 be two words such that each address in u_1 is orthogonal to each address in u_2. Then we have*

$$\begin{aligned}
\mathtt{di}_{u_1} \cdot \mathtt{di}_{u_2} &\equiv^+ \mathtt{di}_{u_2} \cdot \mathtt{di}_{u_1} \\
\mathtt{di}_{u_1} \cdot \mathtt{di}_{u_2} \cdot \mathtt{i}_\varnothing &\equiv^+_{\text{LDI}} \mathtt{i}_\varnothing \cdot \mathtt{di}_{0u_1} \cdot \mathtt{di}_{0u_2} \cdot \mathtt{di}_{1u_1}\, \mathtt{di}_{1u_2} \\
\mathtt{di}_{0u_1} \cdot \mathtt{di}_{0u_2} \cdot \mathtt{d}_\varnothing &\equiv^+ \mathtt{d}_\varnothing \cdot \mathtt{di}_{00u_1} \cdot \mathtt{di}_{00u_2} \cdot \mathtt{di}_{10u_1}\, \mathtt{di}_{10u_2} \\
\mathtt{di}_{10u_1} \cdot \mathtt{di}_{10u_2} \cdot \mathtt{d}_\varnothing &\equiv^+ \mathtt{d}_\varnothing \cdot \mathtt{di}_{01u_1}\, \mathtt{di}_{01u_2} \\
\mathtt{di}_{11u_1} \cdot \mathtt{di}_{11u_2} \cdot \mathtt{d}_\varnothing &\equiv^+ \mathtt{d}_\varnothing \cdot \mathtt{di}_{11u_1}\, \mathtt{di}_{11u_2}
\end{aligned}$$

Proof. Use the induction on $\lg(u_1) + \lg(u_2)$. □

Now we define the heirs of a set B. The heirs are addresses obtained as images of B by an operator DI_u.

Definition. Let B a set of addresses of $\mathbf{A}$, and let u be a word on $\mathbf{A}_{\text{LDI}}$. The set $\operatorname{Heir}(B, u)$ of all *heirs* of addresses in B by the operator DI_u is defined inductively:
(i) $\operatorname{Heir}(B, u)$ exists if and only if $\operatorname{Heir}(\{\beta\}, u)$ exists for each β in B, and, in this case, we have $\operatorname{Heir}(B, u) = \bigcup_{\beta \in B} \operatorname{Heir}(\{\beta\}, u)$;

(ii) For each B we have $\operatorname{Heir}(B, \varepsilon) = B$;
(iii) For each α in $\mathbf{A}_{\text{LDI}}$, $\operatorname{Heir}(\{\beta\}, \alpha)$ is defined as:

$$\operatorname{Heir}(\{\beta\}, \alpha) = \begin{cases} \{\beta\} & \text{for } \beta \perp \alpha \text{ or } \beta = \alpha 11\gamma, \\ \{\alpha 00\gamma, \alpha 10\gamma\} & \text{for } \beta = \alpha 0\gamma, \\ \{\alpha 01\gamma\} & \text{for } \beta = \alpha 10\gamma, \\ \text{is not defined} & \text{for } \beta \sqsubseteq \alpha 1, \end{cases} \qquad \text{for } \alpha \in \mathbf{A}_{\text{LD}},$$

$$\operatorname{Heir}(\{\beta\}, \alpha) = \begin{cases} \{\beta\} & \text{for } \beta \perp \alpha, \\ \{\alpha 0\gamma, \alpha 1\gamma\} & \text{for } \beta = \alpha\gamma, \\ \text{is not defined} & \text{for } \beta \sqsubset \alpha. \end{cases} \qquad \text{for } \alpha \in \mathbf{A}_{\text{I}},$$

(iv) For $u = \alpha \cdot u_0$, we have $\operatorname{Heir}(B, u) = \operatorname{Heir}(\operatorname{Heir}(B, \alpha), u_0)$, if it exists.

The following lemma is easy to see:

Lemma 3.3. *Let u be a word on $\mathbf{A}_{\text{LDI}}$ and let β be an address.*
(i) If $\operatorname{Heir}(\{\beta\}, u)$ is defined then $\operatorname{Heir}(\{\beta\gamma\}, u)$ is also defined, for each address γ, and we have $\operatorname{Heir}(\{\beta\gamma\}, u) = \{\beta'\gamma;\ \beta' \in \operatorname{Heir}(\{\beta\}, u)\}$.
(ii) The elements in every set $\operatorname{Heir}(\{\beta\}, u)$ are pairwise orthogonal.
(iii) Suppose $t' = t \bullet u$ and β in $\operatorname{Skel}(t)$. If $\operatorname{Heir}(\{\beta\}, u)$ is defined then the subterms of t' at all addresses in $\operatorname{Heir}(\{\beta\}, u)$ are equal.

Proposition 3.4. *Suppose that u is a word, that β is an address in $\mathbf{A}$ and that $\operatorname{Heir}(\{\beta\}, u)$ is defined. Then we have*

$$\mathtt{d}_\beta \cdot \mathtt{di}_u \equiv^+ \mathtt{di}_u \cdot \prod_{\beta' \in \operatorname{Heir}(\{\beta\}, u)} \mathtt{d}_{\beta'},$$
$$\mathtt{i}_\beta \cdot \mathtt{di}_u \equiv^+ \mathtt{di}_u \cdot \prod_{\beta' \in \operatorname{Heir}(\{\beta\}, u)} \mathtt{i}_{\beta'}\,.$$

The latter equivalence is true for LDLI only if $\operatorname{Heir}(\{\beta\}, u)$ contains only addresses from $\mathbf{A}_{\text{LI}}$.

Proof. We show the result by the induction on $\lg(u)$. For $u = \varepsilon$, the result is evident. Suppose now $\mathtt{di}_u = \mathtt{i}_\alpha$. The set $\operatorname{Heir}(\{\beta\}, u)$ exists and so α is either orthogonal to β, or it is a prefix of β. The orthogonal case is solved by Lemma 3.2. Suppose $\alpha\gamma = \beta$. But the relation $\mathtt{di}_\beta \cdot \mathtt{i}_\alpha \equiv^+ \mathtt{i}_\alpha \cdot \mathtt{di}_{\alpha 0\gamma} \cdot \mathtt{di}_{\alpha 1\gamma}$ is an LDI-relation of type DI or I.

For $\mathtt{di}_u = \mathtt{d}_\alpha$ there are four possibilities:
a) $\beta \perp \gamma$ is solved by Lemma 3.2 (i);
b) $\beta = \alpha 0\gamma$ gives $\mathtt{di}_\beta \cdot \mathtt{d}_\alpha \equiv^+ \mathtt{d}_\alpha \cdot \mathtt{di}_{\alpha 00\gamma} \cdot \mathtt{di}_{\alpha 10\gamma}$ (types D0 or ID0);
c) $\beta = \alpha 10\gamma$ gives $\mathtt{di}_\beta \cdot \mathtt{d}_\alpha \equiv^+ \mathtt{di}_\alpha \cdot \mathtt{di}_{\alpha 01\gamma}$ (types D10 or ID10), except of

the case $\mathtt{di}_\beta = \mathtt{i}_{\alpha 10}$ where the relation is not true for LDLI since $\mathtt{i}_{\alpha 01}$ is not in $\mathcal{A}_{\text{LDLI}}$;
d) $\beta = \alpha 11\gamma$ gives $\mathtt{di}_\beta \cdot \mathtt{d}_\alpha \equiv^+ \mathtt{d}_\alpha \cdot \mathtt{di}_\beta$ (types D11 or ID11).
No other possibilities occur since the set $\mathrm{Heir}(\{\beta\}, u)$ is defined.

Suppose now $\lg(u) \geq 2$, let us say $u = \alpha \cdot u_0$. By construction, the hypothesis that $\mathrm{Heir}(\{\beta\}, u)$ exists gives the existence of the set $\mathrm{Heir}(\{\beta\}, \alpha)$ and of the set $\mathrm{Heir}(\mathrm{Heir}(\{\beta\}, \alpha), u_0)$ and that the latter is equal to $\mathrm{Heir}(\{\beta\}, u)$. By the induction hypothesis, one has

$$\mathtt{di}_\beta \cdot \mathtt{di}_u \equiv^+ \mathtt{di}_\alpha \cdot \prod_{\beta' \in \mathrm{Heir}(\{\beta\},\alpha)} \mathtt{di}_{\beta'} \cdot \mathtt{di}_{u_0}\,.$$

By the induction hypothesis again, one has, for each β' in $\mathrm{Heir}(\{\beta\}, \alpha)$,

$$\mathtt{di}_{\beta'} \cdot \mathtt{di}_{u_0} \equiv^+ \mathtt{di}_{u_0} \cdot \prod_{\beta'' \in \mathrm{Heir}(\{\beta'\},u_0)} \mathtt{di}_{\beta''},$$

and one gets

$$\mathtt{di}_\beta \cdot \mathtt{di}_u \equiv^+ \mathtt{di}_\alpha \cdot \prod_{\beta' \in \mathrm{Heir}(\{\beta\},\alpha)} \; \prod_{\beta'' \in \mathrm{Heir}(\{\beta'\},u_0)} \mathtt{di}_{\beta''}\,.$$

Now, according to Lemma 3.3 (ii), the addresses β' are pairwise orthogonal, the operators commute and the double product in the formula is equal to the expression $\prod_{\beta' \in \mathrm{Heir}(\{\beta\},u)} \mathtt{di}_{\beta'}$.

One has to be more careful in the case of LDLI. If an address γ ends by 1, there is always a heir of γ that ends by 1. Therefore, if the set $\mathrm{Heir}(\{\beta\}, u)$ contains no address ending by 1, neither does $\mathrm{Heir}(\{\beta\}, \alpha)$. Hence, the induction step is correct for $\mathtt{i}_\beta$ and LDLI-equivalence too. $\square$

Definition. The image of a term t under an element $\mathtt{di}_u$ from $\mathcal{A}^*_{\text{LDI}}$, written $t \bullet \mathtt{di}_u$, is understood to be the term $t \bullet \mathrm{DI}_u$.

Note that, due to Proposition 2.5, two $\equiv^+$-equivalent words have the same action on terms.

We are going to establish now a few relations tied to a distributive action called the uniform distribution.

Definition (uniform distribution)**.** Let t_0, t be two terms. We define the term $t_0 * t$ inductively:

$$t_0 * t = \begin{cases} t_0 \cdot t & \text{for } t \text{ a variable} \\ (t_0 * t_1) \cdot (t_0 * t_2) & \text{when } t = t_1 \cdot t_2. \end{cases}$$

The uniform distribution consist of distributing t_0 into every leaf of t, that means of replacing every variable x in t by $t_0 \cdot x$ [4]. Now we introduce a word δ_t associated with the uniform distribution. More precisely said, any term $t_0 \cdot t$ is sent by δ_t to $t_0 * t$.

Definition. [4] For t a term, we define the word δ_t on $\mathcal{A}_{\text{LD}}$ by:

$$\delta_t = \begin{cases} \varepsilon & \text{when } t \text{ is a variable,} \\ \mathtt{d}_\varnothing \cdot \mathrm{sh}_1(\delta_{t_2}) \cdot \mathrm{sh}_0(\delta_{t_1}) & \text{for } t = t_1 \cdot t_2, \end{cases}$$

where $\mathrm{sh}_\gamma(\mathtt{d}_u)$ stands for $\mathtt{d}_{\gamma u}$ and the symbol ε means in fact $\mathtt{di}_\varepsilon$ (the neutral element of the syntactical monoid).

Proposition 3.5. [4] *For all terms t_0, t in T, we have $t_0 t \bullet \delta_t = t_0 * t$.*

In the following parts of the paper we are going to make some more technical computations. Although many of the calculations can be represented using terms, all the work has to be done in a formal way. To facilitate the comprehension, we use the following notation:

$$\mathtt{di}_u \cdot \underleftarrow{\mathtt{di}_v \cdot \mathtt{di}_\alpha} \cdot \mathtt{di}_w \equiv^+ \mathtt{di}_u \cdot \mathtt{di}_\alpha \cdot \mathtt{di}_{v'} \cdot \mathtt{di}_w \tag{XY}$$

meaning that we want to push the symbol $\mathtt{di}_\alpha$ forward in front of the word $\mathtt{di}_v$. We do this using the relation of type XY which, in this situation, gives $\mathtt{di}_v \cdot \mathtt{di}_\alpha \equiv^+ \mathtt{di}_\alpha \cdot \mathtt{di}_{v'}$.

The first technical proposition says that if we have an action $t \to t'$ and a uniform distribution $t_0 \cdot t \to t_0 * t$ then we can swap them, what means that $t_0 \cdot t \to t_0 * t \to t_0 * t'$ and $t_0 \cdot t \to t_0 \cdot t' \to t_0 * t'$ give the same results.

Proposition 3.6. *Suppose that u is a word, that t is not a variable and that $\mathtt{di}_u$ sends t on t'. Then we have*

$$\delta_t \cdot \mathtt{di}_u \equiv^+ \mathtt{di}_{1u} \cdot \delta_{t'}.$$

Proof. The proof for u in $\mathbf{A}^*_{\text{LD}}$, is done in [4] hence suppose $u \notin \mathbf{A}^*_{\text{LD}}$. We show the result by the induction on $\lg(u)$. For $u = \varepsilon$, the result is vacuously true, for $u = \alpha$, we do the induction on the length of α. Denote $t = t_1 \cdot t_2$. For $\mathtt{di}_\alpha = \mathtt{i}_\varnothing$, we have

$$\begin{aligned} \delta_t \cdot \mathtt{di}_u &= \mathtt{d}_\varnothing \cdot \underleftarrow{1\delta_{t_2} \cdot 0\delta_{t_1} \cdot \mathtt{i}_\varnothing} && \text{(DI)} \\ &\equiv^+_{\text{LDI}} \underleftarrow{\mathtt{d}_\varnothing \cdot \mathtt{i}_\varnothing} \cdot 01\delta_{t_2} \cdot 11\delta_{t_2} \cdot 00\delta_{t_1} \cdot 10\delta_{t_1} && \text{(ID1)} \\ &\equiv^+_{\text{LDI}} \mathtt{i}_1 \cdot \mathtt{d}_\varnothing \cdot \mathtt{d}_1 \cdot \underleftarrow{\mathtt{d}_0 \cdot 01\delta_{t_2} \cdot 11}\delta_{t_2} \underleftarrow{\cdot 00\delta_{t_1} \cdot 10\delta_{t_1}} && (\perp) \\ &\equiv^+_{\text{LDI}} \mathtt{i}_1 \cdot \mathtt{d}_\varnothing \cdot 1\delta_t \cdot 0\delta_t = \mathtt{di}_{1u} \cdot \delta_{t'}. \end{aligned}$$

Now suppose $\mathtt{di}_\alpha = \mathtt{i}_0$. We have, for t_1 a variable,

$$\begin{aligned}
\delta_t \cdot \mathtt{di}_u &= \mathtt{d}_\varnothing \cdot \underleftarrow{1\delta_{t_2} \cdot \mathtt{i}_0} && (\perp)\\
&\equiv^+_{\text{LDLI}} \underleftarrow{\mathtt{d}_\varnothing \cdot \mathtt{i}_0} \cdot 1\delta_{t_2} && (\text{ID10+})\\
&\equiv^+_{\text{LDLI}} \mathtt{i}_{10} \cdot \mathtt{d}_\varnothing \cdot 1\delta_{t_2} \cdot \mathtt{d}_0 = \mathtt{di}_{1u} \cdot \delta_{t'}.
\end{aligned}$$

For $t_1 = t_3 \cdot t_4$, we have

$$\begin{aligned}
\delta_t \cdot \mathtt{di}_u = \mathtt{d}_\varnothing \cdot 1\delta_{t_2} \cdot 0\delta_{t_1} \cdot \mathtt{i}_0 &= \mathtt{d}_\varnothing \cdot \underleftarrow{1\delta_{t_2} \cdot \mathtt{d}_0 \cdot 01\delta_{t_4} \cdot 00\delta_{t_3} \cdot \mathtt{i}_0} && (\text{DI})\\
\equiv^+_{\text{LDLI}} \underleftarrow{\mathtt{d}_\varnothing \cdot \mathtt{i}_0} \cdot 1\delta_{t_2} \cdot \mathtt{d}_{00} \cdot \mathtt{d}_{01} &\cdot 001\delta_{t_4} \cdot 011\delta_{t_4} \cdot 000\delta_{t_3} \cdot 010\delta_{t_3} && (\text{ID10+})\\
\equiv^+_{\text{LDLI}} \mathtt{i}_{10} \cdot \mathtt{d}_\varnothing \cdot 1\delta_{t_2} \cdot \mathtt{d}_0 &\cdot 01\delta_{t_1} \cdot 00\delta_{t_1} = \mathtt{di}_{1u} \cdot \delta_{t'}.
\end{aligned}$$

Suppose now $\alpha = 0\beta$ with β nonempty in $\mathbf{A}_{\mathrm{I}}$, respectively in $\mathbf{A}_{\mathrm{LI}}$. We write $t' = t'_1 \cdot t'_2$ and we know that $\mathtt{di}_\beta$ sends t_1 on t'_1. By the induction hypothesis we have $\delta_{t_1} \cdot \mathtt{i}_\beta \equiv^+ \mathtt{i}_{1\beta} \cdot \delta_{t'_1}$. According to Proposition 3.1, we have $0\delta_{t_1} \cdot \mathtt{i}_{0\beta} \equiv^+ \mathtt{i}_{01\beta} \cdot 0\delta_{t'_1}$ and we find

$$\begin{aligned}
\delta_t \cdot \mathtt{di}_u = \mathtt{d}_\varnothing \cdot \underrightarrow{1\delta_{t_2} \cdot 0\delta_{t_1} \cdot \mathtt{i}_{0\beta}} &\equiv^+ \mathtt{d}_\varnothing \cdot \underleftarrow{0\delta_{t_1} \cdot \mathtt{i}_{0\beta}} \cdot 1\delta_{t_2} && (\perp), (\text{hyp.})\\
&\equiv^+ \underleftarrow{\mathtt{d}_\varnothing \cdot \mathtt{i}_{01\beta}} \cdot \underleftarrow{0\delta_{t'_1} \cdot 1\delta_{t_2}} && (\text{ID10}), (\perp)\\
&\equiv^+ \mathtt{di}_{10\beta} \cdot \mathtt{d}_\varnothing \cdot 1\delta_{t_2} \cdot 0\delta_{t'_1} = \mathtt{di}_{1\alpha} \cdot \delta_{t'}.
\end{aligned}$$

The argument for $\alpha = 1\beta$ is similar and the induction on $\lg(u)$ is simple. □

The following lemma expresses that making $t_0 * (t_1 * t_2)$ is in fact replacing each variable x of the term t_2 by the term $(t_0 * t_1) \cdot (t_0 \cdot x)$.

Lemma 3.7. *For each t_1, t_2, we have*

$$\delta_{t_1 * t_2} \equiv^+ \delta_{t_2} \cdot \prod_{\alpha \in \mathrm{Out}(t_2)} \big(\mathtt{d}_\alpha \cdot \alpha 0 \delta_{t_1}\big).$$

Proof. This product is correctly defined because all the addresses from the outline of t_2 are pairwise orthogonal. We show the lemma by induction on t_2. When t_2 is a variable, one has $\delta_{t_1 * t_2} = \delta_{t_1 \cdot t_2} = \mathtt{d}_\varnothing \cdot 0\delta_{t_1} = \delta_{t_2} \cdot (\mathtt{d}_\varnothing \cdot 0\delta_{t_1})$. Hence suppose $t_2 = t_3 \cdot t_4$. One computes

$$\begin{aligned}
\delta_{t_1 * t_2} &= \delta_{(t_1 * t_3)\cdot(t_1 * t_4)} = \mathtt{d}_\varnothing \cdot 1\delta_{t_1 * t_4} \cdot 0\delta_{t_1 * t_3} && (\text{def.})\\
&\equiv^+ \mathtt{d}_\varnothing \cdot 1\delta_{t_4} \cdot \prod_{\alpha \in \mathrm{Out}(t_4)} (\mathtt{d}_{1\alpha} \cdot 1\alpha 0 \delta_{t_1}) \cdot 0\delta_{t_3} \cdot \prod_{\beta \in \mathrm{Out}(t_3)} (\mathtt{d}_{0\beta} \cdot 0\beta 0 \delta_{t_1}) && (\text{hyp.})\\
&\equiv^+ \delta_{t_2} \cdot \prod_{\alpha \in \mathrm{Out}(t_2)} (\mathtt{d}_\alpha \cdot \alpha 0 \delta_{t_1}) && (\perp)
\end{aligned}$$

and this is the searched form. □

Proposition 3.8. *For each term t and each u a word, we have*

$$\mathrm{di}_{0u}\cdot\delta_t \equiv^{+} \delta_t\cdot\prod_{\alpha\in\mathrm{Out}(t)}\mathrm{di}_{\alpha 0u}\,.$$

Proof. The idea of the proof is the following: let t_0 be a term in the domain of DI_u and let t_0' be its inverse under DI_u. Then the left-hand side encodes $t_0t \to t_0't \to t_0' * t$ and the right hand side encodes $t_0t \to t_0 * t \to t_0' * t$.

More precisely, δ_t applied to t_0t distributes t_0 into every leaf of t. Therefore $\mathrm{Heir}(\{0\},\delta_t)$ is $\{\alpha 0, \alpha\in\mathrm{Out}(t)\}$ and analogously $\mathrm{Heir}(\{0\beta\},\delta_t)$ is $\{\alpha 0\beta,\ \alpha\in\mathrm{Out}(t)\}$ (see [4]). Hence, applying Proposition 3.4, we get the result. □

4. Confluence

In this section we prove the existence of a common right multiple of an arbitrary pair of elements. The geometric idea (which is to be proved further in the section) is the following: let t be a term and let t_1, t_2,... be terms obtained by different positive operators DI_{α_i} with t in its domain. Then there exists a term, denoted by ∂t, and positive words u_1, u_2,... such that ∂t is the image of t_i under DI_{u_i} for each i. The term ∂t is described inductively:

Definition. [14] Let t be a term. We define the terms $\partial_{\mathrm{LDI}}t$ et $\partial_{\mathrm{LDLI}}t$ by:

$$\partial_{\mathrm{LDI}}t = \begin{cases} t\cdot t & \text{if } t \text{ is a variable,}\\ \partial_{\mathrm{LDI}}t_1 * \partial_{\mathrm{LDI}}t_2 & \text{for } t = t_1\cdot t_2,\end{cases}$$

$$\partial_{\mathrm{LDLI}}t = \begin{cases} t & \text{if } t \text{ is a variable,}\\ \partial_{\mathrm{LDI}}t_1 * \partial_{\mathrm{LDLI}}t_2 & \text{for } t = t_1\cdot t_2.\end{cases}$$

We write only ∂t when a statement is declared for both $\partial_{\mathrm{LDI}}t$ and $\partial_{\mathrm{LDLI}}t$.

We translate the geometrical situation introducing the elements Δ_t which send t to ∂t.

Definition. For t a term, we define the elements Δ_t^{LDI} and Δ_t^{LDLI} inductively:

$$\Delta_t^{\mathrm{LDI}} = \begin{cases} \mathrm{i}_\varnothing & \text{when } t \text{ is a variable,}\\ \mathrm{sh}_0(\Delta_{t_1}^{\mathrm{LDI}})\cdot\delta_{t_2}\cdot\Delta_{t_2}^{\mathrm{LDI}} & \text{for } t = t_1\cdot t_2,\end{cases}$$

$$\Delta_t^{\mathrm{LDLI}} = \begin{cases} \varepsilon & \text{when } t \text{ is a variable,}\\ \mathrm{sh}_0(\Delta_{t_1}^{\mathrm{LDI}})\cdot\delta_{t_2}\cdot\Delta_{t_2}^{\mathrm{LDLI}} & \text{for } t = t_1\cdot t_2,\end{cases}$$

Again, we write simply Δ_t when a statement is true for both Δ_t^{LDI} and Δ_t^{LDLI}.

Lemma 4.1. *We define $\hat{\Delta}_t$ as Δ_t, for t a variable, and as $0\hat{\Delta}^{\text{LDI}}_{t_1}\cdot 1\hat{\Delta}_{t_2}\cdot\delta_{\partial t_2}$, for $t = t_1t_2$. Then $\hat{\Delta}_t$ sends t to ∂t, for each t.*

Proof. When t is a variable, the result holds. Suppose now $t = t_1 \cdot t_2$ and

$$\begin{aligned} t \bullet \hat{\Delta}_t = (t_1 \cdot t_2) \bullet (0\hat{\Delta}^{\text{LDI}}_{t_1} \cdot 1\hat{\Delta}_{t_2} \cdot \delta_{\partial t_2}) = (\partial_{\text{LDI}} t_1 \cdot t_2) \bullet (1\hat{\Delta}_{t_2} \cdot \delta_{\partial t_2}) = \\ = (\partial_{\text{LDI}} t_1 \cdot \partial t_2) \bullet \delta_{\partial t_2} = \partial_{\text{LDI}} t_1 * \partial t_2 = \partial t \end{aligned}$$

is obtained. ☐

Lemma 4.2. *Suppose $t = t_1 \cdot t_2$. Then one has*

$$\begin{aligned} &(i) && \Delta_t \equiv^{+} 1\Delta_{t_2} \cdot 0\Delta^{\text{LDI}}_{t_1} \cdot \delta_{\partial t_2}, \\ &(ii) && \Delta_t \equiv^{+} \delta_{t_2} \cdot \prod_{\alpha \in \operatorname{Out}(t_2)} \alpha 0\Delta^{\text{LDI}}_{t_1} \cdot \Delta_{t_2}. \end{aligned}$$

(i) We prove the relation by induction on t_2 as well as $\Delta_t \equiv^{+} \hat{\Delta}_t$ from Lemma 4.1. When t_2 is a variable, the result is true for Δ^{LDLI}_t since $\delta_{t_2} = \delta_{\partial t_2} = \varepsilon$. For Δ^{LDI}_t one has

$$\begin{aligned} \Delta^{\text{LDI}}_t = 0\Delta^{\text{LDI}}_{t_1} \cdot \mathtt{i}_{\varnothing} \equiv^{+}_{\text{LDI}} 0\Delta^{\text{LDI}}_{t_1} \cdot \mathtt{i}_1 \cdot \mathtt{d}_{\varnothing} = 0\Delta^{\text{LDI}}_{t_1} \cdot 1\Delta^{\text{LDI}}_{t_2} \cdot \delta_{\partial t_2} = \hat{\Delta}^{\text{LDI}}_t \qquad &\text{(C)} \\ \equiv^{+}_{\text{LDI}} 1\Delta^{\text{LDI}}_{t_2} \cdot 0\Delta^{\text{LDI}}_{t_1} \cdot \delta_{\partial t_2}. \qquad &(\perp) \end{aligned}$$

When t_2 is not a variable, we use Proposition 3.6, Lemma 4.1 and the induction hypothesis to obtain $\Delta_t \equiv^{+} 0\Delta_{t_1} \cdot 1\Delta_{t_2} \cdot \delta_{\partial t_2} \equiv^{+} \hat{\Delta}_t$. The idea is that Δ_{t_2}, being equivalent to $\hat{\Delta}_{t_2}$, sends t to ∂t.

(ii) This follows from (i) and Proposition 3.8.

Proposition 4.3. *For each t, one has $t \bullet \Delta_t = \partial t$.*

Proof. It was already proved in the proof of Lemma 4.2 (i). ☐

Lemma 4.4. *For each term t we have $\Delta^{\text{LDI}}_t \equiv^{+}_{\text{LDI}} \mathtt{i}_{\varnothing} \cdot 0\Delta^{\text{LDLI}}_t \cdot 1\Delta^{\text{LDLI}}_t$.*

Proof. We use the induction on t. The result holds trivially when t is a variable. For $t = t_1 \cdot t_2$ we obtain

$$\begin{aligned} \Delta^{\text{LDI}}_t &\equiv^{+}_{\text{LDI}} 0\Delta^{\text{LDI}}_{t_1} \cdot \underbrace{\delta_{t_2} \cdot \mathtt{i}_{\varnothing}}_{\uparrow} \cdot 0\Delta^{\text{LDLI}}_{t_2} \cdot 1\Delta^{\text{LDLI}}_{t_2} && \text{(hyp.), (DI)} \\ &\equiv^{+}_{\text{LDI}} \underbrace{0\Delta^{\text{LDI}}_{t_1} \cdot \mathtt{i}_{\varnothing}}_{\uparrow} \cdot 0\delta_{t_2} \cdot \underbrace{1\delta_{t_2} \cdot 0\Delta^{\text{LDLI}}_{t_2}}_{\uparrow} \cdot 1\Delta^{\text{LDLI}}_{t_2} && \text{(I),(DI)}, (\perp) \\ &\equiv^{+}_{\text{LDI}} \mathtt{i}_{\varnothing} \cdot 00\Delta^{\text{LDI}}_{t_1} \cdot 1\underbrace{0\Delta^{\text{LDI}}_{t_1} \cdot 0\delta_{t_2} \cdot 0\Delta^{\text{LDLI}}_{t_2}}_{\uparrow} \cdot 1\delta_{t_2} \cdot 1\Delta^{\text{LDLI}}_{t_2} && (\perp) \\ &\equiv^{+}_{\text{LDI}} \mathtt{i}_{\varnothing} \cdot 0\Delta^{\text{LDLI}}_t \cdot 1\Delta^{\text{LDLI}}_t, \end{aligned}$$

which finishes the proof. ☐

Now we show the geometric idea from the beginning of the section.

Proposition 4.5. *Suppose that t belongs to the domain of DI_α, with α an address. Then there exists a positive word u satisfying $\mathtt{di}_\alpha \cdot \mathtt{di}_u \equiv^+ \Delta_t$.*

Proof. When t is a variable then the result holds by definition. Suppose $t = t_1 \cdot t_2$. We use the induction on α. For $\mathtt{di}_\alpha = \mathtt{d}_\varnothing$, the result follows from Lemma 4.2 (*ii*). For $\mathtt{di}_\alpha = \mathtt{i}_\varnothing$ or $\mathtt{di}_\alpha = \mathtt{i}_0$, we use Lemma 4.4. Suppose now $\alpha = 0\beta$. By definition, the word Δ_t begins by $0\Delta_{t_1}$. By the induction hypothesis, the word Δ_{t_1} is equivalent to $\mathtt{di}_\beta \cdot \mathtt{di}_{u'}$ for a word u'. Hence we obtain $\Delta_t \equiv^+ \mathtt{di}_\alpha \cdot \mathtt{di}_{0u'} \cdot \delta_{t_2} \cdot \Delta_{t_2}$. The argument is similar for $\alpha = 1\beta$ due to Lemma 4.2 (*i*). ☐

Now a few lemmas come in the direction of finding a common right multiple.

Lemma 4.6. *If α sends t on t' then there exists a positive word u such that $\mathtt{di}_\alpha \cdot \Delta_{t'} \equiv^+ \Delta_t \cdot \mathtt{di}_u$.*

Proof. We show the result by the induction on α. For $\mathtt{di}_\alpha = \mathtt{i}_\varnothing$, one has

$$\mathtt{di}_\alpha \cdot \Delta^{\text{LDI}}_{t'} \equiv^+_{\text{LDI}} \underrightarrow{\mathtt{i}_\varnothing \cdot 0\Delta^{\text{LDI}}_t \cdot 1\Delta^{\text{LDI}}_t} \cdot \delta_{\partial_{\text{LDI}} t} \equiv^+_{\text{LDI}} \Delta^{\text{LDI}}_t \cdot \mathtt{i}_\varnothing \cdot \delta_{\partial_{\text{LDI}} t}.$$

For $\mathtt{di}_\alpha = \mathtt{i}_0$ and $t = t_1 \cdot t_2$, one has

$$\begin{aligned}
\mathtt{di}_\alpha \cdot \Delta^{\text{LDLI}}_{t'} &\equiv^+_{\text{LDLI}} \mathtt{i}_0 \cdot 0\Delta^{\text{LDI}}_{t_1 \cdot t_1} \cdot 1\Delta^{\text{LDLI}}_{t_2} \cdot \delta_{\partial_{\text{LDLI}} t_2} && \text{(L 4.2)}\\
&\equiv^+_{\text{LDLI}} \underrightarrow{\mathtt{i}_0 \cdot 00\Delta^{\text{LDI}}_{t_1} \cdot 01\Delta^{\text{LDI}}_{t_1}} \cdot \underleftarrow{0\delta_{\partial_{\text{LDI}} t_1} \cdot 1\Delta^{\text{LDLI}}_{t_2}} \cdot \delta_{\partial_{\text{LDLI}} t_2} && \text{(I, DI)}, (\perp)\\
&\equiv^+_{\text{LDLI}} 0\Delta^{\text{LDI}}_{t_1} \cdot 1\Delta^{\text{LDLI}}_{t_2} \cdot \underleftarrow{\mathtt{i}_0 \cdot 0\delta_{\partial_{\text{LDI}} t_1} \cdot \delta_{\partial_{\text{LDLI}} t_2}} && \text{(P 3.8)}\\
&\equiv^+_{\text{LDLI}} 0\Delta^{\text{LDI}}_{t_1} \cdot 1\Delta^{\text{LDLI}}_{t_2} \cdot \delta_{\partial_{\text{LDLI}} t_2} \cdot \Pi(\mathtt{i}_{\beta 0} \cdot \beta 0 \delta_{\partial_{\text{LDI}} t_1}) && \text{(L 4.2)}\\
&\equiv^+_{\text{LDLI}} \Delta^{\text{LDLI}}_t \cdot \Pi_{\beta \in \text{Out}(\partial_{\text{LDLI}} t_2)}(\mathtt{i}_{\beta 0} \cdot \beta 0 \delta_{\partial_{\text{LDI}} t_1}).
\end{aligned}$$

Suppose $\mathtt{di}_\alpha = \mathtt{d}_\varnothing$ and $t = t_1 \cdot (t_2 \cdot t_3)$. One has

$$\begin{aligned}
\mathtt{d}_\alpha \cdot \Delta_{t'} &= \mathtt{d}_\varnothing \cdot \Delta_{(t_1 \cdot t_2)\cdot(t_1 \cdot t_3)}\\
&\equiv^+ \mathtt{d}_\varnothing \cdot 0\Delta^{\text{LDI}}_{t_1 \cdot t_2} \cdot 1\Delta_{t_1 \cdot t_3} \cdot \delta_{\partial(t_1 \cdot t_3)} && \text{(L 4.2)}\\
&\equiv^+ \underrightarrow{\mathtt{d}_\varnothing \cdot 00\Delta^{\text{LDI}}_{t_1}} \cdot 01\Delta^{\text{LDI}}_{t_2} \cdot 0\delta_{\partial_{\text{LDI}} t_2} \cdot 10(\Delta^{\text{LDI}}_{t_1})\\
&\qquad \cdot 11\Delta_{t_3} \cdot 1\delta_{\partial t_3} \cdot \delta_{\partial_{\text{LDI}} t_1 * \partial t_3} && (\perp, \text{D0, ID0})\\
&\equiv^+ 0\Delta^{\text{LDI}}_{t_1} \cdot \underrightarrow{\mathtt{d}_\varnothing \cdot 01\Delta^{\text{LDI}}_{t_2}} \cdot 11\Delta_{t_3} \cdot 0\delta_{\partial_{\text{LDI}} t_2}\\
&\qquad \cdot 1\delta_{\partial t_3} \cdot \delta_{\partial_{\text{LDI}} t_1 * \partial t_3} && \text{(D10, ID10)}
\end{aligned}$$

$$
\begin{aligned}
&\equiv^{+} 0\Delta^{\mathrm{LDI}}_{t_1} \cdot 10\Delta^{\mathrm{LDI}}_{t_2} \cdot \underbracket{\mathtt{d}_{\varnothing} \cdot 11\Delta_{t_3}}_{\qquad\qquad\Uparrow} \cdot 0\delta_{\partial_{\mathrm{LDI}} t_2} \\
&\qquad\qquad \cdot 1\delta_{\partial t_3} \cdot \delta_{\partial_{\mathrm{LDI}} t_1 * \partial t_3} && \text{(D11, ID11)} \\
&\equiv^{+} 0\Delta^{\mathrm{LDI}}_{t_1} \cdot 10\Delta^{\mathrm{LDI}}_{t_2} \cdot 11\Delta_{t_3} \cdot \mathtt{d}_{\varnothing} \cdot 0\delta_{\partial_{\mathrm{LDI}} t_2} \\
&\qquad\qquad \cdot 1\delta_{\partial t_3} \cdot \delta_{\partial_{\mathrm{LDI}} t_1 * \partial t_3} && \text{(L 3.7)} \\
&\equiv^{+} 0\Delta^{\mathrm{LDI}}_{t_1} \cdot 10\Delta^{\mathrm{LDI}}_{t_2} \cdot 11\Delta_{t_3} \cdot \underbracket{\delta_{\partial_{\mathrm{LDI}} t_2 \cdot \partial t_3} \cdot \delta_{\partial t_3}}_{\Uparrow\qquad\qquad\qquad} \\
&\qquad\qquad \cdot \Pi(\mathtt{d}_{\alpha} \cdot \alpha 0 \delta_{\partial_{\mathrm{LDI}} t_1}) && \text{(P 3.6)} \\
&\equiv^{+} 0\Delta^{\mathrm{LDI}}_{t_1} \cdot 10\Delta^{\mathrm{LDI}}_{t_2} \cdot 11\Delta_{t_3} \cdot 1\delta_{\partial t_3} \\
&\qquad\qquad \cdot \delta_{\partial_{\mathrm{LDI}} t_2 * \partial t_3} \cdot \Pi(\mathtt{d}_{\alpha} \cdot \alpha 0 \delta_{\partial_{\mathrm{LDI}} t_1}) && \text{(L 4.2)} \\
&\equiv^{+} 0\Delta^{\mathrm{LDI}}_{t_1} \cdot 1\Delta_{t_2 \cdot t_3} \cdot \delta_{\partial (t_2 \cdot t_3)} \cdot \Pi(\mathtt{d}_{\alpha} \cdot \alpha 0 \delta_{\partial_{\mathrm{LDI}} t_1}) && \text{(L 4.2)} \\
&\equiv^{+} \Delta_{t_1 \cdot (t_2 \cdot t_3)} \cdot \Pi(\mathtt{d}_{\alpha} \cdot \alpha 0 \delta_{\partial_{\mathrm{LDI}} t_1}).
\end{aligned}
$$

Suppose now $\alpha = 0\beta$ and $t = t_1 \cdot t_2$. Since $\mathtt{di}_{\beta}$ sends t_1 on a term t'_1, the hypothesis gives us $\mathtt{di}_{\beta} \cdot \Delta_{t'_1} \equiv^{+} \Delta_{t_1} \cdot \mathtt{di}_{u'}$ for a word u'. Hence one has

$$
\begin{aligned}
\mathtt{di}_{\alpha} \cdot \Delta_{t'} = \mathtt{di}_{0\beta} \cdot \Delta_{t'_1 \cdot t_2} &= \underbracket{\mathtt{di}_{0\beta} \cdot 0\Delta^{\mathrm{LDI}}_{t'_1}}_{\qquad\qquad\Uparrow} \cdot \delta_{t_2} \cdot \Delta_{t_2} && \text{(hyp)} \\
&\equiv^{+} 0\Delta^{\mathrm{LDI}}_{t_1} \cdot \underbracket{\mathtt{di}_{0u'} \cdot 1\Delta_{t_2} \cdot \delta_{\partial t_2}}_{\qquad\qquad\qquad\Uparrow} && \text{(P 3.8)} \\
&\equiv^{+} 0\Delta^{\mathrm{LDI}}_{t_1} \cdot 1\Delta_{t_2} \cdot \delta_{\partial t_2} \cdot \Pi(\mathtt{di}_{\beta 0 u'}) \\
&\equiv^{+} \Delta_{t} \cdot \Pi(\mathtt{di}_{\beta 0 u'}).
\end{aligned}
$$

Finally, we suppose $\alpha = 1\beta$ and $t' = t_1 \cdot t'_2$. By the induction hypothesis, one has $\mathtt{di}_{\beta} \cdot \Delta_{t'_2} \equiv^{+} \Delta_{t_2} \cdot \mathtt{di}_{u'}$. One finds

$$
\begin{aligned}
\mathtt{di}_{\alpha} \cdot \Delta_{t'} &\equiv^{+} \underbracket{\mathtt{di}_{1\beta} \cdot 1\Delta_{t'_2}}_{\qquad\qquad\Uparrow} \cdot 0\Delta^{\mathrm{LDI}}_{t_1} \cdot \delta_{\partial t'_2} && \text{(hyp)} \\
&\equiv^{+} 1\Delta_{t_2} \cdot \underbracket{\mathtt{di}_{1u'} \cdot 0\Delta^{\mathrm{LDI}}_{t_1} \cdot \delta_{\partial t'_2}}_{\qquad\qquad\qquad\Uparrow} && \text{(P 3.6)} \\
&\equiv^{+} 1\Delta_{t_2} \cdot 0\Delta^{\mathrm{LDI}}_{t_1} \cdot \delta_{\partial t_2} \cdot \mathtt{di}_{u'} \equiv^{+} \Delta_{t} \cdot \mathtt{di}_{u'}
\end{aligned}
$$

where the equality $\mathtt{di}_{1u'} \cdot \delta_{\partial t'_2} \equiv^{+} \delta_{\partial t_2} \cdot \mathtt{d}_{u'}$ follows from Proposition 3.6 since one has $t_2 \bullet (\mathtt{di}_{\beta} \cdot \Delta_{t'_2}) = \partial t'_2 = t_2 \bullet (\Delta_{t_2} \cdot \mathtt{di}_{u'})$ and hence one has $\partial t'_2 = \partial t_2 \bullet \mathtt{di}_{u'}$. $\square$

Lemma 4.7. *Suppose that u is a positive word and that DI_u sends t on t'. Then there exists a positive word u' satisfying*

$$\mathtt{di}_{u} \cdot \Delta_{t'} \equiv^{+} \Delta_{t} \cdot \mathtt{di}_{u'}\,.$$

Proof. We use the induction on $\lg(u)$. For $u = \varepsilon$, the result is trivial. For $\lg(u) = 1$, the result is Lemma 4.6. Suppose now $u = u_1 \cdot u_2$, where neither u_1 nor u_2 are empty. Let $t_1 = t \bullet \mathrm{DI}_u$. By the induction hypothesis, there exist u_1' and u_2' satisfying $\mathtt{di}_{u_1} \cdot \Delta_{t_1} \equiv^+ \Delta_t \cdot \mathtt{di}_{u_1'}$ and $\mathtt{di}_{u_2} \cdot \Delta_{t'} \equiv^+ \Delta_{t_1} \cdot \mathtt{di}_{u_2'}$. We thus deduce $\mathtt{di}_u \cdot \Delta_{t'} \equiv^+ \mathtt{di}_{u_1} \cdot \Delta_{t_1} \cdot \mathtt{di}_{u_2'} \equiv^+ \Delta_t \cdot \mathtt{di}_{u_1'} \cdot \mathtt{di}_{u_2'}$. ☐

The following definition encodes iterative usage of the ∂ operation in the obvious way that ${}^k\Delta_t$ sends t to $\partial^k t$.

Definition. For each term t, we put ${}^0\Delta_t^{\mathrm{LDI}} = \varepsilon$ and ${}^k\Delta_t^{\mathrm{LDI}} = \Delta_t^{\mathrm{LDI}} \cdot \Delta_{\partial_{\mathrm{LDI}} t}^{\mathrm{LDI}} \cdot \cdots \cdot \Delta_{\partial_{\mathrm{LDI}}^{k-1} t}^{\mathrm{LDI}}$ for $k \geq 1$. The word ${}^k\Delta_t^{\mathrm{LDLI}}$ is defined analogously.

Lemma 4.8. *Let u be a positive word of length at most k and let t be a term from the domain of DI_u. Then there exists a positive word v' satisfying $\mathtt{di}_u \cdot \mathtt{di}_{v'} \equiv^+ {}^k\Delta_t$.*

Proof. We use the induction on k. For $k = 0$, the result is trivial. Otherwise, we write $u = u_0 \cdot \alpha$ with α an address. By the induction hypothesis, there exists a positive word v_0' satisfying $\mathtt{di}_{u_0} \cdot \mathtt{di}_{v_0'} \equiv^+ {}^{k-1}\Delta_t$. Let t' be the image of t by DI_{u_0}. By the hypothesis, the term t' belongs to the domain of DI_α and therefore, according to Proposition 4.5, there exists a positive word v satisfying $\mathtt{di}_\alpha \cdot \mathtt{di}_v \equiv^+ \Delta_{t'}$. Applicating Lemma 4.7 on the terms t' and $\partial^{k-1}t$, we see that there exists a positive word v_0'' that satisfies $\mathtt{di}_{v_0'} \cdot \Delta_{\partial^{k-1}t} \equiv^+ \Delta_{t'} \cdot \mathtt{di}_{v_0''}$. We then deduce

$$\begin{aligned}\mathtt{di}_u \cdot \mathtt{di}_v \cdot \mathtt{di}_{v_0''} = \mathtt{di}_{u_0} \cdot \mathtt{di}_\alpha \cdot \mathtt{di}_v \cdot \mathtt{di}_{v_0''} \equiv^+ \mathtt{di}_{u_0} \cdot \Delta_{t'} \cdot \mathtt{di}_{v_0''} \equiv^+ \\ \equiv^+ \mathtt{di}_{u_0} \cdot \mathtt{di}_{v_0'} \cdot \Delta_{\partial^{k-1}t} \equiv^+ {}^{k-1}\Delta_t \cdot \Delta_{\partial^{k-1}t} = {}^k\Delta_t.\end{aligned}$$

Hence we obtain the result putting $v' = v \cdot v_0''$. ☐

Proposition 4.9. *Let u, v be two positive words of the length at most k. Then there exist positive words u', v' satisfying*

$$\mathtt{di}_u \cdot \mathtt{di}_{v'} \equiv^+ \mathtt{di}_v \cdot \mathtt{di}_{u'} \equiv^+ {}^k\Delta_{t_{u,v}^L}.$$

Proof. The intersection of the domains of the operators DI_u and DI_v contains the term $t_{u,v}^L$, due to Proposition 1.7. According to Lemma 4.8, there exist two positive words u' and v' such that $\mathtt{di}_u \cdot \mathtt{di}_{v'}$ and $\mathtt{di}_v \cdot \mathtt{di}_{u'}$ are $\equiv^+$-equivalent to ${}^k\Delta_{t_{u,v}^L}$. ☐

We have just proved that all pairs of elements have common right multiples and we have proved it at once for the $\equiv_{\mathrm{LDI}}^+$ and $\equiv_{\mathrm{LDLI}}^+$ relations as well as for the positive geometry monoids $\mathcal{G}_{\mathrm{LDI}}^+$ and $\mathcal{G}_{\mathrm{LDLI}}^+$.

5. Syntactical monoid

In this section we study the monoid generated by the LDLI-relations using the method of complemented presentations, described in [7]. This method gives an algorithm for resolving the word problem of this monoid and it enables us to say that the monoid is left cancellative and that the left divisibility order forms a lattice. We are not interested in LDI-relations because there are too many of them and the corresponding monoid seems to be less useful than the monoid of LDLI.

Definition. [4] Let A be an alphabet. We say that f is a *complement* on A if f is a partial mapping from $A \times A$ to A^* satisfying $f(x, x) = \varepsilon$, for each x in A, and that $f(x, y)$ exists if $f(y, x)$ exists. We denote by $\equiv_f^+$ the relation generated by the relations $(xf(x, y), yf(y, x))$ with (x, y) in the domain of f. The monoid *associated on the right* is the monoid A^* factored by $\equiv_f^+$.

Let us define the syntactical monoid of LDLI M_{LDLI} as the monoid $(\mathbf{A}_{\mathrm{LDLI}})^*$ factored by the LDLI-relations. It is not immediate to see but after a closer look we observe that the monoid M_{LDLI} is associated to a right complement: we have

$$f(\mathtt{d}_\alpha, \mathtt{di}_\beta) = \begin{cases} \varepsilon & \text{for } \mathtt{di}_\alpha = \mathtt{di}_\beta, \\ \mathtt{di}_\beta & \text{for } \beta \perp \alpha \text{ or } \alpha 11 \sqsubseteq \beta \text{ or } (\beta \sqsubset \alpha \text{ and } \alpha \neq \beta 1), \\ \mathtt{i}_\beta & \text{for } \mathtt{di}_\beta = \mathtt{i}_\beta \text{ and } (\alpha = \beta \text{ or } \alpha = \beta 1), \\ \mathtt{di}_{\alpha 10\gamma} \cdot \mathtt{di}_{\alpha 00\gamma} & \text{for } \beta = \alpha 0\gamma, \\ \mathtt{di}_{\alpha 01\gamma} & \text{for } \beta = \alpha 10\gamma \text{ and } \mathtt{di}_\beta \neq \mathtt{i}_{\alpha 10}, \\ \mathtt{i}_{\alpha 0} & \text{for } \mathtt{di}_\beta = \mathtt{i}_{\alpha 10}, \\ \mathtt{d}_\beta \cdot \mathtt{d}_\alpha & \text{for } \beta = \alpha 1, \\ \mathtt{d}_\beta \cdot \mathtt{d}_\alpha \cdot \mathtt{d}_{\beta 0} & \text{for } \alpha = \beta 1, \end{cases}$$

$$f(\mathtt{i}_\alpha, \mathtt{di}_\beta) = \begin{cases} \varepsilon & \text{for } \mathtt{di}_\alpha = \mathtt{di}_\beta, \\ \mathtt{di}_\beta & \text{for } \beta \perp \alpha \text{ or } (\beta \sqsubset \alpha \text{ and } (\alpha \neq \beta 10 \text{ or } \beta \in \mathbf{A}_{\mathrm{LI}})), \\ \mathtt{di}_{\alpha 0\gamma} \cdot \mathtt{di}_{\alpha 1\gamma} & \text{for } \beta = \alpha\gamma \text{ and } \mathtt{di}_\alpha \neq \mathtt{di}_\beta, \\ \mathtt{d}_\beta \cdot \mathtt{d}_{\beta 0} & \text{for } \alpha = \beta 10 \text{ and } \beta \in \mathbf{A}_{\mathrm{LD}}. \end{cases}$$

The LDI-relations do not give a complement because there is, e.g., the relation $\mathtt{i}_\varnothing \cdot \mathtt{i}_\varnothing \equiv_{\mathrm{LDI}}^+ \mathtt{i}_\varnothing \cdot \mathtt{i}_0 \cdot \mathtt{i}_1$.

The complemented presentations permit the usage of a combinatorial method called the *word reversing*. The reversing consists of iteratively replacing a subword $x^{-1} \cdot y$ by the subword $f(x, y) \cdot f(y, x)^{-1}$.

Definition. **[7]** Let w, w' be two words. We say that w is *reversed to the right* into w', denoted by $w \curvearrowright w'$, if there exists a sequence of words $w = w_1, \dots, w_k = w'$ satisfying, for each $i < k$,

$$w_i = w_i' \cdot x_i^{-1} \cdot y_i \cdot w_i'' \qquad \text{and} \qquad w_{i+1} = w_i' \cdot f(x_i, y_i) \cdot f(y_i, x_i)^{-1} \cdot w_i''$$

where x_i and y_i are letters.

We see that $w \curvearrowright w'$ implies $w \equiv_f w'$. Hence we can possibly obtain by the reversing a word equivalent to w which is a product of a positive word and a negative word. A priori, the reversing needs not to be a deterministic process, at each step we can reverse arbitrary pair of letters $x^{-1}y$. Though, the process is confluent and if we reach a word vu^{-1}, then it is unique.

Proposition 5.1. **[4]** *Each word w can be reversed into at most one word of the form $v \cdot u^{-1}$ with u and v positive words.*

Definition. **[4]** Let u and v be two positive words. We define $u \backslash v$ as the unique word u' such that $v^{-1}u$ is reversed into $v'u'^{-1}$ with u' and v' positive, if such a word exists.

We can see particularly that we have $x \backslash y = f(x, y)$ for all x, y in A. Remark also a "symmetry" of the definition: a word $v^{-1}u$ is reversed always into $(v \backslash u)(u \backslash v)^{-1}$.

If there is $u \backslash v = v \backslash u = \varepsilon$, then we have $u \equiv_f^+ v$. We would like this implication to be an equivalence, that means, we would like to have $u \equiv_f^+ v$ if and only if $u^{-1}v \curvearrowright \varepsilon$.

Definition. **[7]** We say that a complement f on an alphabet A is *right homogeneous* if there exists a mapping $\lambda : A^* \to \mathbb{N}$ satisfying

$$\lambda(xv) > \lambda(v) \quad \text{and} \quad \lambda(u) = \lambda(v)$$

for all x in A and $u \equiv_f^+ v$ positive words.

Proposition 5.2. **[7]** *Let M be a monoid associated with a right homogeneous right complement $f : A \times A \to A^*$. Then the relation $u \equiv_f^+ v$ implies $u^{-1}v \curvearrowright \varepsilon$ if and only if the following condition is satisfied for all x, y, z in A:*

$$((x \backslash y) \backslash (x \backslash z)) \backslash ((y \backslash x) \backslash (y \backslash z)) = \varepsilon \tag{$\star$}$$

We want to show that the monoid M_{LDLI} satisfies the conditions of Proposition 5.2. We start with the homogeneity of the complement f.

Lemma 5.3. *The complement f of the monoid M_{LDLI} is right homogeneous.*

Proof. We define, for each positive word u,

$$\lambda(\mathrm{di}_u) = \lg(t_u^R) - \lg(t_u^L),$$

where the length of a term is the number of all addresses in its skeleton. Since $\mathrm{di}_u \equiv^+_{\text{LDLI}} \mathrm{di}_v$ implies $\mathrm{DI}_u = \mathrm{DI}_v$, we have $t_u^L = t_v^L$ and $t_u^R = t_v^R$ and also $\lambda(\mathrm{di}_u) = \lambda(\mathrm{di}_v)$. By definition, we have $t_{\alpha\cdot u}^R = t_{\alpha\cdot u}^L \bullet \alpha \bullet u$, hence there exists a substitution h satisfying $t_{\alpha\cdot u}^L \bullet \alpha = (t_u^L)^h$ and $t_{\alpha\cdot u}^R = (t_u^R)^h$. We deduce

$$\begin{aligned}\lambda(\mathrm{di}_\alpha\cdot\mathrm{di}_u) &= \lg(t_{\alpha\cdot u}^R) - \lg(t_{\alpha\cdot u}^L)\\ &= \lg(t_{\alpha\cdot u}^R) - \lg(t_{\alpha\cdot u}^L \bullet \alpha) + \lg(t_{\alpha\cdot u}^L \bullet \alpha) - \lg(t_{\alpha\cdot u}^L)\\ &= \lg((t_u^R)^h) - \lg((t_u^L)^h) + \lg(t_{\alpha\cdot u}^L \bullet \alpha) - \lg(t_{\alpha\cdot u}^L)\\ &> \lg((t_u^R)^h) - \lg((t_u^L)^h) \geq \lg(t_u^R) - \lg(t_u^L) = \lambda(\mathrm{di}_u).\end{aligned}$$

Hence f is right homogeneous. ☐

Now we want to prove the condition ($\star$). We need an auxiliary lemma. We write $\mathrm{di}_u =^\perp \mathrm{di}_v$ for two positive words u and v if the word v is obtained from u using only replacements of a subword $\alpha_1\cdot\alpha_2$ by a subword $\alpha_2\cdot\alpha_1$ with $\alpha_1 \perp \alpha_2$.

Lemma 5.4. *Let di_u, di_v and di_w be words on $\mathcal{A}_{\text{LDLI}}$. Then*
(i) $\mathrm{di}_u \backslash(\mathrm{di}_v\cdot\mathrm{di}_w) = \mathrm{di}_u \backslash \mathrm{di}_v \cdot (\mathrm{di}_v \backslash \mathrm{di}_u)\backslash \mathrm{di}_w$,
(ii) $(\mathrm{di}_u\cdot\mathrm{di}_v)\backslash \mathrm{di}_w = \mathrm{di}_v \backslash(\mathrm{di}_u \backslash \mathrm{di}_w)$,
(iii) $\mathrm{di}_u =^\perp \mathrm{di}_v$ *implies* $\mathrm{di}_u \backslash \mathrm{di}_v = \mathrm{di}_v \backslash \mathrm{di}_u = \varepsilon$.

Proof. (*i*) Denote $\mathrm{di}_v^{-1}\cdot\mathrm{di}_u \curvearrowright \mathrm{di}_{v'}\cdot\mathrm{di}_{u'}^{-1}$ and $\mathrm{di}_w^{-1}\cdot\mathrm{di}_{v'} \curvearrowright \mathrm{di}_{w'}\cdot\mathrm{di}_{v''}^{-1}$ Then

$$\mathrm{di}_w^{-1}\cdot\mathrm{di}_v^{-1}\cdot\mathrm{di}_u \curvearrowright \mathrm{di}_w^{-1}\cdot\mathrm{di}_{v'}\cdot\mathrm{di}_{u'}^{-1} \curvearrowright \mathrm{di}_{w'}\cdot\mathrm{di}_{v''}^{-1}\cdot\mathrm{di}_{u'}^{-1}.$$

(*ii*) Denote $\mathrm{di}_w^{-1}\cdot\mathrm{di}_u \curvearrowright \mathrm{di}_{w'}\cdot\mathrm{di}_{u'}^{-1}$ and $\mathrm{di}_{u'}^{-1}\cdot\mathrm{di}_v \curvearrowright \mathrm{di}_{u''}\cdot\mathrm{di}_{v'}^{-1}$. Then

$$\mathrm{di}_w^{-1}\cdot\mathrm{di}_u\cdot\mathrm{di}_v \curvearrowright \mathrm{di}_{w'}\cdot\mathrm{di}_{u'}^{-1}\cdot\mathrm{di}_v \curvearrowright \mathrm{di}_{w'}\cdot\mathrm{di}_{u'}\cdot\mathrm{di}_{v'}^{-1}.$$

(*iii*) For $u = \varepsilon$ the result is trivial. Suppose $u = \alpha\cdot u_0$. The word v is of the form $v_0\cdot\alpha\cdot v_1$, where each address of v_0 is orthogonal to α. Now we have

$$\mathrm{di}_u^{-1}\cdot\mathrm{di}_v \curvearrowright \mathrm{di}_{u_0}^{-1}\cdot\mathrm{di}_{v_0}\cdot\mathrm{di}_\alpha^{-1}\cdot\mathrm{di}_\alpha\cdot\mathrm{di}_{v_1} \curvearrowright \mathrm{di}_{u_0}^{-1}\cdot\mathrm{di}_{v_0}\cdot\mathrm{di}_{v_1} \curvearrowright \varepsilon$$

by the induction hypothesis because $\mathrm{di}_{u_0} =^\perp \mathrm{di}_{v_0}\cdot\mathrm{di}_{v_1}$. ☐

Proposition 5.5. *The complement f of the monoid M_{LDLI} satisfies the condition $(\star)$.*

Proof. We consider all the triples $\mathtt{di}_\alpha$, $\mathtt{di}_\beta$, $\mathtt{di}_\gamma$ from $\boldsymbol{\mathcal{A}}_{\text{LDLI}}$. Since the LDLI-relations are closed under shifts, we can consider that the greatest common prefix of α, β and γ is the address $\varnothing$ or the address 0.
Case 1. *Two elements are equal.* Suppose $\mathtt{di}_\alpha = \mathtt{di}_\beta$. One has

$$\begin{aligned}
(\mathtt{di}_\alpha \setminus \mathtt{di}_\beta)\backslash(\mathtt{di}_\alpha \setminus \mathtt{di}_\gamma) &= \varepsilon\backslash(\mathtt{di}_\alpha \setminus \mathtt{di}_\gamma) = \mathtt{di}_\alpha \setminus \mathtt{di}_\gamma,\\
(\mathtt{di}_\beta \setminus \mathtt{di}_\alpha)\backslash(\mathtt{di}_\beta \setminus \mathtt{di}_\gamma) &= \varepsilon\backslash(\mathtt{di}_\beta \setminus \mathtt{di}_\gamma) = \mathtt{di}_\beta \setminus \mathtt{di}_\gamma;\\
(\mathtt{di}_\beta \setminus \mathtt{di}_\gamma)\backslash(\mathtt{di}_\beta \setminus \mathtt{di}_\alpha) &= (\mathtt{di}_\beta \setminus \mathtt{di}_\gamma)\backslash\varepsilon = \varepsilon,\\
(\mathtt{di}_\gamma \setminus \mathtt{di}_\beta)\backslash(\mathtt{di}_\gamma \setminus \mathtt{di}_\alpha) &= (\mathtt{di}_\gamma \setminus \mathtt{di}_\alpha)\backslash(\mathtt{di}_\gamma \setminus \mathtt{di}_\alpha) = \varepsilon;
\end{aligned}$$

and this suffices because α and β play a symmetrical role.
Case 2. *An address is orthogonal to the greatest common prefix of the other two addresses.* Suppose that γ is orthogonal to the greatest common prefix of α and β. In this case, γ is also orthogonal to each address in the words $\mathtt{di}_\alpha \setminus \mathtt{di}_\beta$ and $\mathtt{di}_\beta \setminus \mathtt{di}_\alpha$. One has

$$\begin{aligned}
(\mathtt{di}_\alpha \setminus \mathtt{di}_\beta)\backslash(\mathtt{di}_\alpha \setminus \mathtt{di}_\gamma) &= (\mathtt{di}_\alpha \setminus \mathtt{di}_\beta)\backslash \mathtt{di}_\gamma = \mathtt{di}_\gamma,\\
(\mathtt{di}_\beta \setminus \mathtt{di}_\alpha)\backslash(\mathtt{di}_\beta \setminus \mathtt{di}_\gamma) &= (\mathtt{di}_\beta \setminus \mathtt{di}_\alpha)\backslash \mathtt{di}_\gamma = \mathtt{di}_\gamma;\\
(\mathtt{di}_\beta \setminus \mathtt{di}_\gamma)\backslash(\mathtt{di}_\beta \setminus \mathtt{di}_\alpha) &= \mathtt{di}_\gamma \backslash(\mathtt{di}_\beta \setminus \mathtt{di}_\alpha) = (\mathtt{di}_\beta \setminus \mathtt{di}_\alpha),\\
(\mathtt{di}_\beta \setminus \mathtt{di}_\gamma)\backslash(\mathtt{di}_\beta \setminus \mathtt{di}_\alpha) &= \mathtt{di}_\gamma \backslash(\mathtt{di}_\beta \setminus \mathtt{di}_\alpha) = (\mathtt{di}_\beta \setminus \mathtt{di}_\alpha);
\end{aligned}$$

and this suffices because α and β play a symmetrical role.
Case 3. *One of the elements is* $\mathtt{i}_0$ *and* 0 *is the greatest common prefix of all addresses.* Suppose $\mathtt{di}_\gamma = \mathtt{i}_0$. We can suppose that the other addresses are different from $\mathtt{i}_0$, otherwise we are in the case 1. We write $\beta = 0\beta_0$ and $\gamma = 0\gamma_0$. One has

$$\begin{aligned}
(\mathtt{di}_\alpha \setminus \mathtt{di}_\beta)\backslash(\mathtt{di}_\alpha \setminus \mathtt{di}_\gamma) &= (\mathtt{di}_{0\alpha_0} \setminus \mathtt{di}_{0\beta_0})\backslash \mathtt{i}_0 = \mathtt{i}_0,\\
(\mathtt{di}_\beta \setminus \mathtt{di}_\alpha)\backslash(\mathtt{di}_\beta \setminus \mathtt{di}_\gamma) &= (\mathtt{di}_{0\beta_0} \setminus \mathtt{di}_{0\alpha_0})\backslash \mathtt{i}_0 = \mathtt{i}_0;\\
(\mathtt{di}_\beta \setminus \mathtt{di}_\gamma)\backslash(\mathtt{di}_\beta \setminus \mathtt{di}_\alpha) &= \mathtt{i}_0 \backslash(\mathtt{di}_{0\beta_0} \setminus \mathtt{di}_{0\alpha_0})\\
&=^{\perp} (\mathtt{di}_{00\beta_0} \setminus \mathtt{di}_{00\alpha_0}) \cdot (\mathtt{di}_{01\beta_0} \setminus \mathtt{di}_{01\alpha_0}),\\
(\mathtt{di}_\gamma \setminus \mathtt{di}_\beta)\backslash(\mathtt{di}_\gamma \setminus \mathtt{di}_\alpha) &= (\mathtt{di}_{00\beta_0} \cdot \mathtt{di}_{01\beta_0})\backslash(\mathtt{di}_{00\alpha_0} \cdot \mathtt{di}_{01\alpha_0})\\
&= (\mathtt{di}_{00\beta_0} \setminus \mathtt{di}_{00\alpha_0}) \cdot (\mathtt{di}_{01\beta_0} \setminus \mathtt{di}_{01\alpha_0});
\end{aligned}$$

Case 4. *An address is a proper prefix of the greatest common prefix of the other two addresses.* We suppose that γ is a prefix of the greatest common prefix γ' of α and β. We can suppose $\mathtt{di}_\gamma = \mathtt{d}_\varnothing$, otherwise we are in the case 3.

Case 4.1. *The address 0 is a prefix of α and of β.* We write $\alpha = 0\alpha_0$ and $\beta = 0\beta_0$. One has

$$\begin{aligned}
(\mathrm{di}_\alpha \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \backslash \mathrm{di}_\gamma) &= \mathrm{sh}_0(\mathrm{di}_{\alpha_0} \backslash \mathrm{di}_{\beta_0})\backslash \mathrm{d}_\varnothing = \mathrm{d}_\varnothing,\\
(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma) &= \mathrm{sh}_0(\mathrm{di}_{\alpha_0} \backslash \mathrm{di}_{\beta_0})\backslash \mathrm{d}_\varnothing = \mathrm{d}_\varnothing;\\
(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha) &= \mathrm{d}_\varnothing \backslash \mathrm{sh}_0(\mathrm{di}_{\beta_0} \backslash \mathrm{di}_{\alpha_0})\\
&=^{\perp} \mathrm{sh}_{10}(\mathrm{di}_{\beta_0} \backslash \mathrm{di}_{\alpha_0}) \cdot \mathrm{sh}_{00}(\mathrm{di}_{\beta_0} \backslash \mathrm{di}_{\alpha_0}),\\
(\mathrm{di}_\gamma \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\gamma \backslash \mathrm{di}_\alpha) &= (\mathrm{di}_{10\beta_0} \cdot \mathrm{di}_{00\beta_0})\backslash(\mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{00\alpha_0})\\
&= \mathrm{sh}_{10}(\mathrm{di}_{\beta_0} \backslash \mathrm{di}_{\alpha_0}) \cdot \mathrm{sh}_{00}(\mathrm{di}_{\beta_0} \backslash \mathrm{di}_{\alpha_0});
\end{aligned}$$

and this suffices because α and β play a symmetrical role.

Case 4.2. *The address* 1 *is a proper prefix of a common prefix of α and β.*

Case 4.2.1. *One of the elements is* i_{10}. We suppose $\mathrm{di}_\beta = \mathrm{i}_{10}$ and $\mathrm{di}_\alpha = \mathrm{di}_{10\alpha_0} \neq \mathrm{i}_{10}$.

$$\begin{aligned}
(\mathrm{di}_\alpha \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \backslash \mathrm{di}_\gamma) &= \mathrm{i}_{10} \backslash \mathrm{d}_\varnothing = \mathrm{d}_\varnothing \cdot \mathrm{d}_0,\\
(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma) &= (\mathrm{di}_{100\alpha_0} \cdot \mathrm{di}_{101\alpha_0})\backslash(\mathrm{d}_\varnothing \cdot \mathrm{d}_0)\\
&= \mathrm{d}_\varnothing \cdot ((\mathrm{di}_{010\alpha_0} \cdot \mathrm{di}_{011\alpha_0})\backslash \mathrm{d}_0) = \mathrm{d}_\varnothing \cdot \mathrm{d}_0;\\
(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha) &= (\mathrm{d}_\varnothing \cdot \mathrm{d}_0)\backslash(\mathrm{di}_{100\alpha_0} \cdot \mathrm{di}_{101\alpha_0})\\
&= \mathrm{d}_0 \backslash(\mathrm{di}_{010\alpha_0} \cdot \mathrm{di}_{011\alpha_0}) = \mathrm{di}_{001\alpha_0} \cdot \mathrm{di}_{011\alpha_0},\\
(\mathrm{di}_\gamma \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\gamma \backslash \mathrm{di}_\alpha) &= \mathrm{i}_0 \backslash \mathrm{di}_{01\alpha_0} = \mathrm{di}_{001\alpha_0} \cdot \mathrm{di}_{011\alpha_0};\\
(\mathrm{di}_\gamma \backslash \mathrm{di}_\alpha)\backslash(\mathrm{di}_\gamma \backslash \mathrm{di}_\beta) &= \mathrm{di}_{01\alpha_0} \backslash \mathrm{i}_0 = \mathrm{i}_0,\\
(\mathrm{di}_\alpha \backslash \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \backslash \mathrm{di}_\beta) &= \mathrm{d}_\varnothing \backslash \mathrm{i}_{10} = \mathrm{i}_0\,.
\end{aligned}$$

Case 4.2.2. *None of the elements is* i_{10}. We write $\alpha = 1e\alpha_0$ and $\beta = 1e\beta_0$ with $e = 0$ or $e = 1$. One has

$$\begin{aligned}
(\mathrm{di}_\alpha \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \backslash \mathrm{di}_\gamma) &= \mathrm{sh}_{1e}(\mathrm{di}_{\alpha_0} \backslash \mathrm{di}_{\beta_0})\backslash \mathrm{d}_\varnothing = \mathrm{d}_\varnothing,\\
(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma) &= \mathrm{sh}_{1e}(\mathrm{di}_{\beta_0} \backslash \mathrm{di}_{\alpha_0})\backslash \mathrm{d}_\varnothing = \mathrm{d}_\varnothing;\\
(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha) &= \mathrm{d}_\varnothing \backslash \mathrm{sh}_{1e}(\mathrm{di}_{\beta_0} \backslash \mathrm{di}_{\alpha_0}) = \mathrm{sh}_{e1}(\mathrm{di}_{\beta_0} \backslash \mathrm{di}_{\alpha_0}),\\
(\mathrm{di}_\gamma \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\gamma \backslash \mathrm{di}_\alpha) &= \mathrm{di}_{e1\beta_0} \backslash \mathrm{di}_{e1\alpha_0} = \mathrm{sh}_{e1}(\mathrm{di}_{\beta_0} \backslash \mathrm{di}_{\alpha_0});
\end{aligned}$$

and this suffices because α and β play a symmetrical role.

Case 4.3. *The address* 1 *is the greatest common prefix of α and β.*

Case 4.3.1. *The addresses α and β are orthogonal.*

Case 4.3.1.1 *One of the elements is* i_{10}. We suppose $\mathrm{di}_\beta = \mathrm{i}_{10}$ and $\alpha = 11\alpha_0$. One has

$$\begin{aligned}
(\mathrm{di}_\alpha \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \backslash \mathrm{di}_\gamma) &= \mathrm{i}_{10} \backslash \mathrm{d}_\varnothing = \mathrm{d}_\varnothing \cdot \mathrm{d}_0,\\
(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma) &= \mathrm{di}_{11\alpha_0} \backslash(\mathrm{d}_\varnothing \cdot \mathrm{d}_0) = \mathrm{d}_\varnothing \cdot \mathrm{d}_0;
\end{aligned}$$

$$\begin{aligned}
(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha) &= (\mathrm{d}_\varnothing \cdot \mathrm{d}_0)\setminus \mathrm{di}_{11\alpha_0} = \mathrm{di}_{11\alpha_0},\\
(\mathrm{di}_\gamma \setminus \mathrm{di}_\beta)\backslash(\mathrm{di}_\gamma \setminus \mathrm{di}_\alpha) &= \mathrm{i}_0 \setminus \mathrm{di}_{11\alpha_0} = \mathrm{di}_{11\alpha_0};\\
(\mathrm{di}_\gamma \setminus \mathrm{di}_\alpha)\backslash(\mathrm{di}_\gamma \setminus \mathrm{di}_\beta) &= \mathrm{di}_{11\alpha_0} \setminus \mathrm{i}_0 = \mathrm{i}_0,\\
(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta) &= \mathrm{d}_\varnothing \setminus \mathrm{i}_{10} = \mathrm{i}_0\,.
\end{aligned}$$

Case 4.3.1.2 *None of the elements is* i_{10}. We suppose $\beta = 10\beta_0$ and $\alpha = 11\alpha_0$.

$$\begin{aligned}
(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma) &= \mathrm{di}_{10\beta_0} \setminus \mathrm{d}_\varnothing = \mathrm{d}_\varnothing,\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma) &= \mathrm{di}_{11\alpha_0} \setminus \mathrm{d}_\varnothing = \mathrm{d}_\varnothing;\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha) &= \mathrm{d}_\varnothing \setminus \mathrm{di}_{11\alpha_0} = \mathrm{di}_{11\alpha_0},\\
(\mathrm{di}_\gamma \setminus \mathrm{di}_\beta)\backslash(\mathrm{di}_\gamma \setminus \mathrm{di}_\alpha) &= \mathrm{di}_{10\beta_1} \setminus \mathrm{di}_{11\alpha_0} = \mathrm{di}_{11\alpha_0};\\
(\mathrm{di}_\gamma \setminus \mathrm{di}_\alpha)\backslash(\mathrm{di}_\gamma \setminus \mathrm{di}_\beta) &= \mathrm{di}_{11\alpha_0} \setminus \mathrm{di}_{10\beta_0} = \mathrm{di}_{10\beta_0},\\
(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta) &= \mathrm{d}_\varnothing \setminus \mathrm{di}_{10\beta_0} = \mathrm{di}_{10\beta_0}\,.
\end{aligned}$$

Case 4.3.2. *The addresses* α *and* β *are comparable.* We can suppose that β is a proper prefix of α, hence $\mathrm{di}_\beta = \mathrm{d}_1$. (The element i_1 does not belong to M_{LDLI}.)

Case 4.3.2.1 *The address* 10 *is a prefix of* α.

Case 4.3.2.1.1. *The element* di_α *is* i_{10}. One has

$$\begin{aligned}
(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma) &= \mathrm{d}_1 \backslash(\mathrm{d}_\varnothing \cdot \mathrm{d}_0) = \mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0 \cdot((\mathrm{d}_1 \cdot \mathrm{d}_\varnothing)\setminus \mathrm{d}_0)\\
&= \mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0 \cdot(\mathrm{d}_\varnothing \setminus \mathrm{d}_0) = \mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0 \cdot \mathrm{d}_{10} \cdot \mathrm{d}_{00},\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma) &= (\mathrm{i}_{100} \cdot \mathrm{i}_{110})\backslash(\mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0)\\
&= \mathrm{d}_\varnothing \cdot((\mathrm{i}_{010} \cdot \mathrm{i}_{110})\backslash(\mathrm{d}_1 \cdot \mathrm{d}_0)) = \mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_{10} \cdot \mathrm{d}_0 \cdot \mathrm{d}_{00};\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha) &= (\mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0)\backslash(\mathrm{i}_{100} \cdot \mathrm{i}_{110}) = (\mathrm{d}_1 \cdot \mathrm{d}_0)\backslash(\mathrm{i}_{010} \cdot \mathrm{i}_{110})\\
&= \mathrm{i}_{00} \cdot \mathrm{i}_{10},\\
(\mathrm{di}_\gamma \setminus \mathrm{di}_\beta)\backslash(\mathrm{di}_\gamma \setminus \mathrm{di}_\alpha) &= (\mathrm{d}_1 \cdot \mathrm{d}_\varnothing)\setminus \mathrm{i}_0 = \mathrm{d}_\varnothing \setminus \mathrm{i}_0 = \mathrm{i}_{10} \cdot \mathrm{i}_{00};\\
(\mathrm{di}_\gamma \setminus \mathrm{di}_\alpha)\backslash(\mathrm{di}_\gamma \setminus \mathrm{di}_\beta) &= \mathrm{i}_0 \backslash(\mathrm{d}_1 \cdot \mathrm{d}_\varnothing) = \mathrm{d}_1 \cdot \mathrm{d}_\varnothing,\\
(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta) &= (\mathrm{d}_\varnothing \cdot \mathrm{d}_0)\setminus \mathrm{d}_1 = \mathrm{d}_0 \backslash(\mathrm{d}_1 \cdot \mathrm{d}_\varnothing) = \mathrm{d}_1 \cdot \mathrm{d}_\varnothing\,.
\end{aligned}$$

Case 4.3.2.1.2. *The element* di_α *is not* i_{10}. We write $\alpha = 10\alpha_0$. One has

$$\begin{aligned}
(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma) &= \mathrm{d}_1 \setminus \mathrm{d}_\varnothing = \mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0,\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma) &= (\mathrm{di}_{110\alpha_0} \cdot \mathrm{di}_{100\alpha_0})\backslash(\mathrm{d}_1 \setminus \mathrm{d}_\varnothing = \mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0)\\
&= \mathrm{d}_1 \setminus \mathrm{d}_\varnothing = \mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0;
\end{aligned}$$

$$
\begin{aligned}
(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha) &= (\mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0)\backslash(\mathrm{di}_{110\alpha_0} \cdot \mathrm{di}_{100\alpha_0})\\
&= (\mathrm{d}_1 \cdot \mathrm{d}_0)\backslash(\mathrm{di}_{110\alpha_0} \cdot \mathrm{di}_{010\alpha_0}) = \mathrm{di}_{101\alpha_0} \cdot \mathrm{di}_{001\alpha_0},\\
(\mathrm{di}_\gamma \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\gamma \backslash \mathrm{di}_\alpha) &= (\mathrm{d}_1 \cdot \mathrm{d}_\varnothing)\backslash \mathrm{di}_{01\alpha_0} = \mathrm{d}_\varnothing \backslash \mathrm{di}_{01\alpha_0} = \mathrm{di}_{101\alpha_0} \cdot \mathrm{di}_{001\alpha_0};\\
(\mathrm{di}_\gamma \backslash \mathrm{di}_\alpha)\backslash(\mathrm{di}_\gamma \backslash \mathrm{di}_\beta) &= \mathrm{di}_{01\alpha_0} \backslash(\mathrm{d}_1 \cdot \mathrm{d}_\varnothing) = \mathrm{d}_1 \cdot \mathrm{d}_\varnothing,\\
(\mathrm{di}_\alpha \backslash \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \backslash \mathrm{di}_\beta) &= \mathrm{d}_\varnothing \backslash \mathrm{d}_1 = \mathrm{d}_1 \cdot \mathrm{d}_\varnothing;
\end{aligned}
$$

Case 4.3.2.2 *The address* 11 *is a proper prefix of* α.

Case 4.3.2.2.1. *The element* di_α *is* i_{110}. One has

$$
\begin{aligned}
(\mathrm{di}_\alpha \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \backslash \mathrm{di}_\gamma) &= (\mathrm{d}_1 \cdot \mathrm{d}_{10})\backslash \mathrm{d}_\varnothing = \mathrm{d}_{10} \backslash(\mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0)\\
&= \mathrm{d}_\varnothing \cdot(\mathrm{d}_{01} \backslash(\mathrm{d}_1 \cdot \mathrm{d}_0)) = \mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0 \cdot \mathrm{d}_{01} \cdot \mathrm{d}_{00},\\
(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma) &= \mathrm{i}_{10} \backslash(\mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0)\\
&= \mathrm{d}_\varnothing \cdot \mathrm{d}_0 \cdot(\mathrm{i}_0 \backslash(\mathrm{d}_1 \cdot \mathrm{d}_0)) = \mathrm{d}_\varnothing \cdot \mathrm{d}_0 \cdot \mathrm{d}_1 \cdot \mathrm{d}_{00} \cdot \mathrm{d}_{01};\\
(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha) &= (\mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0)\backslash \mathrm{i}_{10} = \mathrm{i}_0,\\
(\mathrm{di}_\gamma \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\gamma \backslash \mathrm{di}_\alpha) &= (\mathrm{d}_1 \cdot \mathrm{d}_\varnothing)\backslash \mathrm{i}_{110} = \mathrm{d}_\varnothing \backslash \mathrm{i}_{10} = \mathrm{i}_0;\\
(\mathrm{di}_\gamma \backslash \mathrm{di}_\alpha)\backslash(\mathrm{di}_\gamma \backslash \mathrm{di}_\beta) &= \mathrm{i}_{110} \backslash(\mathrm{d}_1 \cdot \mathrm{d}_\varnothing) = \mathrm{d}_1 \cdot \mathrm{d}_{10} \cdot(\mathrm{i}_{10} \backslash \mathrm{d}_\varnothing)\\
&= \mathrm{d}_1 \cdot \mathrm{d}_{10} \cdot \mathrm{d}_\varnothing \cdot \mathrm{d}_0,\\
(\mathrm{di}_\alpha \backslash \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \backslash \mathrm{di}_\beta) &= \mathrm{d}_\varnothing \backslash(\mathrm{d}_1 \cdot \mathrm{d}_{10}) = \mathrm{d}_1 \cdot \mathrm{d}_\varnothing \cdot((\mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0)\backslash \mathrm{d}_{10})\\
&= \mathrm{d}_1 \cdot \mathrm{d}_\varnothing \cdot((\mathrm{d}_1 \cdot \mathrm{d}_0)\backslash \mathrm{d}_{01}) = \mathrm{d}_1 \cdot \mathrm{d}_\varnothing \cdot \mathrm{d}_{01} \cdot \mathrm{d}_0;
\end{aligned}
$$

and one finds

$$
\begin{aligned}
\mathrm{d}_0^{-1} \cdot \mathrm{d}_\varnothing^{-1} \cdot \mathrm{d}_{10}^{-1} \cdot \mathrm{d}_1^{-1} \cdot \mathrm{d}_1 \cdot \mathrm{d}_\varnothing \cdot \mathrm{d}_{01} \cdot \mathrm{d}_0 &\curvearrowright \mathrm{d}_0^{-1} \cdot \mathrm{d}_\varnothing^{-1} \cdot \mathrm{d}_{10}^{-1} \cdot \mathrm{d}_\varnothing \cdot \mathrm{d}_{01} \cdot \mathrm{d}_0\\
&\curvearrowright \mathrm{d}_0^{-1} \cdot \mathrm{d}_\varnothing^{-1} \cdot \mathrm{d}_\varnothing \cdot \mathrm{d}_{01}^{-1} \cdot \mathrm{d}_{01} \cdot \mathrm{d}_0 \curvearrowright \mathrm{d}_0^{-1} \cdot \mathrm{d}_0 \curvearrowright \varepsilon.
\end{aligned}
$$

Case 4.3.2.1.2. *The element* di_α *is not* i_{110}. We write $\alpha = 11e\alpha_0$. One has

$$
\begin{aligned}
(\mathrm{di}_\alpha \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \backslash \mathrm{di}_\gamma) &= \mathrm{d}_1 \backslash \mathrm{d}_\varnothing = \mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0,\\
(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma) &= \mathrm{di}_{1e1\alpha_0} \backslash(\mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0 = \mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0;\\
(\mathrm{di}_\beta \backslash \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \backslash \mathrm{di}_\alpha) &= (\mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0)\backslash \mathrm{di}_{1e1\alpha_0} = (\mathrm{d}_1 \cdot \mathrm{d}_0)\backslash \mathrm{di}_{e11\alpha_0} = \mathrm{di}_{e11\alpha_0},\\
(\mathrm{di}_\gamma \backslash \mathrm{di}_\beta)\backslash(\mathrm{di}_\gamma \backslash \mathrm{di}_\alpha) &= (\mathrm{d}_1 \cdot \mathrm{d}_\varnothing)\backslash \mathrm{di}_{11e\alpha_0} = \mathrm{d}_\varnothing \backslash \mathrm{di}_{1e1\alpha_0} = \mathrm{di}_{e11\alpha_0};\\
(\mathrm{di}_\alpha \backslash \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \backslash \mathrm{di}_\beta) &= \mathrm{di}_{11e\alpha_0} \backslash(\mathrm{d}_1 \cdot \mathrm{d}_\varnothing) = \mathrm{d}_1 \cdot \mathrm{d}_\varnothing,\\
(\mathrm{di}_\alpha \backslash \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \backslash \mathrm{di}_\beta) &= \mathrm{d}_\varnothing \backslash \mathrm{d}_1 = \mathrm{d}_1 \cdot \mathrm{d}_\varnothing\,.
\end{aligned}
$$

Case 4.3.2.3 *The address α is* 11. This is the most complicated one but we do not need to consider it here because all these three addresses belong to $\mathbf{A}_{\text{LD}}$ and this one is shown in [4].

Case 5. *The address $\varnothing$ is the greatest common prefix of α and β.* We can suppose $\alpha \perp \beta$, otherwise we are in the case 1. We write $\alpha = 0\alpha_0$.

Case 5.1. *The address* 1 *is a proper prefix of β.*

Case 5.1.1. *The element* di_β *is* i_{10}. One has

$$\begin{aligned}
(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma) &= \mathrm{i}_{10} \setminus \mathrm{d}_\varnothing = \mathrm{d}_\varnothing \cdot \mathrm{d}_0,\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma) &= \mathrm{di}_{0\alpha_0} \backslash(\mathrm{d}_\varnothing \cdot \mathrm{d}_0) = \mathrm{d}_\varnothing \cdot \mathrm{d}_0;\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha) &= (\mathrm{d}_\varnothing \cdot \mathrm{d}_0)\backslash \mathrm{di}_{0\alpha_0} = \mathrm{d}_0 \backslash(\mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{00\alpha_0})\\
&= \mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{010\alpha_0} \cdot \mathrm{di}_{000\alpha_0},\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha) &= \mathrm{i}_0 \backslash(\mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{00\alpha_0}) = \mathrm{di}_{00\alpha_0} \cdot \mathrm{di}_{000\alpha_0} \cdot \mathrm{di}_{010\alpha_0};\\
(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta) &= (\mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{00\alpha_0})\backslash \mathrm{i}_0 = \mathrm{i}_0,\\
(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta) &= \mathrm{d}_\varnothing \setminus \mathrm{i}_{10} = \mathrm{i}_0 \,.
\end{aligned}$$

Case 5.1.2. *The element* di_β *is not* i_{10}. We write $\beta = 1e\beta_0$ and we have

$$\begin{aligned}
(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma) &= \mathrm{di}_{1e\beta_0} \setminus \mathrm{d}_\varnothing = \mathrm{d}_\varnothing,\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma) &= \mathrm{di}_{0\alpha_0} \setminus \mathrm{d}_\varnothing = \mathrm{d}_\varnothing;\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha) &= \mathrm{d}_\varnothing \setminus \mathrm{di}_{0\alpha_0} = \mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{00\alpha_0},\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha) &= \mathrm{di}_{e1\beta_0} \backslash(\mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{00\alpha_0}) = \mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{00\alpha_0};\\
(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta) &= (\mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{00\alpha_0})\backslash \mathrm{di}_{e1\beta_0} = \mathrm{di}_{e1\beta_0},\\
(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta) &= \mathrm{d}_\varnothing \setminus \mathrm{di}_{1e\beta_0} = \mathrm{di}_{1e\beta_0} \,.
\end{aligned}$$

Case 5.2. *The address β is equal to* 1. We find

$$\begin{aligned}
(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma) &= \mathrm{d}_1 \setminus \mathrm{d}_\varnothing = \mathrm{d}_\varnothing \cdot \mathrm{d}.\,\mathrm{d}_0,\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma) &= \mathrm{di}_{0\alpha_0} \backslash(\mathrm{d}_\varnothing \cdot \mathrm{d}.\,\mathrm{d}_0) = \mathrm{d}_\varnothing \cdot \mathrm{d}.\,\mathrm{d}_0;\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha) &= (\mathrm{d}_\varnothing \cdot \mathrm{d}_1 \cdot \mathrm{d}_0)\backslash \mathrm{di}_{0\alpha_0} = (\mathrm{d}_1 \cdot \mathrm{d}_0)\backslash(\mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{00\alpha_0})\\
&= \mathrm{di}_{110\alpha_0} \cdot \mathrm{di}_{100\alpha_0} \cdot \mathrm{di}_{010\alpha_0} \cdot \mathrm{di}_{000\alpha_0},\\
(\mathrm{di}_\beta \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\beta \setminus \mathrm{di}_\alpha) &= (\mathrm{d}_1 \cdot \mathrm{d}_\varnothing)\backslash(\mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{00\alpha_0})\\
&= \mathrm{di}_{110\alpha_0} \cdot \mathrm{di}_{010\alpha_0} \cdot \mathrm{di}_{100\alpha_0} \cdot \mathrm{di}_{000\alpha_0};\\
(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta) &= (\mathrm{di}_{10\alpha_0} \cdot \mathrm{di}_{00\alpha_0})\backslash(\mathrm{d}_1 \cdot \mathrm{d}_\varnothing) = \mathrm{d}_1 \cdot \mathrm{d}_\varnothing,\\
(\mathrm{di}_\alpha \setminus \mathrm{di}_\gamma)\backslash(\mathrm{di}_\alpha \setminus \mathrm{di}_\beta) &= \mathrm{d}_\varnothing \setminus \mathrm{d}_1 = \mathrm{d}_1 \cdot \mathrm{d}_\varnothing \,.
\end{aligned}$$

Case 6. *The greatest common prefix of two addresses is a prefix of the third one.* Suppose that the greatest prefix γ' of α and β is a prefix of γ. If we have $\beta = \gamma'$ or $\gamma = \gamma'$, then we are in the case 4 or in the case 5. If α and β are orthogonal, then we are in the case 1. We have considered all the cases and the proof is finished. ☐

We deduce from Proposition 5.2:

Proposition 5.6. *: The word problem of M_{LDLI} is solvable.*

Each monoid satisfying the conditions of Proposition 5.2 has some good properties:

Proposition 5.7. *The monoid M_{LDLI} is left cancellative and the left divisibility order on M_{LDLI} forms a lattice.*

Proof. It is shown in [4] that each monoid satisfying the conditions of Proposition 5.2 is left cancellative, each two elements have a unique greatest common left divisor and each two elements having a common right multiple have also the least one. According to Proposition 4.9, each pair of elements in M_{LDLI} has a common right multiple and hence the left divisibility order on M_{LDLI} forms a lattice. ☐

Remark 5.8. The canonical projection $M_{\text{LDLI}} \twoheadrightarrow \mathcal{G}^{+}_{\text{LDLI}}$ is not injective. We have $\mathrm{i}_0 \cdot \mathrm{i}_{00} \cdot \mathrm{i}_0 \not\equiv^{+}_{\text{LDLI}} \mathrm{i}_0 \cdot \mathrm{i}_0 \cdot \mathrm{d}_0$, because M_{LDLI} is left cancellative and $\mathrm{I}_{00} \bullet \mathrm{I}_0 \neq \mathrm{I}_0 \bullet \mathrm{D}_0$: the operator $\mathrm{I}_{00} \bullet \mathrm{I}_0$ sends $(x{\cdot}y){\cdot}z$ onto $(((x{\cdot}x){\cdot}y){\cdot}((x{\cdot}x){\cdot}y)){\cdot}z$ and the operator $\mathrm{I}_0 \bullet \mathrm{D}_0$ sends $(x \cdot y) \cdot z$ onto $(((x \cdot y) \cdot x) \cdot ((x \cdot y) \cdot y)) \cdot z$. However, we have

$$\begin{aligned}\mathrm{i}_0 \cdot \mathrm{i}_{00} \cdot \mathrm{i}_0 \equiv^{+}_{\text{LDI}} \mathrm{i}_0 \cdot \mathrm{i}_0 \cdot \mathrm{i}_{000} \cdot \mathrm{i}_{010} \equiv^{+}_{\text{LDI}} \mathrm{i}_0 \cdot \mathrm{i}_{01} \cdot \mathrm{d}_0 \cdot \mathrm{i}_{000} \cdot \mathrm{i}_{010} \equiv^{+}_{\text{LDI}} \\ \equiv^{+}_{\text{LDI}} \mathrm{i}_0 \cdot \mathrm{i}_{01} \cdot \mathrm{i}_{00} \cdot \mathrm{d}_0 \equiv^{+}_{\text{LDI}} \mathrm{i}_0 \cdot \mathrm{i}_0 \cdot \mathrm{d}_0\end{aligned}$$

and hence also the equality $\mathrm{I}_0 \bullet \mathrm{I}_{00} \bullet \mathrm{I}_0 = \mathrm{I}_0 \bullet \mathrm{I}_0 \bullet \mathrm{D}_0$.

The solution of the word problem of the LD identity actually uses the geometry group, not the geometry monoid nor the positive geometry monoid. The geometry group is obtained as the fraction group of the positive geometry monoid. In the cases of LDI and LDLI the positive geometry monoids are not cancellative and it is unlikely that their group of fractions could describe the identities well. Nevertheless, the monoid M_{LDLI} is (at least left) cancellative and it has all the important properties of the positive geometry monoid of LDLI. Hence it might be possible to attack the word problem from this direction.

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Received by the editors: 26.02.2007
and in final form 12.04.2007.