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On $\mathfrak{F}$-radicals of finite π-soluble groups

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Abstract. In this paper, we prove that for every local π-saturated Fitting class $\mathcal{F}$ with $char(\mathcal{F}) = \mathbb{P}$, the $\mathcal{F}$-radical of every finite π-soluble groups G has the property: $C_G(G_{\mathcal{F}}) \subseteq G_{\mathcal{F}}$. From this, some well known results are followed and some new results are obtained.

1. Introduction

It is well known that in the class of all finite soluble groups the nilpotent radical (or say $\mathcal{N}$-radical) $F(G)$ of a soluble group G has the important property that $C_G(F(G)) \subseteq F(G)$(cf. [2, Theorem 1.8.18]), and the important property has wide applications in the theory of soluble groups. Developing the result, it was fount that in the class of all π-soluble groups, the π-nilpotent radical (or say $\mathcal{N}^\pi$-radical) $F_\pi(G)$ of a π-soluble group G also has the property that $C_G(F_\pi(G)) \subseteq F_\pi(G)$ (cf. [5, Theorem 4.1.2]).

Observe that the class of all nilpotent groups $\mathcal{N}$ and the class of all π-nilpotent groups $\mathcal{N}^\pi$ are all local Fitting classes. In connection with these results, Vorob'ev N.T. proposed the following problem at Vitebsk seminar in 1996: find out the local Fitting classes $\mathcal{F}$ and the universals $\mathcal{U}$ such that for the $\mathcal{F}$-radical $G_{\mathcal{F}}$ of every group G in $\mathcal{U}$ has the property: $C_G(G_{\mathcal{F}}) \subseteq G_{\mathcal{F}}$. In this paper, we prove that for every π-saturated Fitting class $\mathcal{F}$ with $char(\mathcal{F}) = \mathbb{P}$ and the universal $\mathcal{S}^\pi$ of all finite π-soluble groups, the $\mathcal{F}$-radical of every finite π-soluble group G has the property: $C_G(G_{\mathcal{F}}) \subseteq G_{\mathcal{F}}$. From this, some well known results are followed and some new results are obtained.

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2. Preliminaries

Throughout this paper, all groups considered are finite groups.

Recall that a class of groups $\mathcal{F}$ is called a Fitting class provided the following two conditiond are satisfied: (i) if $G \in \mathcal{F}$ and $N \triangleleft G$, then $N \in \mathcal{F}$; (ii) if $N_1, N_2 \trianglelefteq G$ and $N_1, N_2 \in \mathcal{F}$, then $N_1N_2 \in \mathcal{F}$. It follows from the condition (ii) that if $\mathcal{F}$ is a nonempty Fitting class, then every group G has a unique $\mathcal{F}$-maximal normal subgroup, denoted by $G_{\mathcal{F}}$ and called the $\mathcal{F}$-radical of G. For example, the class of all nilpotent groups $\mathcal{N}$ is a Fitting class and every group G has the $\mathcal{N}$-radical which is just the Fitting subgroup $F(G)$ of G.

If $\mathcal{F}$ and $\mathcal{H}$ are Fitting classes, then the class of groups $\mathcal{FH} = (G : G/G_{\mathcal{F}} \in \mathcal{H})$ is called the product of Fitting classes $\mathcal{F}$ with $\mathcal{H}$. It is well known that the product $\mathcal{FH}$ of two Fitting classes $\mathcal{F}$ with $\mathcal{H}$ is still a Fitting class and the multiplication of Fitting classes satisfies associative law.

Let π be a nonempty set of prime numbers and π' the complement of π in the set of all prime numbers $\mathbb{P}$. A group G is called π-soluble [2], if G has a chief series such that every chief factor in which is either an abelian p-group for $p \in \pi$ or a π'-group. If $\pi = p$, then π-soluble is said to be a p-soluble group. We denote by $\mathcal{S}^{\pi}$ the class of all π-soluble groups; $\mathcal{S}$ denotes the class of all soluble groups; $\mathcal{E}$ denotes the class of all finite groups; $\mathcal{E}_{\pi}$ denotes the class of all finite π-groups; $\mathcal{N}_p$ denotes the class of all p-groups; For a class of groups $\mathcal{F}$, put $\mathcal{F}_{\pi} = \mathcal{F} \cap \mathcal{E}_{\pi}$. In particular, $\mathcal{S}_{\pi} = \mathcal{S} \cap \mathcal{E}_{\pi}$ is the class of all soluble π-groups.

A function f defined by $f : \mathbb{P} \longrightarrow \{\text{Fitting classes}\}$ is called a Hartley function (or in brevity, H-function) (see [6]). Let $\sigma = Supp(f)$ is the support of the function f (see [1, p. 323]) and $LR(f) = \mathcal{E}_{\sigma} \cap (\cap_{p \in \sigma} f(p)\mathcal{N}_p\mathcal{E}_{p'})$. Then, a Fitting class $\mathcal{F}$ is called a local [6], if there exists a H-function f such that $\mathcal{F} = LR(f)$.

Let $\mathcal{F}$ is a class of groups. Then we define $char(\mathcal{F}) = \{p : p \in \mathbb{P} \text{ and } \mathrm{Z_p} \in \mathcal{F}\}$, and call $char(\mathcal{F})$ the characteristic of $\mathcal{F}$. If $char(\mathcal{F}) = \mathbb{P}$, then we say that $\mathcal{F}$ has full characteristic.

Recall that a Fitting class is said to be a Fischer class if the following condition holds: if $K \trianglelefteq G \in \mathcal{F}$ and H/K is a nilpotent subgroup of G/K, then $H \in \mathcal{F}$ (cf. [1, p.601]). The following two known lemmas are useful later on in our paper.

Lemma 2.1[3] *Let $\mathcal{F}$ be a Fitting class of finite groups. If $\mathcal{F}$ a Fischer class. Then $char\mathcal{F} = \pi(\mathcal{F})$.*

Lemma 2.2[7, Lemma 2]. *Every local Fitting class is a Fischer class.*

By using the above Lemmas, we can prove the following lemma.

Lemma 2.3. *If $\mathcal{F}$ is a local Fitting class, then $Supp(f) = \pi(\mathcal{F}) = char\mathcal{F}$.*

Proof. Let $\pi = Supp(f)$ and $p \in \pi$. Then $f(p) \neq \emptyset$. It is clear that $f(p) \subseteq f(p)\mathcal{N}_p \subseteq f(p)\mathcal{N}_p\mathcal{E}_{p'}$, for every $p \in \pi$. It is also easy to see that $Z_p \in f(p)\mathcal{N}_p \subseteq f(p)\mathcal{N}_p\mathcal{E}_{p'}$ and $Z_p \in f(q)\mathcal{N}_q\mathcal{E}_{q'}$ for every $q \neq p$. Hence $Z_p \in \cap_{p\in\pi} f(p)\mathcal{N}_p\mathcal{E}_{p'}$. Moreover, since $p \in \pi$, we have that $Z_p \in \mathcal{E}_\pi$. Hence $Z_p \in \mathcal{E}_\pi \cap (\cap_{p\in\pi} f(p)\mathcal{N}_p\mathcal{E}_{p'}) = \mathcal{F}$. This means that $p \in char(\mathcal{F})$ and hence $p \in \pi(\mathcal{F})$ by Lemma 2.1 and Lemma 2.2. This shows that $Supp(f) \subseteq \pi(\mathcal{F})$. On the other hand, if $p \in \pi(\mathcal{F})$, then there exists a group $G \in \mathcal{F}$ such that $p||G|$. Since $\mathcal{F} = \mathcal{E}_\pi \cap (\cap_{p\in\pi} f(p)\mathcal{N}_p\mathcal{E}_{p'})$, we know that $f(p) \neq \emptyset$. Hence $p \in Supp(f)$. This shows that $\pi(\mathcal{F}) \subseteq Supp(f)$. This completes the proof.

All unexplained notations and terminologies are standard. The reader is refereed to the text of Doerk and Hawkes [1] and Shemetkov [5] if necessary.

3. The main results

Definition 3.1. Let $\emptyset \subset \pi \subseteq \mathbb{P}$. A Fitting class $\mathcal{F}$ is said to be π-saturated if $\mathcal{F}\mathcal{E}_{\pi'} = \mathcal{F}$.

Theorem 3.2. Let $\mathcal{F}$ be a π-saturated Fitting class with full characteristic and $\mathcal{F} = LR(f)$, for some H-function f. If $\mathcal{X}$ is a nonempty Fitting class such that $\mathcal{X} \subseteq \cap f(p)$, then $C_G(G_\mathcal{F}/G_\mathcal{X}) \subseteq G_\mathcal{F}$ for every group $G \in \mathcal{F}\mathcal{S}^\pi$ (in particular, for every π-soluble group).

Proof. Firstly, we note that $\mathcal{X} \subseteq \mathcal{F} = LR(f)$ since $LR(f) = \mathcal{E}_\mathbb{P} \cap (\cap_{p\in\mathbb{P}} f(p)\mathcal{N}_p\mathcal{E}_{p'}) = \cap_p f(p)\mathcal{N}_p\mathcal{E}_{p'}$ and $\mathcal{X} \subseteq \cap_{p\in\sigma} f(p)$. Hence, $G_\mathcal{X} \subseteq G_\mathcal{F}$.

Let $C = C_G(G_\mathcal{F}/G_\mathcal{X})$. Assume that $C \not\subseteq G_\mathcal{F}$. Then the factor group $C/C\cap G_\mathcal{F}$ is non-trivial. It follows that there exists a normal subgroup K of G contained in C such that $K/C\cap G_\mathcal{F}$ is a non-trivial chief factor of G. Obviously, $K/K\cap G_\mathcal{F} = K/C\cap G_\mathcal{F}$. Then, since $K/K\cap G_\mathcal{F} \simeq KG_\mathcal{F}/G_\mathcal{F}$ and $G/G_\mathcal{F}$ is π-soluble, we have that the chief factor $K/K\cap G_\mathcal{F}$ is either a π'-group or an elementary abelian p-group for some $p \in \pi$. If $K/K\cap G_\mathcal{F}$ is a π'-group, then $K \in \mathcal{F}\mathcal{E}_{\pi'}$. Since $\mathcal{F}$ is π-saturated, we have $\mathcal{F}\mathcal{E}_{\pi'} = \mathcal{F}$, and consequently, $K \in \mathcal{F}$. This implies that $K \subseteq G_\mathcal{F}$, which contradicts to that $K/K\cap G_\mathcal{F}$ is non-trivial.

Now, assume that $K/K\cap G_\mathcal{F}$ is an elementary abelian p-group for some $p \in \pi$. Then, the derived subgroup $(K/K \cap G^\mathcal{F})'$ is an identity group. However, since $(K/K\cap G^\mathcal{F})' = K'(K\cap G_\mathcal{F})/K\cap G_\mathcal{F}$, we see that $K'(K\cap G_\mathcal{F}) = K\cap G_\mathcal{F}$ and so $K' \subseteq K\cap G_\mathcal{F}$. Since $K \subseteq C$, $K \subseteq C_G(K\cap G_\mathcal{F}/G_\mathcal{X})$. It follows that $[K', K] \subseteq [K\cap G_\mathcal{F}, K] \subseteq G_\mathcal{X}$ and hence $[(K/G_\mathcal{X})', K/G_\mathcal{X}] = 1$. Therefore, $K/G_\mathcal{X}$ is a nilpotent group with nilpotent class at most 2 and hence $K/G_{\mathcal{X}}$ has a nonidentity normal Sylow p-subgroup $P/G_{\mathcal{X}}$. Obviously, $P \trianglelefteq G$. Then, by [1, A.10.9], we have that $P/G_{\mathcal{X}}$ covers the p-chief factor $(K/G_{\mathcal{X}})/(K \cap G_{\mathcal{F}}/G_{\mathcal{X}})$ of $G/G_{\mathcal{X}}$. This implies that $(P/G_{\mathcal{X}})(K \cap G_{\mathcal{F}}/G_{\mathcal{X}}) \supseteq K/G_{\mathcal{X}}$, and consequently, $P(K \cap G_{\mathcal{F}}) \supseteq K$. Hence $PG_{\mathcal{F}} = KG_{\mathcal{F}}$.

We now prove that $P \in \mathcal{F}$. In fact, by Lemma 2.3, we have that $Supp(f) = \pi(\mathcal{F}) = char\mathcal{F}$ and consequently $\mathcal{F} = \cap_p f(p)\mathcal{N}_p\mathcal{E}_{q'}$. Since $G_{\mathcal{X}} = P \cap G_{\mathcal{X}} = P_{\mathcal{X}}$ and $P/P_{\mathcal{X}} \in \mathcal{N}_p$, we have that $P \in \mathcal{X}\mathcal{N}_p \subseteq f(p)\mathcal{N}_p \subseteq f(p)\mathcal{N}_p\mathcal{E}_{p'}$. Now, if q is a prime and $q \neq p$, then $P/P_{\mathcal{X}} \in \mathcal{E}_{q'} \subseteq f(q)\mathcal{E}_{q'} \subseteq f(q)\mathcal{N}_q\mathcal{E}_{q'}$. This shows that $P \in \cap_p f(p)\mathcal{N}_p\mathcal{E}_{q'} = \mathcal{F}$.

Finally, because $P \in \mathcal{F}$, we have $PG_{\mathcal{F}} = KG_{\mathcal{F}} \in \mathcal{F}$, and consequently, $KG_{\mathcal{F}} \subseteq G_{\mathcal{F}}$. This contradicts that $KG_{\mathcal{F}}/G_{\mathcal{F}}$ is non-trivial. The proof is thus completed.

From Theorem 3.2, we can obtain many applications. For example, firstly, the following well-known results now follows from our Theorem 3.2.

Corollary 3.3([4, 5.4.4]). If G is a soluble group, then $C_G(F(G)) \subseteq F(G)$.

Proof. Let $\pi = \mathbb{P}$. Then the class of all nilpotent groups $\mathcal{N}$ is a π-saturated Fitting class (since $\mathcal{N}\mathcal{E}_{\mathbb{P}'} = \mathcal{N}(1) = \mathcal{N}$) with full characteristic, and $\mathcal{N} = LR(f)$ where $f(p) = (1)$ for all $p \in \mathbb{P}$. Obviously $\cap_p f(p) = (1)$, where (1) is the class consists of identity groups. Then, by using our theorem, we have that $C_G(G_{\mathcal{N}}) \subseteq G_{\mathcal{N}}$, where $G_{\mathcal{N}} = F(G)$.

Corollary 3.4([2, Theorem 1.8.19]). If G is a p-soluble group, then $C_G(F_p(G)) \subseteq F_p(G)$, where $F_p(G)$ is the maximal p-nilpotent normal subgroup of G.

Proof. By the definition of product of Fitting classes, we see that the class of all p-nilpotent groups is exactly the product $\mathcal{E}_{p'}\mathcal{N}_p$ of the class of all finite p'-groups $\mathcal{E}_{p'}$ with the class of all p-groups $\mathcal{N}_p$. Let f is the H-function such that $f(p) = \mathcal{E}_{p'}\mathcal{N}_p$ and $f(q) = \mathcal{E}_{p'}$, for all $q \neq p$. Then $supp(f) = \mathbb{P}$ and $LR(f) = \mathcal{E}_{\mathbb{P}} \cap (\cap_{p\in\mathbb{P}} f(p)\mathcal{N}_p\mathcal{E}_{p'}) = \cap_p f(p)\mathcal{N}_p\mathcal{E}_{p'} = \mathcal{E}_{p'}\mathcal{N}_p\mathcal{E}_{p'} \cap (\cap_{q\neq p}\mathcal{E}_{p'}\mathcal{N}_q\mathcal{E}_{q'}) = \mathcal{E}_{p'}\mathcal{N}_p\mathcal{E}_{p'} \cap \mathcal{E}_{p'}(\cap_q\mathcal{E}_{q'}) = \mathcal{E}_{p'}\mathcal{N}_p\mathcal{E}_{p'} \cap \mathcal{E}_{p'}\mathcal{N}_p = \mathcal{E}_{p'}\mathcal{N}_p$, that is, $LR(f) = \mathcal{E}_{p'}\mathcal{N}_p$. It is clear that $\mathcal{E}_{p'}\mathcal{N}_p$ is p'-saturated. Let $\mathcal{X} = (1)$. Then, by Theorem 3.2, we have that $C_G(G_{\mathcal{E}_{p'}\mathcal{N}_p}) \subseteq G_{\mathcal{E}_{p'}\mathcal{N}_p}$, that is, $C_G(F_p(G)) \subseteq F_p(G)$.

Now, by using Theorem 3.2, we can also obtain some new results. For example, we now apply Theorem 3.2 to the class of groups $\mathcal{E}_\pi\mathcal{E}_{\pi'}$ and the class of groups $\mathcal{N}_\pi\mathcal{E}_{\pi'}$, where $\mathcal{E}_\pi\mathcal{E}_{\pi'}$ is the product of the class of all π-groups $\mathcal{E}_\pi$ with the class of all π'-groups $\mathcal{E}_{\pi'}$ and $\mathcal{N}_\pi\mathcal{E}_{\pi'}$ is the product of the class of all nilpotent π-groups $\mathcal{N}_\pi$ with the class of all π'-groups $\mathcal{E}_{\pi'}$. On the other words, $\mathcal{E}_\pi\mathcal{E}_{\pi'}$ is the class of all extensions of π-groups by π'-groups, and $\mathcal{N}_\pi\mathcal{E}_{\pi'}$ is the class of all extensions of nilpotent π-groups by π'-groups.

Corollary 3.5. If G is a π-soluble group, then $C_G(G_{\mathcal{E}_\pi\mathcal{E}_{\pi'}}) \subseteq G_{\mathcal{E}_\pi\mathcal{E}_{\pi'}}$.

Proof. Let f is a H-function such that $f(p) = \mathcal{E}_\pi$ for $p \in \pi$ and $f(p) = \mathcal{E}_\pi\mathcal{E}_{\pi'}$ for all $p \in \pi'$. Then $Supp(f) = \mathbb{P}$ and $LR(f) = \cap_{p\in\mathbb{P}} f(p)\mathcal{N}_p\mathcal{E}_{p'}) = (\cap_{p\in\pi}\mathcal{E}_\pi\mathcal{N}_p\mathcal{E}_{p'}) \cap (\cap_{p\in\pi'}\mathcal{E}_\pi\mathcal{E}_{\pi'}\mathcal{N}_p\mathcal{E}_{p'}) = \mathcal{E}_\pi\mathcal{E}_{\pi'} \cap \mathcal{E}_\pi\mathcal{E}_{\pi'} = \mathcal{E}_\pi\mathcal{E}_{\pi'}$. This shows that the classes $\mathcal{E}_\pi\mathcal{E}_{\pi'}$ is a local Fitting classes and clearly it is a π-saturated Fitting class with full characteristic. Let $\mathcal{X} = (1)$. Then, by using our Theorem, we obtain that $C_G(G_{\mathcal{S}_\pi\mathcal{S}_{\pi'}}) \subseteq G_{\mathcal{S}_\pi\mathcal{S}_{\pi'}}$. The proof is completed.

Corollary 3.6. If G is a π-soluble group, then $C_G(G_{\mathcal{N}_\pi\mathcal{S}_{\pi'}}) \subseteq G_{\mathcal{N}_\pi\mathcal{S}_{\pi'}}$.

Proof. Let h is a H-function such that $h(p) = (1)$ for all $p \in \pi$ and $h(p) = \mathcal{N}_\pi\mathcal{E}_{\pi'}$ for all $p \in \pi'$. Then, $LR(h) = \cap_{p\in\mathbb{P}} h(p)\mathcal{N}_p\mathcal{E}_{p'} = (\cap_{p\in\pi}\mathcal{N}_p\mathcal{E}_{p'})\cap(\cap_{p\in\pi'}\mathcal{N}_\pi\mathcal{E}_{\pi'}\mathcal{N}_p\mathcal{E}_{p'}) = \mathcal{N}_\pi\mathcal{E}_{\pi'}\cap\mathcal{N}_\pi\mathcal{E}_{\pi'}(\cap_{p\in\pi'}\mathcal{N}_p\mathcal{E}_{p'}) = \mathcal{N}_\pi\mathcal{E}_{\pi'}\cap \mathcal{N}_\pi\mathcal{E}_{\pi'}\mathcal{N}_\pi\mathcal{E}_{\pi'} = \mathcal{N}_\pi\mathcal{E}_{\pi'}$. This shows that the classes $\mathcal{N}_\pi\mathcal{E}_{\pi'}$ are a π-saturated local Fitting classes with full characteristic. Let $\mathcal{X} = (1)$. Then, by using our Theorem, we obtain that $C_G(G_{\mathcal{N}_\pi\mathcal{E}_{\pi'}}) \subseteq G_{\mathcal{N}_\pi\mathcal{E}_{\pi'}}$. The proof is thus completed.

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