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RESEARCH ARTICLE

On the dimension of Kirichenko space

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Abstract. We introduce a notion of the Kirichenko space which is connected with the notion of Gorenstein matrix (see [2], ch. 14). Every element of Kirichenko space is an $n \times n$ matrix, whose elements are solutions of the equations $a_{i,j} + a_{j,\sigma(i)} = a_{i,\sigma(i)}$; $a_{1,i} = 0$ for $i, j = 1, \ldots, n$ determined by a permutation σ which has no cycles of the length 1. We give a formula for the dimension of this space in terms of the cyclic type of σ.

Introduction

We remind some definitions and notions from [2], ch 14. Denote by $M_n(B)$ the ring of all $n \times n$ matrices over the ring B. Let $\mathbb{Z}$ be a ring of all integers.

An integer matrix $\mathcal{E} = (a_{ij})$ is called
an exponent matrix, if $a_{ij} + a_{jk} \geq a_{ik}$ for all i, j, k;
an reduced exponent, if $a_{ij} + a_{ji} > 0$ for all i, j.

A reduced exponent matrix $\mathcal{E}$ will be called Gorenstein, if there exists a permutation σ of $\{1, \ldots, n\}$ such that $a_{i,k} + a_{k,\sigma(i)} = a_{i,\sigma(i)}$.

We will denote σ by $\sigma(\mathcal{E})$. As usual, we will call $\sigma(\mathcal{E})$ the Kirichenko permutation. Note that $\sigma(\mathcal{E})$ of Gorenstein matrix has no cycles of length 1. We will name relations of the type $a_{i,k} + a_{k,\sigma(i)} = a_{i,\sigma(i)}$ **Gorenstein relations**. Exponent matrices are widely used in the theory of tiled orders over a discrete valuation ring. Many properties of tiled orders and their quivers are fully determined by their exponent matrices. Gorenstein matrices appeared at the first time in [5].

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It is easy to see, that from the condition $a_{i,k} + a_{i,\sigma(k)} = a_{i,\sigma(i)}$ for $i = k$ we obtain $a_{k,k} = 0$ for every k, $1 \le k \le n$.

Theorem. (see [1], p. 17) Let permutation σ of the set $\{1, 2, \dots n\}$ be arbitrary permutation without fixed elements. Then there exists Gorenstein matrix A with σ as Kirichenko permutation, such that all elements of A belong to the set $\{0, 1, 2\}$.

Proof. Matrix A, which has σ as correspondent Kirichenko permutation may be constructed in direct way:

$a_{i,i} = 0$ and $a_{i,\sigma(i)} = 2$ for $i = 1, \dots, n$;

$a_{i,j} = 1$ for all $i, j \in [1, n]$, such that $i \ne j$ and $j \ne \sigma(i)$.

It is obvious that such matrix will be Gorenstein and σ will be Kirichenko permutation. □

Exponent matrices A, B are called **equivalent**, if one may be obtained from another by the following transformations:

1) adding an integer to all elements of some row with simultaneous subtracting it from the elements of the column with the same number.

2) simultaneous interchanging of two rows and equally numbered columns,

or by compositions of such transformations.

Theorem (see [1], p. 15): Under the transformations of the first type Gorenstein matrix goes to Gorenstein one with the same Kirichenko permutation.

Note that transformations of the first type define free commutative group with n generators.

If one consider matrices

$$H_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \text{ and } G_{2m} = \left(\begin{array}{|c|c|} H_m & H_m^{(1)} \\ \hline H_m^{(1)} & H_m \end{array} \right),$$

where $H_m^{(1)} = E + H_m$, then it will be easy to see, that H_n is Gorenstein matrix with Kirichenko permutation $\sigma(H_n) = (n, n-1, \dots 2, 1)$, and G_{2m} is Gorenstein one with Kirichenko permutation $\sigma(G_{2m}) = \prod_{i=1}^{m} (i, m+i)$.

A matrix $B = (b_{i,j})$ is called (0, 1) matrix, if $b_{i,j}$ is either zero or one.

Theorem (see [1], p. 15): Gorenstein (0, 1) matrix is equivalent either H_n or G_{2m}.

Theorem (see [1], p. 15): If A is Gorenstein matrix with Kirichenko permutation σ, and B is obtained from it by transformation of the second type, defined by transposition τ, then B is Gorenstein matrix with the permutation $\tau\sigma\tau$.

According to the last theorem, without bounding of generality we may suppose that for the cyclic index $\{l_1, \ldots, l_q\}$ of correspond Kirichenko permutation the conditions $2 \leq l_1 \leq \ldots \leq l_q$ and $l_1 + \ldots + l_q = n$ take place. Denote $n_i = \sum\limits_{k=1}^{i} l_k$ for every $1 \leq i \leq q$. Then σ will look as

$$\sigma = (1, 2 \ldots n_1)(n_1 + 1, n_1 + 2 \ldots, n_2) \cdots (n_{q-1} + 1, n_{q-1} + 2, \ldots, n_q).$$

Further we will consider only Gorenstein matrices and under **equivalence of matrices** we will consider possibility of obtaining one from another only by transformations of the first type.

One may come to a question of describing of Gorenstein matrices in inverse way to the one, presented in the definition. In the set of square matrices of order n over the field K of characteristic 0 consider linear space $K(n, \sigma)$ of all matrices $A = (a_{i,j})$ such that Gorenstein relations take place for A.

When a problem of describing of such matrices will be solved, it will become possible to talk about restrictions, generated by inequalities from the definition of reduced exponent matrix.

If $A \in K(n, \sigma)$, B is equivalent to it, then $B \in K(n, \sigma)$, and every equivalence class contains the unique matrix with zeros at the first line, which is canonical representor of the equivalence class. Call matrix is called **Kirichenko matrix**. More over, it's easy to see, that for Kirichenko matrix from the condition $a_{i,k} + a_{k,\sigma(i)} = a_{i,\sigma(i)}$ for $i = 1$ one may obtain $a_{k,\sigma(1)} = 0$ for all k. It is easy to prove that the set of Kirichenko matrices is linear space and we will name it **Kirichenko space**.

We need the next notations. Let $A = (a_{i,j})$ be Kirichenko matrix of order n with Kirichenko permutation σ, and σ is the product of q cycles of length $l_1, l_2, \ldots, l_q$, respectively $2 \leq l_i \leq l_{i+1}$, $1 \leq i \leq q-1$ and $l_1 + \ldots + l_q = n$.

The main result of this work is calculating the dimension of Kirichenko space. To do this, some elements of Kirichenko matrix were considered as parameters, and represent all another elements of matrix in terms of these parameters. The relations which represent the elements of matrix through parameters we name **element relations**, and relations between parameters we name **count relations**. We were succeed to find the set of parameters, element and count relations in the form such that it is possible to prove the equivalence of the set of Gorenstein relations to the set of element relations and count relations. After this, we were succeed to find the formula for the defect of the system of count relations i.e. the dimension of Kirichenko space may be represented in terms of cyclic index of Kirichenko permutation σ as

$$2 - 2q + \sum_{r=1}^{q} \left[\frac{l_r}{2}\right] + \frac{1}{2}\sum_{r\neq s}(l_s, l_r),$$

where (a, b) denotes the greatest common divisor of numbers a and b, and $[x]$ denotes integer part of a number x which is the greatest integer, not larger then x.

Definition 1. *Denote $x_k = a_{k,1}$ for every k, $2 \leq k \leq n$. For arbitrary r, $0 < r < q$, and for every k, $n_r + 2 \leq k \leq n$, denote as well $z_{k,r} =$ $= a_{n_r+1,k}$. Variables x_k and $z_{k,r}$ we name* ***parameters****.*

Definition 2. ***Element relation*** *is formula which is corollary of Gorenstein relations and expresses some element of Gorenstein matrix as linear function of parameters.* ***Full set*** *of element relations is set of relations which contains the formula for every element of matrix A.*

Example 1. Relations

$$a_{k,\sigma(k)} = x_k, \quad 2 \leq k \leq n \tag{1}$$

are **element**.

Really, if we substitutes value $i = 1$ to the equality $a_{k,i} + a_{i,\sigma(k)} = a_{k,\sigma(k)}$ and take into account the fact that the first row of A is zero, we obtain the necessary relation.

Example 2. Relations

$$\begin{array}{ll} a_{k,2} = 0, & 1 \leq k \leq n \\ a_{k,k} = 0, & 1 \leq k \leq n \\ a_{1,k} = 0, & 1 \leq k \leq n \end{array} \tag{2}$$

and definition of parameters are element relations.

Element relations from two previous examples we name **trivial**.

1. Element relations are corollaries of Gorenstein relations

Proposition 1. *Equalities (1)-(11) are full set of element relations, where*

$$a_{2,k} = \begin{cases} x_{n_{s+1}}, & \text{if } k = n_s + 1 \text{ for some } s \\ x_{k-1}, & \text{if } k \neq n_s + 1 \text{ for any } s \end{cases} \tag{3}$$

For every $m,\ 3 \leq m \leq n_1$, every $s,\ 1 \leq s \leq q-1$ and every $k,\ n_s + 1, \leq k \leq n_{s+1}$ or $m+1 \leq k \leq n_1$ write out

$$a_{k,m} = \begin{cases} \sum\limits_{i=2}^{m-1} x_i - \sum\limits_{i=0}^{m-3} x_{n_{s+1}-i}, \ \text{if } k = n_s + 1; \\ \sum\limits_{i=2}^{m-1} x_i - \sum\limits_{i=1}^{l-1} x_{k-i} - \sum\limits_{i=0}^{m-l-2} x_{n_{s+1}-i}, \ \text{if } k = n_s + l,\ 2 \leq l \leq m-2; \\ \sum\limits_{i=2}^{m-1} x_i - \sum\limits_{i=1}^{m-2} x_{k-i}, \ \text{if } n_s + m - 1 \leq k \leq n_{s+1}, \text{or } m < k \leq n_1 \end{cases} \tag{4}$$

and

$$a_{m,k} = \begin{cases} \sum\limits_{i=0}^{m-2} x_{n_{s+1}-i} - \sum\limits_{i=2}^{m-1} x_i, \ \text{if } k = n_s + 1, \\ \sum\limits_{i=1}^{l-1} x_{k-i} + \sum\limits_{i=0}^{m-l-1} x_{n_{s+1}-i} - \sum\limits_{i=2}^{m-1} x_i, \ \text{if } k = n_s + l,\ 2 \leq l < m \\ \sum\limits_{i=1}^{m-1} x_{k-i} - \sum\limits_{i=2}^{m-1} x_i, \ \text{if } n_s + m \leq k \leq n_{s+1}, \text{or } m < k \leq n_1 \end{cases} \tag{5}$$

Writing out element relations for other elements of matrix A, we will write indices of its elements as $a_{k,m}$ and $a_{m,k}$, where $k > m$ for $n_r + 1 \leq m \leq n_{r+1}$ and some $r,\ 1 \leq r \leq q-1$. in this case for every $s,\ r \leq s \leq q-1$, and $k,\ n_s + 1 \leq k \leq n_{s+1}$ relations (6)-(11) look like:

$$a_{k,n_r+1} = \begin{cases} x_k - z_{n_s+1,r}, & \text{if } k = n_{s+1},\ s > r \\ x_k - z_{k+1,r}, & \text{if } n_s + 1 \leq k < n_{s+1},\ s > r \end{cases} \tag{6}$$

$$a_{k,n_r+1} = \begin{cases} x_k, & \text{if } k = n_{r+1}. \\ x_k - z_{k+1,r}, & \text{if } n_r + 1 < k < n_{r+1} \end{cases} \tag{7}$$

$$a_{k,n_r+2} = x_{n_r+1} - z_{k,r},\ \text{if } k > n_r + 2 \tag{8}$$

$$a_{n_r+2,k} = \begin{cases} x_{n_{s+1}} - x_{n_r+1} + z_{n_{s+1},r}, \ \text{if } k = n_s + 1,\ s > r \\ x_{k-1} - x_{n_r+1} + z_{k-1,r}, \ \text{if } n_s + 2 \leq k \leq n_{s+1},\ k > n_r + 2 \end{cases} \tag{9}$$

$$a_{k,n_r+m} = \begin{cases} -z_{n_{s+1}-m+3,r} + \sum\limits_{i=1}^{m-1} x_{n_r+i} - \sum\limits_{i=0}^{m-3} x_{n_{s+1}-i}, \\ \qquad\qquad \text{if } k = n_s+1 > n_r+m; \\ -z_{n_{s+1}-m+l+2,r} + \sum\limits_{i=1}^{m-1} x_{n_r+i} - \sum\limits_{i=0}^{m-l-2} x_{n_{s+1}-i} - \sum\limits_{i=1}^{l-1} x_{k-i}, \\ \qquad\qquad \text{if } k = n_s+l > n_r+m \quad 2 \le l \le m-2; \\ -z_{k-m+2,r} + \sum\limits_{i=1}^{m-1} x_{n_r+i} - \sum\limits_{i=1}^{m-2} x_{k-i}, \\ \qquad\qquad \text{if } n_s+m-1 \le k \le n_{s+1},\ k > n_r+m; \end{cases} \tag{10}$$

$$a_{n_r+m,k} = \begin{cases} z_{n_{s+1}-m+2,r} - \sum\limits_{i=1}^{m-1} x_{n_r+i} + \sum\limits_{i=0}^{m-2} x_{n_{s+1}-i}, \\ \qquad\qquad \text{if } k = n_s+1 > n_r+m; \\ z_{n_{s+1}-m+l+1,r} - \sum\limits_{i=1}^{m-1} x_{n_r+i} + \sum\limits_{i=1}^{l-1} x_{k-i} + \sum\limits_{i=0}^{m-l-1} x_{n_{s+1}-i}, \\ \qquad\qquad \text{if } k = n_s+l > n_r+m, \quad 2 \le l \le m-1; \\ z_{k-m+1,r} - \sum\limits_{i=1}^{m-1} x_{n_r+i} + \sum\limits_{i=1}^{m-1} x_{k-i}, \\ \qquad\qquad \text{if } n_s+m \le k \le n_{s+1},\ k > n_r+m. \end{cases} \tag{11}$$

For proving this proposition we need the following lemmas.

Lemma 1. *If* $3 \le m \le n_1$, $\max(n_s+1, m+1) \le k \le n_{s+1}$, *then relations (4) and (5) are corollaries of Gorenstein ones.*

Proof. Consider equality $a_{2,k} + a_{k,3} = x_2$, whence, using (3) obtain, that $a_{k,3} = x_2 - x_{n_{s+1}}$, if $k = n_s+1$, and $a_{k,3} = x_2 - x_{k-1}$ if $n_s+2 \le k \le n_{s+1}$, which is equality (4) for $m = 3$.

Consider $a_{k,3} + a_{3,k+1} = x_k$, $k \ne n_{s+1}$. For use (4), consider situations $k = n_s+1$ and $k \ne n_s+1$. For $k = n_s+1$, obtain $(x_2 - x_{n_{s+1}}) + a_{3,n_s+2} = x_{n_s+1}$, whence $a_{3,p} = x_{p-1} + x_{n_{s+1}} - x_2$, for $p = n_s+2$. Consider situation, when $n_s+2 \le k < n_{s+1}$. In this case $(x_2 - x_{k-1}) + a_{3,k+1} = x_k$, whence $a_{3,k+1} = x_k + x_{k-1} - x_2$, that is $a_{3,p} = x_{p-1} + x_{p-2} - x_2$ for $n_s+3 \le p \le n_{s+1}$. Now, substituting $k = n_{s+1}$ to an equality $a_{k,3} + a_{3,\sigma(k)} = x_k$, we obtain equality (5) for $m = 3$.

Let us prove the validity of relations (4) and (5) for $3 \le m \le n_1$ by induction. The induction base is just proved. Suppose these relations to be valid for some m, $3 \le m < n_1$,. Let us proof their validity for some $m+1$ too.

Consider equality $a_{m,k}+a_{k,m+1}=x_m,\ m<n_1$, whence $a_{k,m+1}=x_m-a_{m,k}$.

For $k=n_s+1$ we obtain $a_{k,m+1}=x_m-\left(\sum\limits_{i=0}^{m-2}x_{n_{s+1}-i}-\sum\limits_{i=2}^{m-1}x_i\right)=\sum\limits_{i=2}^{(m+1)-1}x_i-\sum\limits_{i=0}^{(m+1)-3}x_{n_{s+1}-i}$, which is necessary.

Fix $l\in[2,m-1]$. Then for $k=n_s+l$ we obtain $a_{k,m+1}=x_m-a_{m,k}=$

$$x_m-\left(\sum_{i=1}^{l-1}x_{k-i}+\sum_{i=0}^{m-l-1}x_{n_{s+1}-i}-\sum_{i=2}^{m-1}x_i\right)=\sum_{i=2}^{(m+1)-1}x_i-\sum_{i=1}^{l-1}x_{k-i}-$$

$$-\sum_{i=0}^{(m+1)-l-2}x_{n_{s+1}-i} \text{ for } k=n_s+l,\ 2\le l\le(m+1)-2, \text{ which is necessary.}$$

Consider equality $a_{m,k}+a_{k,\sigma(m)}=x_m$ for $k,\ n_s+m\le k\le n_{s+1}$ and obtain $a_{k,m+1}=x_m-a_{m,k}=x_m-\left(\sum\limits_{i=1}^{m-1}x_{k-i}-\sum\limits_{i=2}^{m-1}x_i\right)=\sum\limits_{i=2}^{(m+1)-1}x_i-\sum\limits_{i=1}^{(m+1)-2}x_{k-i},\ n_s+(m+1)-1\le k\le n_{s+1}$.

Thus we have proved that $a_{k,m+1}=$

$$=\begin{cases}\sum\limits_{i=2}^{(m+1)-1}x_i-\sum\limits_{i=0}^{(m+1)-3}x_{n_{s+1}-i},\ if k=n_s+1,\\ \sum\limits_{i=2}^{(m+1)-1}x_i-\sum\limits_{i=1}^{l-1}x_{k-i}-\sum\limits_{i=0}^{(m+1)-2-l}x_{n_{s+1}-i},\ if k=n_s+l,\quad 2\le l<m\\ \sum\limits_{i=2}^{(m+1)-1}x_i-\sum\limits_{i=1}^{(m+1)-2}x_{k-i},\ if k>n_s+m-1\end{cases}.$$

Consider equality $a_{k,m+1}+a_{m+1,k+1}=x_k,\ k\neq n_s$.

Let $k=n_s+1$. Then $a_{m+1,n_s+2}=x_{n_s+1}-a_{n_s+1,m+1}=x_{n_s+1}-\left(\sum\limits_{i=2}^{m}x_i-\sum\limits_{i=0}^{m-2}x_{n_{s+1}-i}\right)=x_{p-1}-\sum\limits_{i=2}^{(m+1)-1}x_i+\sum\limits_{i=0}^{(m+1)-3}x_{n_{s+1}-i}=a_{m+1,p}$, for $p=n_s+l,\ l=2$.

Let $k=n_s+l,\ 2\le l\le m-1$. Then $a_{m+1,n_s+(l+1)}=x_{n_s+l}-$ $-a_{n_s+l,m+1}=x_{n_s+l}-\left(\sum\limits_{i=2}^{m}x_i-\sum\limits_{i=1}^{l-1}x_{k-i}-\sum\limits_{i=0}^{m-l-1}x_{n_{s+1}-i}\right)=-\sum\limits_{i=2}^{(m+1)-1}x_i+\sum\limits_{i=1}^{(l+1)-1}x_{(k+1)-i}+\sum\limits_{i=0}^{(m+1)-(l+1)-1}x_{n_{s+1}-i}$, which is the same as $a_{m+1,n_s+l}=$

$$=-\sum_{i=2}^{(m+1)-1}x_i+\sum_{i=1}^{l-1}x_{k-i}+\sum_{i=0}^{(m+1)-l-1}x_{n_{s+1}-i}, \text{ for } 3\le l\le(m+1)-1.$$

Let $n_s + m \le k \le n_{s+1}$. Then $a_{m+1,k+1} = x_k - a_{k,m+1} = x_k - \left(\sum_{i=2}^{m} x_i - \sum_{i=1}^{m-1} x_{k-i}\right) = -\sum_{i=2}^{m} x_i + \sum_{i=1}^{m} x_{k+1-i}$, for $n_s + m \le k \le n_{s+1}$, which is the same as $a_{m+1,k} = -\sum_{i=2}^{(m+1)-1} x_i + \sum_{i=1}^{(m+1)-1} x_{k-i}$ for $n_s + (m+1) \le k \le n_{s+1}$.

Consider equality $a_{k,m+1} + a_{m+1,\sigma(k)} = x_k$ for $k = n_{s+1}$, and obtain $a_{m+1,n_s+1} = x_{n_{s+1}} - a_{n_{s+1},m+1} = x_{n_{s+1}} - \left(\sum_{i=2}^{m} x_i - \sum_{i=1}^{m-1} x_{n_{s+1}-i}\right) = \sum_{i=0}^{(m+1)-2} x_{n_{s+1}-i} - \sum_{i=2}^{(m+1)-1} x_i = a_{m+1,p}$, where $p = n_s + 1$.

thus we have proved that $a_{m+1,k} =$

$$= \begin{cases} \sum\limits_{i=0}^{(m+1)-2} x_{n_{s+1}-i} - \sum\limits_{i=2}^{(m+1)-1} x_i, \text{ if } k = n_s + 1, \\ \sum\limits_{i=1}^{l-1} x_{k-i} + \sum\limits_{i=0}^{(m+1)-l-1} x_{n_{s+1}-i} - \sum\limits_{i=2}^{(m+1)-1} x_i, \text{ if } k = n_s + l, \quad 2 \le l < m+1 \\ \sum\limits_{i=1}^{(m+1)-1} x_{k-i} - \sum\limits_{i=2}^{(m+1)-1} x_i, \text{ if } n_s + (m+1) \le k \le n_{s+1} \end{cases},$$

and it finishes the proof of relations (4) and (5). □

Fix arbitrary r, $1 \le r \le q-1$, arbitrary s, $r \le s \le q-1$, and consider k, $n_s + 1 \le k \le n_{s+1}$. For convenience instead of $z_{k,r}$ we will write z_k.

Lemma 2. *For r, s and k from the intervals noted above, equalities (6)-(9) take place.*

Proof. Consider equality $a_{k,n_r+1} + a_{n_r+1,\sigma(k)} = x_k$. For $k \ne n_{s+1}$ obtain $a_{k,n_r+1} + z_{k+1} = x_k$, whence $a_{k,n_r+1} = x_k - z_{k+1}$.

Considering $k = n_{r+1}$ obtain $a_{n_{r+1},n_r+1} + a_{n_r+1,n_r+1} = x_{n_{r+1}}$, whence $a_{k,n_r+1} = x_k$, that is

$$a_{k,n_r+1} = \begin{cases} x_k, & \text{if } k = n_{r+1}. \\ x_k - z_{k+1}, & \text{if } n_r + 1 < k < n_{r+1} \end{cases},$$

which coincides with (7).

For $k = n_{s+1}, s > r$ obtain $a_{n_{s+1},n_r+1} + a_{n_r+1,n_s+1} = x_{n_{s+1}}$, whence $a_{k,n_r+1} = x_k - z_{n_s+1}$, that is

$$a_{k,n_r+1} = \begin{cases} x_k - z_{n_s+1}, & \text{if } k = n_{s+1}. \\ x_k - z_{k+1}, & \text{if } n_s + 1 \le k < n_{s+1} \end{cases},$$

which coincides with (6).

Consider equality $a_{n_r+1,k} + a_{k,n_r+2} = x_{n_r+1}$, whence, according to denotations, $a_{k,n_r+2} = x_{n_r+1} - a_{n_r+1,k} = x_{n_r+1} - z_k$, which is (8).

If $l_r > 2$, then consider equality $a_{k,n_r+2}+a_{n_r+2,k+1} = x_k$ for $k \neq n_{s+1}$. Whence $a_{n_r+2,k+1} = x_k - a_{k,n_r+2} = x_k - x_{n_r+1} + z_k$, that is

$$a_{n_r+2,p} = x_{p-1} - x_{n_r+1} + z_{p-1} \text{ for } p \neq n_s + 1.$$

For $k = n_{s+1}$ obtain $a_{n_{s+1},n_r+2} + a_{n_r+2,n_s+1} = x_{n_{s+1}}$, and so

$$a_{n_r+2,k} = x_{n_{s+1}} - a_{n_{s+1},n_r+2} = x_{n_{s+1}} - (x_{n_r+1} - z_{n_{s+1}})$$

for $k = n_s + 1$, whence

$$a_{n_r+2,p} = \begin{cases} x_{n_{s+1}} - x_{n_1+1} + z_{n_{s+1}}, & \text{if } k = n_s + 1, \qquad \text{which is (9).} \\ x_{k-1} - x_{n_1+1} + z_{k-1}, & \text{if } n_s + 2 < k \leq n_{s+1} \end{cases} \quad \square$$

Leaving in force denotations for boundaries for r, s and k, consider arbitrary m, $3 \leq m \leq l_r$.

Lemma 3. *For every r, s, k and m from noted intervals, relations (10) and (11) are corollaries of Gorenstein relations.*

Proof. Proof by induction for m.

Consider the equality $a_{n_r+2,k} + a_{k,n_r+3} = x_{n_r+2}$, whence

$$a_{k,n_r+3} = x_{n_r+2} - a_{n_r+2,k}, \text{ i.e.}$$

$$a_{k,n_r+3} = \begin{cases} x_{n_r+1} + x_{n_r+2} - x_{n_{s+1}} - z_{n_{s+1}}, & \text{if } k = n_s + 1. \\ x_{n_r+1} + x_{n_r+2} - x_{k-1} - z_{k-1}, & \text{if } k \neq n_s + 1 \end{cases}, \text{ and so}$$

obtain induction base for (10).

Consider equality $a_{k,n_r+3} + a_{n_r+3,k+1} = x_k$ for $k \neq n_{s+1}$, whence $a_{n_r+3,p} = x_{p-1} - a_{p-1,n_r+3}$, for $n_s + 2 \leq p \leq n_{s+1}$.

For $k \neq n_s + 2$ then the last equality is equivalent to $a_{n_r+3,k} = x_{k-1} - (x_{n_r+1} + x_{n_r+2} - x_{k-2} - z_{k-2}) = x_{k-1} + x_{k-2} - x_{n_r+1} - x_{n_r+2} + z_{k-2}$.

For $k = n_s+2$ obtain $a_{n_r+3,k} = x_{k-1} - (x_{n_r+1} + x_{n_r+2} - x_{n_{s+1}} - z_{n_{s+1}}) = x_{k-1} - x_{n_r+1} - x_{n_r+2} + x_{n_{s+1}} + z_{n_{s+1}}$

Substitute $k = n_{s+1}$ to equality $a_{k,n_r+3} + a_{n_r+3,\sigma(k)} = x_k$, and obtain $a_{n_r+3,n_s+1} = x_{n_{s+1}} - a_{n_{s+1},n_r+3} = x_{n_{s+1}} - (x_{n_r+1} + x_{n_r+2} - x_{n_{s+1}-1} - z_{n_{s+1}-1}) = x_{n_{s+1}} + x_{n_{s+1}-1} - x_{n_r+1} - x_{n_r+2} + z_{n_{s+1}-1}$. Whence obtain

$$a_{n_r+3,k} = \begin{cases} x_{n_{s+1}} + x_{n_{s+1}-1} - x_{n_r+1} - x_{n_r+2} + z_{n_{s+1}-1}; & \text{if } k = n_s + 1, \\ x_{k-1} - x_{n_r+1} - x_{n_r+2} + x_{n_{s+1}} + z_{n_{s+1}}; & \text{if } k = n_s + 2, \\ x_{k-1} + x_{k-2} - x_{n_r+1} - x_{n_r+2} + z_{k-2}, & \text{if } n_s + 3 \leq k \leq n_{s+1}, \end{cases}$$

which gives induction base for (11).

Assume that relations (10) and (11) are valid for some $3 \le m \le l_r - 1$, and proof, that in this case they will be valid for $m+1$ also.

Consider equality $a_{n_r+m,k} + a_{k,n_r+m+1} = x_{n_r+m}$, whence $a_{k,n_r+m+1} = x_{n_r+m} - a_{n_r+m,k}$, that is

$$a_{k,n_r+m+1} = x_{n_r+m} - \left(z_{n_{s+1}-m+2} - \sum_{i=1}^{m-1} x_{n_r+i} + \sum_{i=0}^{m-2} x_{n_{s+1}-i} \right)$$

if $k = n_s + 1$; For the case $k = n_s + l, \ \ 2 \le l \le m-1$ the formula for a_{k,n_r+m+1} will be $a_{k,n_r+m+1} =$

$$= x_{n_r+m} - \left(z_{n_{s+1}-m+l+1} - \sum_{i=1}^{m-1} x_{n_r+i} + \sum_{i=1}^{l-1} x_{k-i} + \sum_{i=0}^{m-l-1} x_{n_{s+1}-i} \right),$$

and if $n_s + m \le k \le n_{s+1}$, then

$$a_{k,n_r+m+1} = x_{n_r+m} - \left(z_{k-m+1} - \sum_{i=1}^{m-1} x_{n_r+i} + \sum_{i=1}^{m-1} x_{k-i} \right),$$

which is necessary.

Consider equality $a_{k,n_r+m+1} + a_{n_r+m+1,\sigma(k)} = x_k$, whence for $k \ne n_{s+1}$ obtain $a_{n_r+m+1,k+1} = x_k - a_{k,n_r+m+1}$.

Substitute $k = n_s + 1$ to the last equality and obtain

$$a_{n_r+m+1,n_s+2} = x_{n_s+1} - a_{n_s+1,n_r+m+1} = x_{n_s+1} - \left(\sum_{i=1}^{m} x_{n_r+i} - \right.$$
$$\left. - \sum_{i=0}^{m-2} x_{n_{s+1}-i} - z_{n_{s+1}-m+2} \right) = z_{n_{s+1}-m+2} + x_{n_s+1} - \sum_{i=1}^{m} x_{n_r+i} +$$
$$+ \sum_{i=0}^{m-2} x_{n_{s+1}-i}, \text{ that is } x_{n_r+m+1,p} = z_{n_{s+1}-(m+1)+3} + x_{p-1} -$$
$$- \sum_{i=1}^{(m+1)-1} x_{n_r+i} + \sum_{i=0}^{(m+1)-3} x_{n_{s+1}-i} \text{ for } p = n_s + 2, \text{ which is necessary.}$$

Substitute $k = n_s + l, \ 2 \le l \le m-1$ to equality $a_{n_r+m+1,k+1} = x_k - a_{k,n_r+m+1}$, and obtain $a_{n_r+m+1,k+1} = x_k - \left(-z_{n_{s+1}-m+l+1} + \sum\limits_{i=1}^{m} x_{n_r+i} - \right.$ $\left. - \sum\limits_{i=1}^{l-1} x_{k-i} - \sum\limits_{i=0}^{m-1-l} x_{n_{s+1}-i} \right) = z_{n_{s+1}-m+l+1} - \sum\limits_{i=1}^{m} x_{n_r+i} + \sum\limits_{i=1}^{l} x_{k+1-i} +$

$\sum\limits_{i=0}^{m-l-1} x_{n_{s+1}-i}$, that is $a_{n_r+m+1,p} =$

$$= z_{n_{s+1}-(m+1)+l+2} - \sum_{i=1}^{(m+1)-1} x_{n_r+i} + \sum_{i=1}^{l-1} x_{p-i} + \sum_{i=0}^{m-l-1} x_{n_{s+1}-i}$$

for $p = n_s + l$, $3 \leq l \leq (m+1) - 1$, which is necessary. Substitute $n_s + m \leq k \leq n_{s+1} - 1$ to an equality $a_{k,n_r+m+1} + a_{n_r+m+1,\sigma(k)} = x_k$, whence, according to assumption of induction, obtain $a_{n_r+m+1,k+1} =$

$$= x_k - a_{k,n_r+m+1} = x_k - \left(\sum_{i=1}^{m} x_{n_r+i} - z_{k-m+1} - \sum_{i=1}^{m-1} x_{k-i}\right) =$$

$$= z_{k-m+1} - \sum_{i=1}^{m} x_{n_r+i} + \sum_{i=0}^{m-1} x_{k-i},$$

that is

$$a_{n_r+m+1,p} = z_{p-(m+1)+2} - \sum_{i=1}^{(m+1)-1} x_{n_r+i} + \sum_{i=1}^{(m+1)-1} x_{p-i}$$

for $n_s + (m+1) \leq p \leq n_{s+1}$, which is necessary.

Substitute $k = n_{s+1}$ to an equality $a_{n_r+m+1,\sigma(k)} = x_k - a_{k,n_r+m+1}$,

and obtain $a_{n_r+m+1,n_s+1} = x_{n_{s+1}} - a_{n_{s+1},n_r+m+1} = x_{n_{s+1}}-$

$$- \left(\sum_{i=1}^{m} x_{n_r+i} - z_{n_{s+1}-m+1} - \sum_{i=1}^{m-1} x_{n_{s+1}-i}\right) = z_{n_{s+1}-(m+1)+2} -$$

$$- \sum_{i=1}^{(m+1)-1} x_{n_r+i} + \sum_{i=0}^{(m+1)-2} x_{n_{s+1}-i},$$ which is necessary to prove. □

2. Count relations are corollaries of Gorenstein relations

Definition 3. *Untrivial relation between parameters is called **count relation**, if it is provided by Gorenstein relations.*

Proposition 2. *Relations (12)-(18) are count relations.*

If $n_1 = 2$, then for every s, $1 \leq s \leq q-1$, equalities

$$\begin{cases} x_{k-1} + x_k = x_2, & \text{if } n_s + 2 \leq k \leq n_{s+1}, \\ x_{n_{s+1}} + x_k = x_2, & \text{if } k = n_s + 1 \end{cases} \tag{12}$$

take place.

For every $m,\ 1 \le m \le n_1 - 1,$ *equalities*

$$x_{m+1} = x_{n_1-m+1} \tag{13}$$

take place.

If $n_1 > 2,$ *then for arbitrary* $s,\ 1 \le s \le q-1,$ *equalities*

$$x_{n_s+1} + \sum_{i=0}^{n_1-2} x_{n_{s+1}-i} = \sum_{i=2}^{n_1} x_i, \tag{14}$$

$$\begin{cases} x_{n_s+p} = x_{n_{s+1}-n_1+p}, & \text{if } 1 \le p \le n_1 \\ x_{n_s+p} = x_{n_s+p-n_1}, & \text{if } n_1 < p \le l_s \end{cases} \tag{15}$$

take place.

For every $r,\ 1 \le r \le q-1,$ *and* $m,\ 2 \le m \le l_r - 1,$ *equalities (16)-(18)*

$$z_{n_r+2,r} = x_{n_r+1}, \tag{16}$$

$$\sum_{i=1}^{m} x_{n_r+i} - \sum_{i=0}^{m-2} x_{n_{r+1}-i} = z_{n_{r+1}-m+2,r} + z_{n_r+m+1,r}, \tag{17}$$

$$\begin{cases} z_{n_s+1} - z_{n_{s+1}-(l_r-1)} = \sum\limits_{i=n_{s+1}-(l_r-1)}^{n_{s+1}} x_i - \sum\limits_{i=n_r+1}^{n_{r+1}} x_i =: f_0 \\ z_{n_s+p+1} - z_{n_{s+1}-l_r+p+1} = \\ = \sum\limits_{i=0}^{p-1} x_{(n_s+p)-i} + \sum\limits_{i=0}^{l_r-p-1} x_{n_{s+1}-i} - \sum\limits_{i=n_r+1}^{n_{r+1}} x_i =: f_p \\ \text{for } 1 \le p \le l_r - 1 \\ z_{n_s+p+1} - z_{n_s+p-(l_r-1)} = \sum\limits_{i=0}^{l_r-1} x_{n_s+p-i} - \sum\limits_{i=n_r+1}^{n_{r+1}} x_i =: f_p \\ \text{for } l_r \le p \le l_s - 1 \end{cases} \tag{18}$$

take place.

The proof of this proposition is broken to some natural parts which we will formulate as lemmas. For convenience of further calculations we will write z_k instead of $z_{k,r}$, if it well be obvious $r,\ 0 < r < q$, which is under consideration.

Lemma 4. *If* $n_1 = 2$, *then relation (12) takes place for every* s, $1 \le s \le q-1$.

Proof. Assume that $n_1 = 2$ and consider the equality $a_{2,k} + a_{k,1} = a_{2,1}$.

For $n_s + 2 \le k \le n_{s+1}$, according to (3) obtain $x_{k-1} + x_k = x_2$, which coincides with the first line of (12).

According to (3) for $k = n_s + 1$ we obtain $x_{n_{s+1}} + x_{n_s} = x_2$, which coincides with second line of (12). □

Lemma 5. *The equality (13) takes place for every* m, $1 \le m \le n_1 - 1$.

Proof. We will prove this lemma by induction for m.

Consider equality $a_{k,2} + a_{2,\sigma(k)} = x_k$ for $k = n_1$, and obtain $a_{n_1,2} + a_{2,1} = x_{n_1}$. According to definition and according to (2) we have $a_{n_1,2} = 0$ and $a_{2,1} = x_2$, whence $x_2 = x_{n_1}$, that is (13) for $m = 1$ and gives the induction base.

Let for some m, $1 < m \le n_1 - 1$, the equality (13) takes place for every $i < m$. Show, that in this case it well take place for $i = m$ also.

Substitute $k = n_1$ into equality $a_{k,m+1} + a_{m+1,\sigma(k)} = a_{k,\sigma(k)}$, and obtain $a_{n_1,m+1} + x_{m+1} = x_{n_1}$. To express $a_{n_1,m+1}$ we use (4) substituting respectively n_1 and $m+1$ instead of k and m, whence using the third line of (4) we obtain $\left(\sum\limits_{i=2}^{m} x_i - \sum\limits_{i=1}^{m-1} x_{n_1-i}\right) + x_{m+1} = x_{n_1}$, that is $\sum\limits_{i=2}^{m+1} x_i = \sum\limits_{i=0}^{m-1} x_{n_1-i}$, whence, using induction base obtain that $x_{m+1} = x_{n_1-m+1}$, which is necessary. □

Lemma 6. *If* $n_1 > 2$, *then for arbitrary* s, $1 \le s \le q-1$ *the equality (14) takes place.*

Proof. For arbitrary s, $1 \le s \le q-1$ consider equality $a_{n_1,n_s+1} + a_{n_s+1,1} = a_{n_1,1}$. Then according to (5) for $m := n_1$ and $k = n_s + 1$ (for calculating a_{n_1,n_s+1}, according to the first line of this relation), and notation, obtain $\sum\limits_{i=0}^{n_1-2} x_{n_{s+1}-i} - \sum\limits_{i=2}^{n_1-1} x_i + x_{n_s+1} = x_{n_1}$, that is

$$\sum_{i=0}^{n_1-2} x_{n_{s+1}-i} + x_{n_s+1} = \sum_{i=2}^{n_1} x_i,$$

which is (14). □

Lemma 7. *For every* $n_1 < p \le l_s$ *the equalities, which are second line of (15), are corollaries of element relations and Gorenstein ones.*

Proof. Fix arbitrary value of p, $n_1 < p \le l_s$, and consider equality $a_{n_1,k} + a_{k,1} = a_{n_1,1}$ for $k = n_s + p - 1$ and $k = n_s + p$, whence obtain $\sum\limits_{i=0}^{n_1-1} x_{n_s+p-1-i} = \sum\limits_{i=2}^{n_1} x_i$, that is $\sum\limits_{i=1}^{n_1} x_{n_s+p-i} = \sum\limits_{i=2}^{n_1} x_i$ and $\sum\limits_{i=0}^{n_1-1} x_{n_s+p-i} = \sum\limits_{i=2}^{n_1} x_i$, whence, subtracting the last equality from previous one, obtain

$$x_{n_s+p} = x_{n_s+p-n_1},$$

that proves lemma. □

Lemma 8. *For every $2 \le s \le q-1$ and p, $1 \le p \le n_1$ the equalities, defined by the first line of (15) are corollaries of Gorenstein relations and element relations.*

Proof. We prove this lemma by induction for p.

According to (2,5), for $k = n_s + p$, $1 < p < n_1$ equality $a_{n_1,n_s+p} + a_{n_s+p,1} = a_{n_1,1}$ may be changed to $\sum_{i=1}^{p-1} x_{k-i} + \sum_{i=0}^{n_1-p-1} x_{n_{s+1}-i} - \sum_{i=2}^{n_1-1} x_i + x_{n_s+p} = x_{n_1}$, whence

$$\sum_{i=0}^{p-1} x_{n_s+p-i} + \sum_{i=0}^{n_1-p-1} x_{n_{s+1}-i} = \sum_{i=2}^{n_1} x_i \tag{19}$$

Consider equality $a_{n_1,n_s+1} + a_{n_s+1,1} = a_{n_1,1}$, whence, according to (5), obtain $\left(\sum_{i=0}^{n_1-2} x_{n_{s+1}-i} - \sum_{i=2}^{n_1-1} x_i\right) + x_{n_s+1} = x_{n_1}$ whence (19) is valid for every p, $1 \le p < n_1$.

Using (19) for $p = 2$ obtain $x_{n_s+1} + x_{n_s+2} + \sum_{i=0}^{n_1-3} x_{n_{s+1}-i} = \sum_{i=2}^{n_1} x_i$. Subtracting from it equality for $p = 1$, obtain

$$x_{n_s+2} = x_{n_{s+1}-(n_1-2)},$$

and so, the induction base, that is the first line of (15) for $p = 2$ is proved.

Consider arbitrary arbitrary p, from the interval $[3, n_1-1]$. Substitute $k = n_s + p - 1$ and $k = n_s + p$ into equality $a_{n_1,k} + a_{k,1} = a_{n_1,1}$, and using (5) for $m = n_1$ obtain

$$\left(\sum_{i=1}^{(p-1)-1} x_{n_s+p-1-i} + \sum_{i=0}^{n_1-(p-1)-1} x_{n_{s+1}-i} - \sum_{i=2}^{n_1-1} x_i\right) + x_{n_s+p-1} = x_{n_1},$$

that is

$$\sum_{i=1}^{p-1} x_{n_s+p-1-i} + \sum_{i=0}^{n_1-(p-1)-1} x_{n_{s+1}-i} = \sum_{i=2}^{n_1} x_i$$

and

$$\sum_{i=0}^{p-1} x_{n_s+p-i} + \sum_{i=0}^{n_1-p-1} x_{n_{s+1}-i} = \sum_{i=2}^{n_1} x_i,$$

whence, subtracting the last equality from previous one, obtain $x_{n_s+p} = x_{n_{s+1}-(n_1-p)}$, which proves the validity of lemma for $3 \le p \le n_1 - 1$.

For $k = n_s + n_1$, equality $a_{n_1,k} + a_{k,1} = a_{n_1,1}$ according to (5) one may change to the form $\sum_{i=1}^{n_1-1} x_{n_s+n_1-i} - \sum_{i=2}^{n_1-1} x_i + x_{n_s+n_1} = x_{n_1}$, that is

$\sum\limits_{i=0}^{n_1-1} x_{n_s+n_1-i} = \sum\limits_{i=2}^{n_1} x_i$. Subtracting this equality from (19) for $p = n_1 - 1$ obtain (15) for $p = n_1$.

Let's return to equality $a_{n_1,k} + a_{k,1} = x_{n_1}$. According to (5), it may be rewritten in the form $\left(\sum\limits_{i=1}^{n_1-1} x_{k-i} - \sum\limits_{i=2}^{n_1-1} x_i\right) + x_k = x_{n_1}$, that is $\sum\limits_{i=0}^{n_1-1} x_{k-i} = \sum\limits_{i=2}^{n_1} x_i$. From the second line of equality (15) (which is proved in previous lemma) obtain, that for arbitrary s, $2 \leq s \leq q-1$ equalities $x_{n_s+1} = x_{n_s+n_1+1}$, $x_{n_s+2} = x_{n_s+n_1+2}$ and so on up to $x_{n_{s+1}-n_1} = x_{n_{s+1}}$ take place, so sequence $\{x_{n_s+1}, \ldots, x_{n_{s+1}}\}$ is periodic with period n_1. It means that $\sum\limits_{i=0}^{n_1-1} x_{k-i}$ is sum of n_1 elements of this sequence, which are equal to it's last elements, whence obtain equality $\sum\limits_{i=0}^{n_1-1} x_{k-i} = \sum\limits_{i=2}^{n_1} x_i$, which according to (14) one may change to the form $\sum\limits_{i=0}^{n_1-1} x_{n_{s+1}-i} = \sum\limits_{i=0}^{n_1-2} x_{n_{s+1}-i} + x_{n_s+1}$, whence $x_{n_{s+1}-n_1+1} = x_{n_s+1}$, which is (15) for $p = 1$, and finishes the proof of equalities (15) in general. □

Substitute $k = n_r + 1$ into equality $a_{k,n_r+1} + a_{n_r+1,\sigma(k)} = x_k$, and obtain $z_{n_r+2} = x_{n_r+1}$, which is (16).

Lemma 9. *For every $2 \leq m \leq l_r - 1$ equalities (17) are corollaries of element relations and Gorenstein ones.*

Proof. Proof by induction for m.

Substitute $k = n_{r+1}$ into equality $a_{k,n_r+2} + a_{n_r+2,\sigma(k)} = x_k$ and obtain $a_{n_{r+1},n_r+2} + a_{n_r+2,n_r+1} = x_{n_{r+1}}$, whence, according to (8), and (9), obtain $(x_{n_r+1} - z_{n_{r+1}}) + (x_{n_r+2} - z_{n_r+3}) = x_{n_{r+1}}$, that is

$$z_{n_{r+1}} + z_{n_r+3} = x_{n_r+1} + x_{n_r+2} - x_{n_{r+1}},$$

which is the equality (17) for $m = 2$ and so induction base is proved.

For every m, $3 \leq m \leq l_r - 1$, substitute $k = n_{r+1}$ into equality $a_{k,n_r+m} + a_{n_r+m,\sigma(k)} = x_k$, and obtain $a_{n_{r+1},n_r+m} + a_{n_r+m,n_r+1} = x_{n_r+1}$, whence, according to (7) and (10) obtain

$$\left(\sum_{i=1}^{m} x_{n_r+i} - \sum_{i=1}^{m-1} x_{n_{r+1}-i} - z_{n_{r+1}-m+1}\right) + (x_{n_r+m+1} - z_{n_r+m+2}) = x_{n_{r+1}},$$

whence

$$\sum_{i=1}^{m} x_{n_r+i} - \sum_{i=0}^{m-2} x_{n_{r+1}-i} = z_{n_{r+1}-m+2} + z_{n_r+m+1},$$

and lemma is proved. □

Lemma 10. *For arbitrary p, $1 \le p \le l_r - 1$, equalities, which are second line of (18) are corollaries of element relations and Gorenstein ones.*

Proof. Consider equality $a_{n_{r+1},k} + a_{k,n_r+1} = a_{n_{r+1},n_r+1} = x_{n_{r+1}}$ for $k = n_s + p,\ p > 0$.

For $p = 1$ obtain $a_{n_{r+1},k}$, using (11) for $k = n_r + 1$ and $m = l_r$. More over, according to (6), $a_{k,n_r+1} = x_k - z_{k+1},\ k \ne n_s$, whence obtain $\left(-\sum_{i=1}^{l_r-1} x_{n_r+i} + \sum_{i=0}^{l_r-2} x_{n_{s+1}-i} + z_{n_{s+1}-l_r+2}\right) + (x_{n_s+1} - z_{n_s+2}) = x_{n_{r+1}}$, that is

$$\sum_{i=0}^{l_r-2} x_{n_{s+1}-i} + z_{n_{s+1}-l_r+2} + x_{n_s+1} - z_{n_s+2} = \sum_{i=n_1+1}^{n_{r+1}} x_i$$

For $2 \le p \le l_r - 1$ obtain $a_{n_{r+1},k}$ using (11), for $k = n_s + l$, substituting $l = p$; $m = l_r$, whence

$$\left(-\sum_{i=1}^{l_r-1} x_{n_r+i} + \sum_{i=1}^{p-1} x_{n_s+p-i} + \sum_{i=0}^{l_r-p-1} x_{n_{s+1}-i} + z_{n_{s+1}-l_r+p+1}\right) + (x_{n_s+p} - z_{n_s+p+1}) = x_{n_{r+1}}, \text{ that is}$$

$$\sum_{i=0}^{p-1} x_{n_s+p-i} + \sum_{i=0}^{l_r-p-1} x_{n_{s+1}-i} + z_{n_{s+1}-l_r+p+1} - z_{n_s+p+1} = \sum_{i=n_r+1}^{n_{r+1}} x_i. \quad \square$$

For $p = n_{s+1} - n_s$ the formula (6) gives $a_{n_{s+1},n_r+1} = x_{n_{s+1}} - z_{n_s+1}$ whence obtain

$$\sum_{i=0}^{l_r-1} x_{n_{s+1}-i} + z_{n_{s+1}-l_r+1} - z_{n_s+1} = \sum_{i=n_r+1}^{n_{r+1}} x_i,$$

which gives the first line of (18).

Lemma 11. *For every p, $l_r \le p \le l_s - 1$ equalities, which are second line of (18) are corollaries of element relations and Gorenstein relations.*

Proof. Consider equality $a_{n_{r+1},k} + a_{k,n_r+1} = a_{n_{r+1},n_r+1}$. Let $k = n_s + p$.

For $l_r-1<p<l_s$ from the formula (11) for k, $n_s+m\le k\le n_{s+1}$, and $m=n_{r+1}-n_r$ obtain $a_{n_{r+1},k}=-\sum\limits_{i=1}^{l_r-1}x_{n_r+i}+\sum\limits_{i=1}^{l_r-1}x_{k-i}+$ $+z_{k-l_r+1}$, and as $n_s+k<n_{s+1}$, then $a_{k,n_r+1}=x_k-z_{k+1}$. Thus obtain $\left(-\sum\limits_{i=1}^{l_r-1}x_{n_r+i}+\sum\limits_{i=1}^{l_r-1}x_{n_s+p-i}+z_{n_s+p-l_r+1}\right)+(x_{n_s+p}-z_{n_s+p+1})=x_{n_{r+1}}$, that is

$$\sum_{i=0}^{l_r-1}x_{n_s+p-i}+z_{n_s+p-l_r+1}-z_{n_s+p+1}=\sum_{i=n_r+1}^{n_{r+1}}x_i,$$

which is necessary. □

3. Gorenstein relations are corollaries of element relations and count ones

For an arbitrary i, $1\le i\le n$ the equality $a_{i,k}+a_{k,\sigma(i)}=a_{i,\sigma(i)}$ for $k=i$ is valid. Later for every i we will consider equalities $a_{i,k}+a_{k,\sigma(i)}=a_{i,\sigma(i)}$ and $a_{k,i}+a_{i,\sigma(k)}=a_{k,\sigma(k)}$ for $k>i$, which coincides with the whole set of Gorenstein relations.

Lemma 12. *Equalities $a_{i,k}+a_{k,\sigma(i)}=a_{i,\sigma(i)}$ and $a_{k,i}+a_{i,\sigma(k)}=a_{k,\sigma(k)}$ for $0<i<n_1$ and $i<k\le n_1$ are corollaries of element relations and count ones.*

Proof. For $i=1$ Consider equalities $a_{k,i}+a_{i,\sigma(k)}=a_{k,\sigma(k)}$. As the first line of matrix is zero, these equalities go to $a_{k,1}=a_{k,\sigma(k)}$ which is corollary of (1).

For $i=2$ consider equalities $a_{k,i}+a_{i,\sigma(k)}=a_{k,\sigma(k)}$, which are equivalent to equalities $a_{k,2}+a_{2,\sigma(k)}=a_{k,\sigma(k)}$, which take place in order to (1), (2) and denotations.

For $3\le m\le n_1$ consider equality $a_{k,m}+a_{m,\sigma(k)}=a_{k,\sigma(k)}$, which is $a_{k,m}+a_{m,\sigma(k)}=x_k$. If $k<n_1$, then according to (3) and (4) this equality is equivalent to $\left(\sum\limits_{i=2}^{m-1}x_i-\sum\limits_{i=1}^{m-2}x_{k-i}\right)+\left(\sum\limits_{i=1}^{m-1}x_{k+1-i}-\sum\limits_{i=2}^{m-1}x_i\right)=x_k$, which is identity.

For $i=1$ consider equalities $a_{i,k}+a_{k,\sigma(i)}=a_{i,\sigma(i)}$. they take place in order to (2) and because the first line of matrix is zero.

For $i=2$ consider equalities $a_{i,k}+a_{k,\sigma(i)}=a_{i,\sigma(i)}$.

Consider case, when $n_1>2$. In this case $\sigma(2)=3$, and for finding out the formula for $a_{k,\sigma(i)}$ one may use (4), whence $a_{k,3}=x_2-x_{k-1}$, and according to (3) we obtain $a_{2,k}=x_{k-1}$, whence these equalities are valid.

For $n_1=2$ one obtain equalities $a_{2,k}+a_{k,1}=a_{2,1}$. According to (3) for $k\ne n_s+1$ they are $x_{n_{s+1}}+x_{n_s}=x_2$, which coincides with (12)

For $3 \le m \le n_1$ the equality $a_{m,k}+a_{k,\sigma(m)} = a_{m,\sigma(m)}$ according to (1) is equivalent to $a_{m,k} + a_{k,m+1} = x_m$.

As $k \le n_1$, then according to (4) and (5) it is equivalent to identity

$$\left(\sum_{i=1}^{m-1} x_{k-i} - \sum_{i=2}^{m-1} x_i\right) + \left(\sum_{i=2}^{(m+1)-1} x_i - \sum_{i=1}^{(m+1)-2} x_{k-i}\right) = x_m.$$

For $k = n_1$ consider equality $a_{k,2} + a_{2,\sigma(k)} = a_{k,\sigma(k)}$, that is $a_{n_1,2} + a_{2,1} = x_{n_1}$, whence according to denotations $x_2 = x_{n_1}$, and it is corollary of (13).

For $k = n_1$ and m, $3 \le m \le n_1 - 1$ consider equality $a_{k,m} + a_{m,\sigma(k)} = a_{k,\sigma(k)}$ for m, $3 \le m \le n_1$, that is $a_{n_1,m} + x_m = x_{n_1}$. And according to (4) one may rewrite it in the form $\left(\sum\limits_{i=2}^{m-1} x_i - \sum\limits_{i=1}^{m-2} x_{n_1-i}\right) + x_m = x_{n_1}$, that is $\sum\limits_{i=2}^{m} x_i = \sum\limits_{i=0}^{m-2} x_{n_1-i}$, which is the same as $\sum\limits_{i=1}^{m-1} (x_{1+i} - x_{n_1-i+1}) = 0$, which is corollary of (13). □

Lemma 13. *For every i, $1 \le i \le n_1$ and s, $1 \le s \le q-1$ equalities $a_{i,k} + a_{k,\sigma(i)} = a_{i,\sigma(i)}$ and $a_{k,i} + a_{i,\sigma(k)} = a_{k,\sigma(k)}$ for k, $n_s < k \le n_{s+1}$ are corollaries of element relations and count ones.*

Proof. For $i = 1$ consider equalities $a_{i,k} + a_{k,\sigma(i)} = a_{i,\sigma(i)}$ and $a_{k,i} + a_{i,\sigma(k)} = a_{k,\sigma(k)}$. They are equivalent to equalities $a_{1,k} + a_{k,2} = a_{1,2}$ and $a_{k,1} + a_{1,\sigma(k)} = a_{k,\sigma(k)}$. The former of these equalities are valid in order to (2) and because the first line of matrix is zero. Last is valid in order to definitions, (1), and because the first line of matrix is zero.

Let $n_1 > 2$. Then $\sigma(2) = 3$. For $i = 2$ consider equalities $a_{i,k} + a_{k,\sigma(i)} = a_{i,\sigma(i)}$ and $a_{k,i} + a_{i,\sigma(k)} = a_{k,\sigma(k)}$. They are equal to equalities $a_{2,k} + a_{k,3} = a_{2,3}$ and $a_{k,2} + a_{2,\sigma(k)} = a_{k,\sigma(k)}$. According to (4), one obtain $a_{k,3} = \begin{cases} x_2 - x_{n_{s+1}}, & \text{if } k = n_s + 1 \\ x_2 - x_{k-1}, & \text{if } k \ne n_s + 1 \end{cases}$, and according to (3) $a_{2,k} = \begin{cases} x_{n_{s+1}}, & \text{if } k = n_s + 1 \\ x_{k-1}, & \text{if } k \ne n_s + 1 \end{cases}$, whence the first equality takes place. Second equality takes place in order to (1), (2) and denotations.

In the case $n_1 = 2$, equality $a_{i,k} + a_{k,\sigma(i)} = a_{i,\sigma(i)}$ for $i = 2$ is equivalent to $a_{2,k} + a_{k,1} = x_2$ one. From (3) and denotations, for $k = n_s + 1$ we obtain $x_{n_{s+1}} + x_k = x_2$, which is corollary of (12). If $n_s + 2 \le k \le n_{s+1}$, then the former equality is equivalent to $x_{k-1} + x_k = x_2$, which is corollary of (12).

For every m, $3 \le m \le n_1 - 1$ equality $a_{m,k} + a_{k,\sigma(m)} = a_{m,\sigma(m)}$ according to (1) is equivalent to $a_{m,k} + a_{k,m+1} = x_m$ one. Fix arbitrary $0 < s < q - 1$. For using (4) and (5) consider some different cases for k.

For $k = n_s + 1$ obtain the identity

$$\left(\sum_{i=0}^{m-2} x_{n_{s+1}-i} - \sum_{i=2}^{m-1} x_i\right) + \left(\sum_{i=2}^{m} x_i - \sum_{i=0}^{m-2} x_{n_{s+1}-i}\right) = x_m.$$

For $k = n_s + l,\ 2 \le l \le m-1$ equality $a_{m,k} + a_{k,m+1} = x_m$ is equivalent to $\left(\sum\limits_{i=1}^{l-1} x_{k-i} + \sum\limits_{i=0}^{m-l-1} x_{n_{s+1}-i} - \sum\limits_{i=2}^{m-1} x_i\right) + \left(\sum\limits_{i=2}^{m} x_i - \sum\limits_{i=1}^{l-1} x_{k-i} - \right.$ $\left. - \sum\limits_{i=0}^{m-1-l} x_{n_{s+1}-i}\right) = x_m$ one, which is identity.

For $k,\ n_s+m \le k \le n_{s+1}$ equality $a_{m,k}+a_{k,m+1} = x_m$ is equivalent to $\left(\sum\limits_{i=1}^{m-1} x_{k-i} - \sum\limits_{i=2}^{m-1} x_i\right) + \left(\sum\limits_{i=2}^{m} x_i - \sum\limits_{i=1}^{m-1} x_{k-i}\right) = x_m$ one, which is identity also.

For $m = n_1$ the equality $a_{m,k} + a_{k,\sigma(m)} = a_{m,\sigma(m)}$ according to (1) is equivalent to $a_{n_1,k} + x_k = x_{n_1}$. Consider cases.

Let $k = n_s + 1$. Then according to (5) this equality is equivalent to $\left(\sum\limits_{i=0}^{n_1-2} x_{n_{s+1}-i} - \sum\limits_{i=2}^{n_1-1} x_i\right) + x_{n_s+1} = x_{n_1}$ one, that is $\sum\limits_{i=0}^{n_1-2} x_{n_{s+1}-i} + x_{n_s+1} = \sum\limits_{i=2}^{n_1} x_i$, and coincides with (14).

For $k = n_s+l$ and arbitrary $l,\ 2 \le l \le n_1-1$ the equality $a_{n_1,k}+x_k = x_{n_1}$ is equivalent to $\left(\sum\limits_{i=1}^{l-1} x_{n_s+l-i} + \sum\limits_{i=0}^{n_1-l-1} x_{n_{s+1}-i} - \sum\limits_{i=2}^{n_1-1} x_i\right) + x_{n_s+l} = x_{n_1}$, that is $\sum\limits_{i=1}^{l} x_{n_s+i} + \sum\limits_{i=0}^{n_1-l-1} x_{n_{s+1}-i} = \sum\limits_{i=2}^{n_1} x_i$, whence $x_{n_s+1} + \sum\limits_{i=2}^{l} x_{n_s+i} + \sum\limits_{i=0}^{n_1-2} x_{n_{s+1}-i} - \sum\limits_{i=n_1-l}^{n_1-2} x_{n_{s+1}-i} = \sum\limits_{i=2}^{n_1} x_i$. Subtracting (14) from this equality one may obtain equivalent equality $\sum\limits_{i=2}^{l} x_{n_s+i} - \sum\limits_{i=n_1-l}^{n_1-2} x_{n_{s+1}-i} = 0$, that is $\sum\limits_{i=2}^{l} x_{n_s+i} - \sum\limits_{i=2}^{l} x_{n_{s+1}-n_1+i} = 0$, whence $\sum\limits_{i=2}^{l} \left(x_{n_s+i} - x_{n_{s+1}-n_1+i}\right) = 0$, which is corollary of (15).

For $n_s+n_1 \le k \le n_{s+1}$ the equality $a_{n_1,k}+x_k = x_{n_1}$ according to (5) is equivalent to $\left(\sum\limits_{i=1}^{n_1-1} x_{k-i} - \sum\limits_{i=2}^{n_1-1} x_i\right) + x_k = x_{n_1}$ one, that is $\sum\limits_{i=0}^{n_1-1} x_{k-i} = \sum\limits_{i=2}^{n_1} x_i$. From the second line of equality (15) it follows that equalities $x_{n_s+1} = x_{n_s+n_1+1}$, $x_{n_s+2} = x_{n_s+n_1+2}$ and so on up to $x_{n_{s+1}-n_1} = x_{n_{s+1}}$ take place, whence the sequence is periodic with period n_1. It means that $\sum\limits_{i=0}^{n_1-1} x_{k-i-1}$ is the sum of n_1 elements of this sequence, and this sum equals the sum of n_1 last elements, whence the equality $\sum\limits_{i=0}^{n_1-1} x_{k-i} = \sum\limits_{i=1}^{n_1-1} x_{1+i}$ according to (14) one may transform to $\sum\limits_{i=0}^{n_1-1} x_{n_{s+1}-i} = \sum\limits_{i=0}^{n_1-2} x_{n_{s+1}-i} + x_{n_s+1}$, whence $x_{n_{s+1}-n_1+1} = x_{n_s+1}$, which is (15) for $p=1$. Now for every m, $3 \le m \le n_1 - 1$ consider equality $a_{k,m} + a_{m,\sigma(k)} = a_{k,\sigma(k)}$.

For $k = n_s + 1$, using (4) and (5) one obtains $\left(\sum\limits_{i=2}^{m-1} x_i - \sum\limits_{i=0}^{m-3} x_{n_{s+1}-i}\right) + \left(\sum\limits_{i=1}^{2-1} x_{(k+1)-i} + \sum\limits_{i=0}^{m-2-1} x_{n_{s+1}-i} - \sum\limits_{i=2}^{m-1} x_i\right) = x_{n_s+1}$, which is identity.

For $k = n_s + l$, and every l, $2 \le l \le m-1$ according to (4) and (5), the equality $a_{k,m} + a_{m,k+1} = a_{k,k+1}$ may be transformed to $\left(\sum\limits_{i=2}^{m-1} x_i - \sum\limits_{i=1}^{l-1} x_{k-i} - \sum\limits_{i=0}^{m-l-2} x_{n_{s+1}-i}\right) + \left(\sum\limits_{i=1}^{l} x_{k+1-i} + \sum\limits_{i=0}^{m-l-2} x_{n_{s+1}-i} - \sum\limits_{i=2}^{m-1} x_i\right) = x_{n_s+l}$ one, which is identity.

For $n_s + m \le k \le n_{s+1}$ equality $a_{k,m} + a_{m,k+1} = a_{k,k+1}$ may be transformed to $\left(\sum\limits_{i=2}^{m-1} x_i - \sum\limits_{i=1}^{m-2} x_{k-i}\right) + \left(\sum\limits_{i=1}^{m-1} x_{k+1-i} - \sum\limits_{i=2}^{m-1} x_i\right) = x_k$ one, which is identity.

Consider equality $a_{k,n_1} + a_{n_1,\sigma(k)} = x_k$, $n_s + 1 \le k \le n_{s+1}$.

For $k = n_s + 1$ obtain $a_{n_s+1,n_1} + a_{n_1,n_s+2} = x_{n_s+1}$ and using (4) and (5), this is equivalent to

$\left(\sum\limits_{i=2}^{n_1-1} x_i - \sum\limits_{i=0}^{n_1-3} x_{n_{s+1}-i}\right) + \left(\sum\limits_{i=1}^{1} x_{n_s+2-i} + \sum\limits_{i=0}^{n_1-3} x_{n_{s+1}-i} - \sum\limits_{i=2}^{n_1-1} x_i\right) = x_{n_s+1}$, which is identity.

For $k = n_s + p$, $1 < p < n_1 - 1$ one may obtain $a_{n_s+p,n_1} = \sum\limits_{i=2}^{n_1-1} x_i - \sum\limits_{i=1}^{p-1} x_{n_s+p-i} - \sum\limits_{i=0}^{n_1-p-2} x_{n_{s+1}-i}$, and $a_{n_1,n_s+p+1} = \sum\limits_{i=1}^{p} x_{n_s+p-i+1} + \sum\limits_{i=0}^{n_1-p-2} x_{n_{s+1}-i} - \sum\limits_{i=2}^{n_1-1} x_i$ from the relations (4) and (5). Substituting these values to the equality $a_{n_s+p,n_1} + a_{n_1,n_s+p+1} = x_{n_s+p}$, obtain equivalent equality $\left(\sum\limits_{i=2}^{n_1-1} x_i - \sum\limits_{i=1}^{p-1} x_{n_s+p-i} - \sum\limits_{i=0}^{n_1-p-2} x_{n_{s+1}-i}\right) + \left(\sum\limits_{i=1}^{p} x_{n_s+p-i+1} + \sum\limits_{i=0}^{n_1-p-2} x_{n_{s+1}-i} - \sum\limits_{i=2}^{n_1-1} x_i\right) = x_{n_s+p}$, which is identity also.

For $k = n_s + p$, $n_1 \le p$ one may obtain $\left(\sum\limits_{i=2}^{n_1-1} x_i - \sum\limits_{i=1}^{n_1-2} x_{n_s+p-i}\right) + \left(\sum\limits_{i=1}^{n_1-1} x_{n_s+p+1-i} - \sum\limits_{i=2}^{n_1-1} x_i\right) = x_{n_s+p}$, which is identity, in order to the equality $a_{k,n_1} + a_{n_1,\sigma(k)} = x_k$ according to (4) and (5). □

Lemma 14. *For arbitrary* r, $0 < r < q-1$, *equalities* $a_{n_r+m,k} + a_{k,\sigma(n_r+m)} = a_{n_r+m,\sigma(n_r+m)}$ *and* $a_{k,n_r+m} + a_{n_r+m,\sigma(k)} = a_{k,\sigma(k)}$ *for*

$$m,\ 1 \le m \le l_r \text{ and } k,\ n_r + 1 < k \le n_{r+1}$$

are corollaries of element relations and count ones.

Proof. Consider equality $a_{n_r+1,k} + a_{k,n_r+2} = x_{n_r+1}$. Using denotations and (8) one may obtain, that it is equivalent to $z_k + (x_{n_r+1} - z_k) = x_{n_r+1}$, which is identity.

Consider equality $a_{n_r+2,k} + a_{k,\sigma(n_r+2)} = x_{n_r+2}$.

If $l_r > 2$, then using (9) and (10) one may obtain that it is equivalent to $(x_{k-1} - x_{n_r+1} + z_{k-1}) + (x_{n_r+1} + x_{n_r+2} - x_{k-1} - z_{k-1}) = x_{n_r+2}$, which is identity.

If $l_r = 2$, then equality $a_{n_r+2,k} + a_{k,\sigma(n_r+2)} = x_{n_r+2}$ is equivalent to $a_{n_r+2,k} + a_{k,n_r+1} = x_{n_r+2}$. As $n_r + 1 < k \le n_{r+1}$ then according to conditions of the lemma this equivalence is possible to consider only in the case when $k = n_{r+1} = n_r + 1$, but in this case it is trivial.

For arbitrary m, $3 \le m \le l_r - 1$ consider equality

$$a_{n_r+m,k} + a_{k,\sigma(n_r+m)} = x_{n_r+m}.$$

As $n_r + m < k < n_{r+1}$, this equality is $a_{n_r+m,k} + a_{k,n_r+m+1} = x_{n_r+m}$ and according to (10) and (11) is equivalent to $\Big(-\sum\limits_{i=1}^{m-1} x_{n_r+i} + \sum\limits_{i=1}^{m-1} x_{k-i} + z_{k-m+1} \Big) + \Big(\sum\limits_{i=1}^{m} x_{n_r+i} - \sum\limits_{i=1}^{m-1} x_{k-i} - z_{k-m+1} \Big) = x_{n_r+m}$ one, which is identity.

Consider equality $a_{k,n_r+1} + a_{n_r+1,\sigma(k)} = x_k$ for $k \ne n_{r+1}$. Using (6) and denotations one may obtain that it is equivalent to $x_k - z_{k+1} + z_{\sigma(k)} = x_k$, which is identity.

Consider equality $a_{k,n_r+2} + a_{n_r+2,\sigma(k)} = x_k$ for $k \ne n_{r+1}$. Using (8) and (9) one may obtain that it is equivalent to $(x_{n_r+1} - z_k) + (x_k - x_{n_r+1} + z_k) = x_k$ one, which is identity.

For arbitrary m, $3 \le m \le l_r$ consider equality $a_{k,n_r+m} + a_{n_r+m,\sigma(k)} = a_{k,\sigma(k)}$ for $n_r + m < k < n_{r+1}$. According to (11) and (10) it is equivalent to $\Big(\sum\limits_{i=1}^{m-1} x_{n_r+i} - \sum\limits_{i=1}^{m-2} x_{k-i} - z_{k-m+2} \Big) + \Big(-\sum\limits_{i=1}^{m-1} x_{n_r+i} + \sum\limits_{i=1}^{m-1} x_{k+1-i} + z_{k-m+2} \Big) = x_k$ one, which is identity.

For $k = n_{r+1}$ consider equalities $a_{k,n_r+m} + a_{n_r+m,\sigma(k)} = a_{k,\sigma(k)}$, that is $a_{n_{r+1},n_r+m} + a_{n_r+m,n_r+1} = x_{n_{r+1}}$ for $1 \le m < l_r$.

For $m = 1$ one obtains $a_{n_{r+1},n_r+1} + a_{n_r+1,n_r+1} = a_{n_{r+1},n_r+1}$, which is valid.

For $m=2$ the equality $a_{n_{r+1},n_r+m}+a_{n_r+m,n_r+1}=x_{n_{r+1}}$ is $a_{n_{r+1},n_r+2}+a_{n_r+2,n_r+1}=x_{n_{r+1}}$ and according to (6) and (8) it is equivalent to $(x_{n_r+1}-z_{n_{r+1}})+(x_{n_r+2}-z_{n_r+3})=x_{n_{r+1}}$ one, which follows from (17) for $m=2$.

For $3\le m\le l_r-1$ consider equality $a_{n_{r+1},n_r+m}+a_{n_r+m,n_r+1}=x_{n_{r+1}}$, and, according to (10) and (6) this equality is equivalent to $\left(\sum\limits_{i=1}^{m-1} x_{n_r+i}-\right.$ $\left.-\sum\limits_{i=1}^{m-2} x_{n_{r+1}-i}-z_{n_{r+1}-m+2}\right)+(x_{n_r+m}-z_{n_r+m+1})=x_{n_{r+1}}$, whence $\sum\limits_{i=1}^{m} x_{n_r+i}-\sum\limits_{i=0}^{m-2} x_{n_{r+1}-i}=z_{n_{r+1}-m+2}+z_{n_r+m+1}$, which follows from (17) for $3\le m\le l_r-1$. For $m=l_r$ equality $a_{n_{r+1},n_{r+1}}+a_{n_{r+1},n_r+1}=x_{n_{r+1}}$ is corollary of (7). □

Lemma 15. *For an arbitrary r, $0<r<q-1$, equalities $a_{n_r+m,k}+a_{k,\sigma(n_r+m)}=a_{n_r+m,\sigma(n_r+m)}$ and $a_{k,n_r+m}+a_{n_r+m,\sigma(k)}=a_{k,\sigma(k)}$ for every s, $r<s<q$; m, $1\le m\le l_r$ and k, $n_s+1<k\le n_{s+1}$ are corollaries of element relations and count ones.*

Consider equality $a_{n_r+1,k}+a_{k,n_r+2}=x_{n_r+1}$. According to denotations and (8) it is equivalent to $z_k+(x_{n_r+1}-z_k)=x_{n_r+1}$, which is identity.

Consider equality $a_{k,n_r+1}+a_{n_r+1,\sigma(k)}=x_k$. According to (6) and denotations it is equivalent to $\begin{cases} x_k-z_{n_s+1}, & \text{if } k=n_{s+1}\\ x_k-z_{k+1}, & \text{if } k\ne n_s\end{cases}+z_{\sigma(k)}=x_k$, which is identity.

Consider equality $a_{n_r+2,k}+a_{k,\sigma(n_r+2)}=x_{n_r+2}$.

If $l_r>2$, then $\sigma(n_r+2)=n_r+3$, and according to (9) and (10) it is equivalent to $\begin{cases} x_{n_{s+1}}-x_{n_r+1}+z_{n_{s+1}}, & \text{if } k=n_s+1\\ x_{k-1}-x_{n_r+1}+z_{k-1}, & \text{if } k\ne n_s+1\end{cases}+$ $+\begin{cases} x_{n_r+1}+x_{n_r+2}-x_{n_{s+1}}-z_{n_{s+1}}, & \text{if } k=n_s+1.\\ x_{n_r+1}+x_{n_r+2}-x_{k-1}-z_{k-1}, & \text{if } k\ne n_s+1\end{cases}=x_{n_r+2}$, which is identity.

If $l_r=2$, then equality $a_{n_r+2,k}+a_{k,\sigma(n_r+2)}=x_{n_r+2}$ is equivalent to $a_{n_r+2,k}+a_{k,n_r+1}=x_{n_r+2}$. Let $n_{s+1}-n_s>2$. Then according to (6) and (9) for $k=n_{s+1}$ it is the same as $(x_k-z_{n_s+1})+(x_{k-1}-x_{n_r+1}+z_{k-1})=x_{n_r+2}$. For $k=n_s+1$ it is the same as $(x_{n_{s+1}}-x_{n_r+1}+$ $+z_{ns+1})+(x_k-z_{k+1})=x_{n_r+2}$. And for $n_s+1<k<n_{s+1}$ the equality $a_{n_r+2,k}+a_{k,n_r+1}=x_{n_r+2}$ is equivalent to $(x_{k-1}-x_{n_r+1}+z_{k-1})+(x_k-z_{k+1})=x_{n_r+2}$. Note, that all of these equalities are corollaries of (18).

Consider equality $a_{k,n_r+2}+a_{n_r+2,\sigma(k)}=x_k$. According to (9) it is equivalent to $a_{n_r+2,\sigma(k)}=\begin{cases} x_{n_s}-x_{n_r+1}+z_{n_s}, & \text{if } k=n_s\\ x_k-x_{n_r+1}+z_k, & \text{if } k\ne n_s\end{cases}$, whence, according to (8) it is equivalent to

$(x_{n_r+1} - z_k) + \begin{cases} x_{n_s} - x_{n_r+1} + z_{n_s}, & \text{if } k = n_s \\ x_{(k+1)-1} - x_{n_r+1} + z_k, & \text{if } k \neq n_s \end{cases} = x_k$ which is identity.

For arbitrary $3 \leq m \leq l_r$ consider equality $a_{k,n_r+m} + a_{n_r+m,\sigma(k)} = a_{k,\sigma(k)}$. Consider cases for k.

For $k = n_s+1$ according to (1) it is $a_{n_s+1,n_r+m} + a_{n_r+m,n_s+2} = x_{n_s+1}$, whence, according to (10) and (11) it is equivalent to $\left(\sum\limits_{i=1}^{m-1} x_{n_r+i} - \right.$ $\left. - \sum\limits_{i=0}^{m-3} x_{n_{s+1}-i} - z_{n_{s+1}-m+3}\right) + \left(- \sum\limits_{i=1}^{m-1} x_{n_r+i} + \sum\limits_{i=1}^{2-1} x_{(n_s+2)-i} + \right.$ $\left. \sum\limits_{i=0}^{m-2-1} x_{n_{s+1}-i} + z_{n_{s+1}-m+2+1}\right) = x_{n_s+1}$, which is identity.

For $k = n_s + l$, and $l,\ 2 \leq l \leq m-1$, according to (10) and (11) the equality $a_{n_s+l,n_r+m} + a_{n_r+m,n_s+l+1} = x_{n_s+l}$ is equivalent to $\left(\sum\limits_{i=1}^{m-1} x_{n_r+i} - \right.$ $\left. - \sum\limits_{i=0}^{m-l-2} x_{n_{s+1}-i} - \sum\limits_{i=1}^{l-1} x_{k-i} - z_{n_{s+1}-m+l+2}\right) + \left(\sum\limits_{i=1}^{l} x_{(k+1)-i} - \sum\limits_{i=1}^{m-1} x_{n_r+i} + \right.$ $\left. \sum\limits_{i=0}^{m-l-2} x_{n_{s+1}-i} + z_{n_{s+1}-m+l+2}\right) = x_{n_s+l}$, which is identity.

For every $l,\ n_s+m \leq l \leq n_{s+1}-1$ equality $a_{k,n_r+m} + a_{n_r+m,k+1} = x_k$ according to (10) and (11) is equivalent to
$\left(\sum\limits_{i=1}^{m-1} x_{n_r+i} - \sum\limits_{i=1}^{m-2} x_{k-i} - z_{k-m+2}\right) + \left(- \sum\limits_{i=1}^{m-1} x_{n_r+i} + \sum\limits_{i=1}^{m-1} x_{(k+1)-i} + \right.$ $\left. + z_{k-m+2}\right) = x_k$, which is identity.

Consider equality $a_{n_r+m,k} + a_{k,\sigma(n_r+m)} = x_{n_r+m}$ for $m = l_r$ and obtain $a_{n_{r+1},k} + a_{k,n_r+1} = x_{n_{r+1}}$.

For $k = n_s + 1$ obtain $a_{n_{r+1},n_s+1} + a_{n_s+1,n_r+1} = x_{n_{r+1}}$, whence, according to (11) and (6), it is equivalent to $\left(- \sum\limits_{i=1}^{l_r-1} x_{n_r+i} + \sum\limits_{i=0}^{l_r-2} x_{n_{s+1}-i} + \right.$ $\left. z_{n_{s+1}-l_r+2}\right) + (x_{n_s+1} - z_{n_s+2}) = x_{n_{r+1}}$, whence $\sum\limits_{i=0}^{l_r-2} x_{n_{s+1}-i} + z_{n_{s+1}-l_r+2} +$ $x_{n_s+1} - z_{n_s+2} = \sum\limits_{i=1}^{l_r} x_{n_r+i}$ which is (18) for $p = 1$.

For $k = n_s + l$ and $l,\ 2 \leq l \leq l_r - 1$, the equivalence $a_{n_{r+1},k} + a_{k,n_r+1} = x_{n_{r+1}}$ is equivalent to $\left(- \sum\limits_{i=1}^{l_r-1} x_{n_r+i} + \sum\limits_{i=1}^{l-1} x_{k-i} + \sum\limits_{i=0}^{l_r-l-1} x_{n_{s+1}-i} + \right.$ $\left. + z_{n_{s+1}-l_r+l+1}\right) + (x_{n_s+l} - z_{n_s+l+1}) = x_{n_{r+1}}$, whence $\sum\limits_{i=0}^{l-1} x_{n_s+l-i} +$

$+\sum\limits_{i=0}^{l_r-l-1} x_{n_{s+1}-i} + z_{n_{s+1}-l_r+l+1} - z_{n_s+l+1} = \sum\limits_{i=1}^{l_r} x_{n_r+i} = \sum\limits_{i=n_r+1}^{n_{r+1}} x_i$, which is (18) for $2 \le p \le l_r - 1$.

For $k = n_s+l$ and l, $l_r \le l \le l_s-1$, the equivalence $a_{n_{r+1},k}+a_{k,n_r+1} = x_{n_{r+1}}$ according to (11) and (6) is equivalent to $\left(-\sum\limits_{i=1}^{l_r-1} x_{n_r+i}+\right.$
$\left.+\sum\limits_{i=1}^{l_r-1} x_{n_s+l-i} + z_{n_s+l-l_r+1}\right) + (x_{n_s+l} - z_{n_s+l+1}) = x_{n_{r+1}}$, whence $\sum\limits_{i=0}^{l_r-1} x_{n_s+l-i} + z_{n_s+l-l_r+1} - z_{n_s+l+1} = \sum\limits_{i=n_r+1}^{n_{r+1}} x_i$, which is (18) for $l_r \le p \le l_s - 1$.

For $k = n_{s+1}$ the equality $a_{n_{r+1},k} + a_{k,n_r+1} = x_{n_{r+1}}$ according to (11) and (6) is equivalent to $\left(-\sum\limits_{i=1}^{l_r-1} x_{n_r+i} + \sum\limits_{i=1}^{l_r-1} x_{n_{s+1}-i} + z_{n_{s+1}-l_r+1}\right)+$
$+\left(x_{n_{s+1}} - z_{n_s+1}\right) = x_{n_{r+1}}$, whence $\sum\limits_{i=0}^{l_r-1} x_{n_{s+1}-i} + z_{n_{s+1}-l_r+1} - z_{n_s+1} = \sum\limits_{i=n_r+1}^{n_{r+1}} x_i$, which coincides with the first line of (18).

4. Lemma about permutation

Let's prove lemma for counting the number of cycles the permutation, whose bottom line of standard representation is $(l+1, \ldots, n, 1, \ldots, l)$ is decomposed to. This lemma will be used for counting the dimension of Kirichenko space.

Lemma 16. *The permutation*

$$\pi = \begin{pmatrix} 1 & \ldots & n-l & n-l+1 & \ldots & n \\ l+1 & \ldots & n & 1 & \ldots & l \end{pmatrix}$$

is decomposed to (n,l) cycles, where (n,l) is greatest common divisor of numbers n and l. In this case each cycle consists of numbers which give the same remainder in division by (n,l).

Proof. Remark, that if n is divisible by l, then lemma is obvious. Let n be indivisible by l.

Denote by $\xi(\pi) = \xi(n,l)$ the quantity of cycles in the decomposition of the permutation of the type, specified above. Remark, that this quantity is equal to one for permutation

$$\pi^{-1} = \begin{pmatrix} 1 & \ldots & l & l+1 & \ldots & n \\ n-l+1 & \ldots & n & 1 & \ldots & n-l \end{pmatrix},$$

whence obtain relation $\xi(n,l)=\xi(n,n-l)$.

Consider the cases $n>2l$ and $n<2l$. Let $n>2l$.

For $n>3l$ we will show that $\xi(n,l)=\xi(n-l,l)$ which will give us a reason to reduce this case to $2l<n<3l$ one. Write out the permutation π, decomposing it into blocks, as

$$\pi=\left(\boxed{\begin{matrix}1 & \dots & l\\ l+1 & & 2l\end{matrix}}\ \boxed{\begin{matrix}l+1 & \dots & 2l\\ 2l+1 & & 3l\end{matrix}}\ \boxed{\begin{matrix}2l+1 & \dots & n-l\\ 3l+1 & & n\end{matrix}}\right.$$

$$\left.\boxed{\begin{matrix}n-l+1 & \dots & n\\ 1 & & l\end{matrix}}\right).$$

Pay attention to that cycles of π which contain elements of the first block, which is interval $[1,l]$. Consider some element $x_i\in[1,l]$ of some cyclic trajectory, and it's obvious that for the previous element x_{i-1} of this trajectory we have enclosure $x_{i-1}\in[n-l+1,n]$, which means, that x_{i-1} belongs to the last block of π. It is also obvious that for the next element of trajectory x_{i+1} we have inclosure $x_{i+1}\in[l+1,2l]$ which means, that is belongs to the second block of π. More over, as lengthes of the first, second and the last cycles are equal to each other, then for arbitrary element $x_{i-1}\in[n-l+1]$, which belongs to some cyclic trajectory, an element x_{i+1} of the same trajectory will belong to interval $[l+1,2l]$.

That is why the quantity of cycles, which the permutation π is decomposed to, is equal to one for permutation π_1, where

$$\pi_1=\left(\boxed{\begin{matrix}l+1 & \dots & 2l\\ 2l+1 & & 3l\end{matrix}}\ \boxed{\begin{matrix}2l+1 & \dots & n-l\\ 3l+1 & & n\end{matrix}}\ \boxed{\begin{matrix}n-l+1 & \dots & n\\ l+1 & & 2l\end{matrix}}\right).$$

Reducing all numbers, which figure in the record of the permutation π_1 on l we will not change the permutation itself and so we will not change the quantity of cycles it is decomposed to, and obtain

$$\pi_2=\left(\boxed{\begin{matrix}1 & \dots & l\\ l+1 & & 2l\end{matrix}}\ \boxed{\begin{matrix}l+1 & \dots & n-2l\\ 2l+1 & & n-l\end{matrix}}\ \boxed{\begin{matrix}n-2l+1 & \dots & n-l\\ 1 & & l\end{matrix}}\right).$$

So, $\xi(\pi)=\xi(\pi_2)$, whence $\xi(n,l)=\xi(n-l,l)$. Repeating some times, if necessary, these reasonings, obtain $\xi(n,l)=\xi(n-pl,l)$, where $2l<n-pl<3l$. Then $\xi(\pi_2)=\xi(\widetilde{\pi}_2)$, where

$$\widetilde{\pi}_2=\left(\boxed{\begin{matrix}1 & \dots & l\\ l+1 & & 2l\end{matrix}}\ \boxed{\begin{matrix}l+1 & \dots & n_1-l\\ 2l+1 & & n_1\end{matrix}}\ \boxed{\begin{matrix}n_1-l+1 & \dots & n_1\\ 1 & & l\end{matrix}}\right),$$

and $n_1=n-pl$. Let us consider the permutation $\pi_3=\widetilde{\pi}_2^{-1}$, and denote $k=n_1-l$. As $2l<n_1<3l$, then $3k<2n_1<4k$. So the permutation π_3

looks like

$$\pi_3 = \left(\boxed{\begin{matrix} 1 & \dots & n_1-k \\ k+1 & & n_1 \end{matrix}} \ \boxed{\begin{matrix} n_1-k+1 & \dots & k \\ 1 & & 2k-n_1 \end{matrix}} \ \boxed{\begin{matrix} k+1 & \dots & n_1 \\ 2k-n_1+1 & & k \end{matrix}} \right).$$

Notice that number k does not belong to the first line of the first block, i.e. $k > n_1 - k$ (because $4k > 2n_1$), which gives possibility to decompose the permutation π_3 to three blocks, taking k as a border between second and third block.

We may note also, that number $2n_1 - 3k$ belongs to the first block, because $2n_1 - 3k < n_1 - k$ (as $2n_1 < 4k$), and the number $2n_1 - 2k$ belongs to third block, because $k+1 < 2n_1 - 2k + 1$ (as $2n_1 > 3k$). That is why, the permutation π_3 may be represented as

$$\pi_3 = \left(\boxed{\begin{matrix} 1 & \dots & 2n_1-3k \\ k+1 & & 2n_1-2k \end{matrix}} \ \boxed{\begin{matrix} 2n_1-3k+1 & \dots & n_1-k \\ 2n_1-2k+1 & & n_1 \end{matrix}} \ \boxed{\begin{matrix} n_1-k+1 & \dots & k \\ 1 & & 2k-n_1 \end{matrix}} \ \boxed{\begin{matrix} k+1 & \dots & 2n_1-2k \\ 2k-n_1+1 & & n_1-k \end{matrix}} \ \boxed{\begin{matrix} 2n_1-2k+1 & \dots & n_1 \\ n_1-k+1 & & k \end{matrix}} \right),$$

that is both the first and third blocks are decomposed to two blocks.

Pay attention on elements of the forth block. Like the way we have considered the permutation π, it is easy to see that each cycle which contains some element x_i of the forth block contains some element x_{i-1} of the first block, more over, each cyclic trajectory which contains some element of the first block contains some element of the forth block on the next place.

Pay attention on elements of the fifth block. Each cycle, which contains some element $x_i \in [2n_1 - 2k + 1, n_1]$, contains also $x_{i-1} \in [2n_1 - 3k + 1, n_1 - k]$ (from the second block) and $x_{i+1} \in [1, 2k - 1]$ (from the third block), that is why, like in considering the permutation π, it is possible to regard that elements of interval $[2n_1 - 3k + 1, n_1 - k]$ (second block) go accordingly to elements of interval $[1, 2k - n_1]$ (third block) which gives us possibility to state that the number of cycles, the permutation π_3 is decomposed to, is equal to one for permutation

$$\pi_4 = \left(\boxed{\begin{matrix} 1 & \dots & 2n_1-3k \\ 2k-n_1+1 & & n_1-k \end{matrix}} \ \boxed{\begin{matrix} 2n_1-3k+1 & \dots & n_1-k \\ 1 & & 2k-n_1 \end{matrix}} \right).$$

So one may see that
$\xi(n_1, n_1 - k) = \xi(n_1 - k, 2k - n_1) = \xi(l, n_1 - 2l) = \xi(l, n_1 \mod l) =$
$= \xi(l, n \mod l)$. Let $n < 2l$. then the permutation π looks like

$$\pi = \left(\boxed{\begin{array}{ccc} 1 & \dots & n-l \\ l+1 & & n \end{array}} \; \boxed{\begin{array}{ccc} n-l+1 & \dots & l \\ 1 & & 2l-n \end{array}} \; \boxed{\begin{array}{ccc} l+1 & \dots & n \\ 2l-n+1 & & l \end{array}} \right).$$

Reasoning like in consideration of the permutation in the case $n > 3l$ it is possible to state that permutation π is decomposed to the same quantity of cycles as permutation π_5

$$\pi_5 = \left(\boxed{\begin{array}{ccc} 1 & \dots & n-l \\ 2n-l+1 & & l \end{array}} \; \boxed{\begin{array}{ccc} n-l+1 & \dots & l \\ 1 & & 2l-n \end{array}} \right),$$

whence $\xi(n, l) = \xi(l, 2l - n) = \xi(l, l - (2l - n)) = \xi(l, n - l) = \xi(l, n \, mod \, l)$.

So we have obtained that for every n and l the equality $\xi(n, l) = \xi(l, n \, mod \, l)$ takes place, which is the equality used in the algorithm of finding the greatest common divisor which, as known, may be obtained by this step, after using it some number of times.

Show, that all the numbers which belong to the same cycle give the same remainders in division by (n, l). Let x be an element of some cycle. If $x + l \leq n$, then the next element of this cycle is $x + l$, which, obviously, gives the same remainder in dividing by l as x does and so the same one in dividing by (n, l).

If $x + l > n$ then the next element of the cycle is $x + l - n$, which also gives the same remainder in dividing by (n, l) as x does. $\square$

5. The dimension of Kirichenko space.

The purpose of this section is to count the defect of the matrix of count relations which is the dimension of Kirichenko space.

Lemma 17. *Consider the relation*

$$l_1 \sum_{i=1}^{l_s} x_{n_s+i} = l_s \sum_{i=2}^{l_1} x_i. \tag{14a}$$

Systems of relations (14; 15) and (14a; 15) are equivalent.

Proof. Consider the system of relations

$$\sum_{i=0}^{n_1-2} x_{n_{s+1}-i} + x_{n_s+1} = \sum_{i=2}^{n_1} x_i, \tag{14}$$

$$\begin{cases} x_{n_s+p} = x_{n_{s+1}-(n_1-p)} & 1 \le p \le n_1 \\ x_{n_s+p} = x_{n_s+p-n_1} & n_1 < p \le n_{s+1} - n_s \end{cases} \tag{15}$$

According to the first line of the last relation, the relation (14) is equivalent to

$$\sum_{i=1}^{n_1} x_{n_s+i} = \sum_{i=2}^{n_1} x_i \tag{14b}$$

one.

From second line of equalities (15) we may obtain $x_{n_s+n_1+1} = x_{n_s+1}$, $x_{n_s+n_1+2} = x_{n_s+2}$ and so on up to $x_{n_{s+1}} = x_{n_{s+1}-n_1}$, that is the set of parameters $\{x_{n_s+1}, \ldots, x_{n_{s+1}}\}$ is periodic with period n_1. Here consider the periodicity in the sense, that if $i-j$ is divisible by n_1, then $x_i = x_j$. Without bounding of generality one may consider this set as consisted of elements $\{x_{n_s+1}, \ldots, x_{n_s+n_1}\}$.

It goes from the first line of equalities (15) that the last n_1 elements of a set $\{x_{n_s+1}, \ldots, x_{n_{s+1}}\}$ coincide with $\{x_{n_s+1}, x_{n_s+2}, \ldots, x_{n_s+n_1}\}$ correspondingly.

Taking into account the fact that the set $\{x_{n_s+1}, \ldots, x_{n_{s+1}}\}$, is periodic, it is easy to see that the problem of calculating the quantity of different parameters of the set $\{x_{n_s+1}, \ldots, x_{n_s+n_1}\}$ is equivalent to one of finding the quantity of different cycles, the permutation π,

$$\pi = \begin{pmatrix} 1 & \ldots & n_1 - l & n_1 - l + 1 & \ldots & n_1 - 1 & n_1 \\ l+1 & \ldots & n_1 & 1 & \ldots & l-1 & l \end{pmatrix},$$

which the first line corresponds to the last n_1 elements of a set $\{x_{n_s+1}, \ldots, x_{n_{s+1}}\}$ (which are equal to former ones), and second line corresponds to the last elements of a set $\{x_{n_s+1}, \ldots, x_{n_{s+1}}\}$ is decomposed to.

According to lemma 16 (about permutation), the set of parameters $\{x_{n_s+1}, \ldots, x_{n_{s+1}}\}$ is decomposed to (l_s, l_1) sets of pairwise equal ones, and indices of elements of each cycle give the same remainders in division by (l_s, l_1).

Consider ones more the equality (14b) which is $\sum\limits_{i=1}^{n_1} x_{n_s+i} = \sum\limits_{i=2}^{n_1} x_i$. Taking into account facts proved above, sum of first n_1 parameters is equal to sum of all (l_s, l_1) pairwise unequal ones taken $\dfrac{l_1}{(l_s, l_1)}$ times. So, it is obvious, that $\dfrac{(l_s, l_1)}{l_1} \sum\limits_{i=1}^{l_1} x_{n_s+i}$ is sum of one gang of pairwise unequaled parameters, whence $\sum_{i=1}^{l_1} x_{n_s+i} = \frac{l_1}{l_s}\sum_{i=1}^{l_s} x_{n_s+i}$, which is

$$l_1 \sum_{i=1}^{l_s} x_{n_s+i} = l_s \sum_{i=2}^{l_1} x_i. \tag{14a}$$

Using the same steps (with revers order) one may prove that the system of equations (14; 15) is a corollary of (14a; 15). □

Lemma 18. *If $n_1 = 2$, then the system of equations*

$$\begin{cases} x_{k-1} + x_k = x_2 & n_s + 2 \le k \le n_{s+1}, \\ x_{n_{s+1}} + x_k = x_2 & k = n_s + 1 \end{cases} \tag{12}$$

is a partial case of relations system (14a; 15).

Proof. Write out the matrix of relation (12) and obtain

$$\left(\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & \cdots & 0 & x_2 \\ 0 & 1 & 1 & 0 & \cdots & 0 & x_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & x_2 \\ 0 & 0 & \cdots & 0 & 1 & 1 & x_2 \\ 1 & 0 & \cdots & \cdots & 0 & 1 & x_2 \end{array}\right)$$

If one subtract from each line (except the first) the previous one, and add all the lines to the last one then the matrix which determine the system of equations it will appear. This matrix is equivalent to former one, where (just for convenience) it is possible to add the last line of a former matrix to penultimate place, and obtain matrix

$$\left(\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & \cdots & 0 & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{-1} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 2 & 2 & \cdots & \cdots & 2 & 2 & l_s x_2 \end{array}\right)$$

where bold type is used for added line. It's obvious, that this system coincides with the system of relations (14a, 15) for $n_1 = 2$. □

Thus, the problem of calculating the dimension of Kirichenko space has reduced to finding the defect of matrix K whose columns correspond to parameters $\{x_2, \dots x_n, z_{n_1+2,1}, z_{n_1+3,1} \dots z_{n,1}, z_{n_2+2,2}, \dots z_{n,2} \dots$ $\dots, z_{n_{q-1}+2,q-1}, \dots z_{n,q-1}\}$ and lines correspond to relations (13; 14a; 15-18) ordered in a natural way. It's obvious, that this matrix looks like

$$K = \begin{pmatrix} X & 0 \\ X' & Z \end{pmatrix} = \begin{pmatrix} X & 0 & 0 & \cdots & 0 \\ X_1 & Z_1 & 0 & \cdots & 0 \\ X_2 & 0 & Z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ X_{q-1} & 0 & \cdots & 0 & Z_{q-1} \end{pmatrix},$$

where matrices $X, X_1, \dots, X_{q-1}, Z_1, \dots Z_{q-1}$ are block matrices, matrix X corresponds to relations (13), and each block $\begin{pmatrix} X_r & 0 & Z_r & 0 \end{pmatrix}$ corresponds to count relations (14a; 15-18) for every r, $1 \le r \le q-1$. Blocks which correspond to different values of r have similar form and in some sense are differ only by dimension. Consider their structure in more details.

Like in proving relations, we may fix arbitrary r, $1 \le r \le n-1$ and so, for admissible k instead of $z_{k,r}$ we will write z_k.

Consider the system of relations

$$\sum_{i=1}^{m} x_{n_r+i} - \sum_{i=0}^{m-2} x_{n_{r+1}-i} = z_{n_r+m+1} + z_{n_{r+1}-m+2}, \quad 2 \le m \le l_r - 1. \quad (17)$$

Fix arbitrary m_0, $2 \le m_0 \le l_r - 1$. It's obvious that parameter $z_{n_r+m_0+1}$ will be met in the equality for $m = m_0$ and in the equality for such $m = m_1$, that $n_{r+1} - m_1 + 2 = n_r + m_0 + 1$, because $z_{n_r+m_0+1}$ may be met not only as first, but also as second item in relations (17). Let's find indices for z, which will appear in relations for $m = m_1$. Sure, one of them will be $n_r + m_0 + 1$. As $n_{r+1} - m_1 + 2 = n_r + m_0 + 1$, then $m_1 = n_{r+1} - n_r - m_0 + 1$, whence the index, we need, is $n_r + m_1 + 1 = n_{r+1} - m_0 + 2$, which coincides with index of second element in the relation for $m = m_0$. It means, for any relation in (17), there is one more, which has the same right side. Show, that these two relations coincide, that is they are one relation, written out two times. This statement may be formulated as such lemma.

Lemma 19. *For every r, $1 \le r \le q-1$ and every m, $2 \le m \le l_r - 1$ denote*

$$f(m) = \sum_{i=1}^{m} x_{n_r+i} - \sum_{i=0}^{m-2} x_{n_{r+1}-i}.$$

Then $f(m) = f(n_{r+1} - n_r - m + 1)$.

Proof. Let's prove, that $f(m) - f(n_{r+1} - n_r - m + 1) = 0$.

$$f(m) - f(n_{r+1} - n_r - m + 1) = \left(\sum_{i=1}^{m} x_{n_r+i} - \sum_{i=0}^{m-2} x_{n_{r+1}-i}\right) - \\ - \left(\sum_{i=1}^{n_{r+1}-n_r-m+1} x_{n_r+i} - - \sum_{i=0}^{n_{r+1}-n_r-m-1} x_{n_{r+1}-i}\right) = \left(\sum_{i=n_r+1}^{n_r+m} x_i - \\ - \sum_{i=n_{r+1}-m+2}^{n_{r+1}} x_i\right) - \left(\sum_{i=n_r+1}^{n_{r+1}-m+1} x_i - \sum_{i=n_r+m+1}^{n_{r+1}} x_i\right) = 0 \quad \square$$

Thus, lines of matrix, which correspond to fixed r and relation (17) looks like

$$\left(\begin{array}{cc|cccccccc} \mathbf{X_{r,2}} & \mathbf{0} & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & \mathbf{0} \\ \mathbf{X_{r,3}} & \mathbf{0} & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \mathbf{0} \\ \mathbf{X_{r,4}} & \mathbf{0} & 0 & 0 & 1 & \cdots & 1 & 0 & 0 & \mathbf{0} \\ \vdots & \vdots & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & \mathbf{0} \\ \hline \mathbf{X_{r,l_r-3}} & \mathbf{0} & 0 & 0 & 1 & \ddots & 1 & 0 & 0 & \mathbf{0} \\ \mathbf{X_{r,l_r-2}} & \mathbf{0} & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \mathbf{0} \\ \mathbf{X_{r,l_r-1}} & \mathbf{0} & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & \mathbf{0} \end{array}\right)$$

where by the bold font we type block-lines, and the first half of lines of this part of K are pairwise different, but others are copies of some first lines.

The problem of calculating the defect of this part of K is reduced to calculating the quantity of independent parameters in the set (17). This set is decomposed into independent pairs. Each such pair is relation, which looks like sum $z_i + z_j$ equals to some expression, depends only on the set $x_2, \ldots, x_n$. It is obvious, that if $n_r+m_0+1 \neq n_{r+1}+m_0+2$ for any m_0, then one of parameters $z_{n_r+m_0+1}$, $z_{n_{r+1}+m_0+2}$ we may consider as independent, and the other as expressed through independent ones, and so in this case, the relation (17) for $m = m_0$ gives us one independent parameter. If for some m_0 we have equality $n_r+m_0+1 = n_{r+1}+m_0+2$, then $z_{n_r+m_0+1}$ is expressed through parameters of the set $x_2, \ldots, x_n$ and is dependent. That's why the quantity of independent parameters equals $\dfrac{l_r-2}{2}$, if l_r is even, and equals $\dfrac{l_r-3}{2}$, if l_r is odd, which is $\left[\dfrac{l_r-2}{2}\right]$ in general case and is equal to the defect of matrix, determined by (17).

Consider the part of matrix K which corresponds to the system of relations (18), and looks like $\begin{pmatrix} X_{(r)} & 0 & Z_{(r)} & 0 \end{pmatrix}$ (indices are bracketed to make difference with denotations for blocks, which appeared in some previous propositions) which may be written in the matrix form as

$$Z_{(r)} = \left(\begin{array}{cccccccc|c} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & f_0 \\ 0 & 1 & \cdots & 0 & 0 & -1 & \cdots & 0 & f_1 \\ \vdots & \vdots & \ddots & & & & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & 0 & 0 & \cdots & -1 & f_{l_s-l_r-1} \\ -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & f_{l_s-l_r} \\ 0 & -1 & \cdots & 0 & 0 & 1 & \cdots & 0 & f_{l_s-l_r+1} \\ \vdots & \vdots & \ddots & & & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 1 & f_{l_s-1} \end{array}\right).$$

Let's show, that by means of interchanging lines and columns, this matrix may be reduced to block-diagonal form. The possibility of such reducing means that the set of parameters, which correspond to columns of this matrix may by decomposed into such disjunct join, that there is no relation, which connects parameters from different sets. Write out the connections between parameters $z_{n_s+1}, \dots, z_{n_s+l_s}$ in the form of permutation π, where equality $\pi(i) = j$ will mean that there is an equation $z_{n_s+j} - z_{n_s+i} = f_{j+1}$ in the system (18). From the fact that for each i, $1, \leq i \leq l_s$, parameter z_{n_s+i} is met only two times i.e. once with positive sign and once with negative one, we obtain that the quantity of blocks the left part of $Z_{(r)}$ may be decomposed to is the quantity of cycles the permutation π is decomposed to.

The first two lines of system (18) give that π will be like

$$\pi = \begin{pmatrix} \cdots & l_s - l_r + 1 & l_s - l_r + 2 & \cdots & l_s \\ \cdots & 1 & 2 & \cdots & l_r \end{pmatrix}.$$

And in the same time, the last line of (18) gives, that π looks like

$$\begin{pmatrix} 1 & \cdots & l_s - l_r & \cdots \\ l_r + 1 & \cdots & l_s & \cdots \end{pmatrix},$$

and in general π is

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & l_s - l_r & l_s - l_r + 1 & \cdots & l_s \\ l_r + 1 & l_r + 2 & \cdots & l_s & 1 & \cdots & l_r \end{pmatrix}.$$

According to lemma 16 it is decomposed into composition of $(l_s, l_r) = t$ cycles, and each cycle in this decomposition contains exactly $w = \dfrac{l_s}{t}$ elements. Note, that all elements from each cycle in this decomposition give the same remainders in division by t.

Make the simultaneous interchanging of lines and columns of matrix Z_r determined by permutation τ which looks as

$$\left(\begin{array}{ccccc|cccc|c} 1 & \pi(1) & \pi^2(1) & \dots & \pi^{w-1}(1) & 2 & \pi(2) & \cdots & \pi^{w-1}(2) & \cdots \\ 1 & 2 & 3 & \dots & w & w+1 & w+2 & \cdots & 2w & \cdots \end{array}\right.$$

$$\left.\begin{array}{c|ccc} \cdots & t & \cdots & \pi^{w-1}(t) \\ \cdots & l_s-w+1 & \cdots & l_s \end{array}\right).$$

After this matrix $Z_{(r)}$ will be transformed into

$$\left(\begin{array}{ccc|c} \boxed{Z_{r,1}} & \cdots & 0 & F_{r,1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \boxed{Z_{r,t}} & F_{r,t} \end{array}\right)$$

where $F_{r,i}$ is column of functions $\widetilde{f}_{(i-1)t+1}, \widetilde{f}_{(i-1)t+2}, \dots \widetilde{f}_{(i-1)t+w}$, obtained during transformations, and matrices $Z_{r,i}$ are equal and look like

$$Z_{r,i} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let's do transformations with lines of each block of obtained matrix. In each block of obtained matrix add all previous lines to each one except the first line (because it has no previous), and obtain matrix

$$\widetilde{Z}_{r,i} = \left(\begin{array}{ccccc|c} 1 & -1 & 0 & \cdots & 0 & \widetilde{f}_{(i-1)t+1} \\ 1 & 0 & -1 & \cdots & 0 & \widetilde{f}_{(i-1)t+1} + \widetilde{f}_{(i-1)t+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & -1 & \sum\limits_{j=1}^{w-1} \widetilde{f}_{(i-1)t+j} \\ 0 & 0 & \cdots & 0 & 0 & \sum\limits_{j=1}^{w} \widetilde{f}_{(i-1)t+j} \end{array}\right).$$

Now, my means of addition of all columns of matrix $\widetilde{Z}_{r,i}$ except the first one to other columns of matrix K with necessary coefficients we may make zero all elements of $\widetilde{Z}_{r,i}$ except ones, belongs to it's first sub diagonal, which will stay be equal to -1.

Show that the sum $\sum\limits_{j=1}^{w} \widetilde{f}_{(i-1)t+j}$ doesn't depend on i. The last proposition is a corollary of next lemma.

Lemma 20. *For every common divisor t of numbers l_r and l_s and for every b, $1 \le b \le t$ the equality* $\sum\limits_{i=0}^{(l_s/t)-1} \widetilde{f}_{b-1+it} = \frac{l_r}{t}\sum\limits_{i=1}^{l_s} x_{n_s+i} - \frac{l_s}{t}\sum\limits_{i=1}^{l_r} x_{n_r+i}$ *takes place.*

Proof. As functions $\widetilde{f}_0, \dots \widetilde{f}_{l_s-1}$ are obtained from the set of ones $\{f_0, \dots f_{l_s-1}\}$ after interchanging of lines and columns of matrix, then for every p, $0 \le p \le l_q - 1$ the equality $\widetilde{f}_p = \tau^*(f_{\tau(p+1)-1})$, where τ^* is automorphism of the set $\{x_{n_s+1}, \dots x_{n_{s+1}}\}$ such that $\tau^*(x_{n_s+i}) = x_{n_s+\tau(i)}$ for every i, $1 \le i \le l_s$ takes place. As for every b, $1 \le b \le t$ the set $\{b-1+it,\ 1 \le b \le t\}$ is invariant for τ, then the proposition of lemma is equivalent to $\sum\limits_{i=0}^{(l_s/t)-1} f_{b-1+it} = \frac{l_r}{t}\sum\limits_{i=1}^{l_s} x_{n_s+i} - \frac{l_s}{t}\sum\limits_{i=1}^{l_r} x_{n_r+i}$, which we will prove. Denote $g_k = f_k - \sum\limits_{i=n_r+1}^{n_{r+1}} x_i$. It is obvious that it is necessary and enough for proving lemma to prove that $\sum\limits_{i=0}^{(l_s/t)-1} g_{b-1+it} = \frac{l_r}{t}\sum\limits_{i_1}^{l_s} x_{n_s+i}$, because items $-\sum\limits_{i=n_r+1}^{n_{r+1}} x_i$ are met exactly once in each f_k, and that is why alter taking the sum of f_k we will obtain that quantity of appearing of $-\sum\limits_{i=n_r+1}^{n_{r+1}} x_i$ is $\frac{l_s}{t}$. Prove, that $\sum\limits_{i=0}^{(l_s/t)-1} g_{b-1+it} = \frac{l_r}{t}\sum\limits_{i_1}^{l_s} x_{n_s+i}$. Let us write out the matrix G, which columns correspond to parameters $x_{n_s+1}, \dots, x_{n_{s+1}}$ and l_s lines correspond to relations $g_0, \dots g_{l_s-1}$.

When k runs through the interval $[0, l_r - 1]$, then lines of matrix G which correspond to relations g_k look like

$$G^{**} = \left(\begin{array}{ccccc|ccccccc}
0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 1 & \ddots & \ddots & \cdots & 0 & 0 & \ddots & \ddots & 1 & 1 & 1 \\
1 & 1 & \ddots & 0 & \cdots & 1 & 0 & 0 & \ddots & \ddots & 1 & 1 \\
1 & 1 & \dots & 1 & \cdots & 1 & \ddots & 0 & \ddots & 0 & \ddots & 1 \\
1 & 1 & \dots & 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1
\end{array}\right), \text{ or}$$

$$G^{**} = \left(\begin{array}{cccccc|ccccccc}
0 & 0 & 0 & \cdots & \cdots & \mathbf{0} & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 0 & 0 & \cdots & \cdots & \mathbf{0} & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 1 & 0 & \cdots & \cdots & \mathbf{0} & 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\
1 & 1 & \ddots & \ddots & \cdots & \mathbf{0} & 0 & 0 & \ddots & \ddots & 1 & 1 & 1 \\
1 & 1 & \dots & 1 & 0 & \mathbf{0} & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\
1 & 1 & \dots & 1 & 1 & \mathbf{0} & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right)$$

which depends on whether $2l_r \geq l_s$, or not, where the first line of G^{**} contains l_r identities and $l_s - l_r$ zeros. This matrix is decomposed in the natural way to blocks Gl and Gr, the former of which corresponds to parameters $x_{n_s+1}, \ldots, x_{n_{s+1}-l_r}$ and the last - to $x_{n_{s+1}-l_r+1}, \ldots, n_{s+1}$ ones.

When k runs through interval $[l_r, l_s - 1]$, then the part of matrix G, whose lines correspond to relations g_k for these k looks like

$$G^* = \begin{pmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \cdots & & & & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 \end{pmatrix}$$

where the former l_r elements of the first line are equal to identity, and the last ones are zero.

The statement of this lemma in terms of matrix G is the following. If we write out the lines of g with the step t, and arbitrary beginning b, $1 \leq b \leq t$ then the sum of ones in each column will be the same and will be equal to $\frac{l_r}{t}$. Note that as the quantity of lines of matrix G^{**} is equal to l_r which is divisible by t, then such writing out the lines means that we obtain a matrix G_b, which consists of writing out one above another matrices G_b^* and G_b^{**}, each of them is obtained by writing out the lines of matrices G^{**} and G^* with the step t starting from b.

Denote by Gl_b and Gr_b those parts of matrix G_b^{**} which are obtained during writing out the lines with step t. Let's show that matrix G_b will be column-block matrix, i.e. it will be consisted of such vertical blocks with $\frac{l_s}{t}$ lines that each of them will have the same element in each line. We will say about such matrix, that it satisfies column-block condition. Let's show, that for G_b^{**} the correspond widthes of blocks will be $b-1,\ t,\ t, \ldots,\ t,\ t-b+1,\ b-1,\ t,\ t,\ \ldots,\ t,\ t-b+1$, and the first blocks of matrices Gl_b and Gr_b have width $b-1$.

For proving this proposition, consider matrices Gl_b, Gr_b and G_b^* separately.

Consider matrix Gl_b. The first line of this matrix have $b-1$ ones and zeros at the other places, and satisfies column-blocks condition. Each next line of this matrix will have t ones more at the beginning and t zeros less at the end and so, satisfies the column-block condition also.

Consider matrix Gr_b. The first line of this matrix have $b-1$ zeros and ones at the other places, and satisfies the column-block condition. Each next line of this matrix will have t zeros more at the beginning and t ones less at the end and so, satisfies the column-block condition also.

Consider matrix G_b^*. The first line of this matrix have $b-1$ zeros at the beginning, then l_r ones, and at last $(l_s - l_r - t) + (t - b + 1)$ zeros at the end, and satisfies the column-block condition. Each next line of this matrix has t zeros more at the beginning, and t zeros less at the end, and so, satisfies the column-block condition also.

Let us count the quantity of ones in the first column of matrix G_b that is equal to this quantity for the first block-column. If $b = 1$ then each of $\frac{l_r}{t}$ lines except the first one of matrix Gl_b will have some number of ones at its beginning, and the first one will have zero there, so the quantity of ones in the first block of matrix Gl_b is $\frac{l_r}{t} - 1$. At the same time, the first line of the matrix G_b^* has one at its beginning and other ones have zero at their beginnings, so the quantity of ones of the first block of matrix G_b is $\frac{l_r}{t}$.

If $b > 1$, then the first element of each line of matrix Gl_b will be equal to one, but the first element of each line of matrix G_b^* will be equal to zero, whence the quantity of ones at first places of all lines of G_b is equal to $\frac{l_r}{t}$.

It's easy to see that every next block of Gl_b has exactly 1 one less, but every next block of G_b^* has 1 one more, so general quantity of ones is not changed and is equal to $\frac{l_r}{t}$ for all columns from the first up to one number $l_s - l_r$.

Count the quantity of ones in the last column of matrix G_b. The last element of each of $\frac{l_r}{t}$ lines of matrix Gr_b is equal to one, and in the came time the last element of each line of G_b^* is equal to zero. That is why the quantity of ones in the last column of G_b is equal to $\frac{l_r}{t}$.

It's easy to see that moving left from last block of Gr_b each block we meet, will have 1 one less then previous in matrix Gr_b and 1 one more in the matrix G_b^*, saving the general quantity of ones in columns with numbers from $l_s - l_r + 1$ up to l_s. So the quantity of ones in each column of G_b is equal to $\frac{l_r}{t}$. □

Corollary 1 of lemma: The part $\begin{pmatrix} X_{(k)} & 0 & Z_{(k)} & 0 \end{pmatrix}$ of matrix K which corresponds to count relations (18) for fixed r by transforming lines and columns may be reduced to being like

$$N_s = \begin{pmatrix} l_r \sum\limits_{i_1}^{l_s} x_{n_s+i} - l_s \sum\limits_{i_1}^{l_r} x_{n_r+i} & 0 & 0 & 0 \\ 0 & 0 & E & 0 \end{pmatrix},$$

where E is identity matrix of order $l_s - (l_s, l_r)$.

Corollary 2 of lemma: Using transformations of lines and columns, matrix $K = \begin{pmatrix} X & 0 \\ X' & Z \end{pmatrix}$ may be reduced to being like $\begin{pmatrix} X & 0 & 0 \\ X'' & 0 & 0 \\ 0 & E & 0 \end{pmatrix}$, where E in such identity matrix that $\begin{pmatrix} E & 0 \end{pmatrix}$ has defect $\sum\limits_{s=2}^{q-1} \sum\limits_{r=s+1}^{q} (l_s, l_r) + \sum\limits_{s=2}^{q-2} \left[\dfrac{l_s - 2}{2}\right]$, and matrix X'' looks like

$$X'' = \begin{pmatrix}
0 & L_3 & -L_2 & 0 & & & \\
0 & L_4 & 0 & -L_2 & & & \\
 & \vdots & & & \ddots & & \\
0 & L_{q-1} & 0 & 0 & \cdots & -L_2 & 0 \\
0 & L_q & 0 & 0 & \cdots & 0 & -L_2 \\
0 & 0 & L_{s+1} & -L_s & 0 & & \\
0 & 0 & L_{s+2} & 0 & -L_s & & \\
 & & \vdots & & & \ddots & \\
0 & 0 & L_q & 0 & 0 & & -L_s \\
 & & & \vdots & & & \\
0 & 0 & & & L_{q-1} & -L_{q-2} & 0 \\
0 & 0 & & & L_q & 0 & -L_{q-2} \\
0 & 0 & & & 0 & L_q & -L_{q-1}
\end{pmatrix}$$

where all elements at full places are zero, and by L_i we denote a matrix which has one line all whose elements are equal to l_i. In this case the quantity of blocks-columns of matrix is equal to q, the width of first of them is equal to $l_1 - 1$, and the widthes of others are equal to l_i, $i > 1$.

Let's go to describing a matrix X.

According to relations (13; 14a; 15), it is like $X = \begin{pmatrix} A \\ B \end{pmatrix}$, where

$$A = \left(\begin{array}{cccccc|cccc}
1 & 0 & \cdots & 0 & 0 & -1 & & & & \\
0 & 1 & \cdots & 0 & -1 & 0 & & & & \\
 & & \ddots & & & & & & & \\
0 & -1 & \cdots & 0 & 1 & 0 & & & & \\
-1 & 0 & \cdots & 0 & 0 & 1 & & & & \\ \hline
 & & & & & & X^3 & & & \\
 & & & & & & & X^4 & & \\
 & & & & & & & & \ddots & \\
 & & & & & & & & & X^q
\end{array}\right)$$

where the first block has $l_1 - 1$ columns, and each next one has $l_i,\ i > 1$, columns, and X^i is like

$$X^i = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & & & \ddots & \vdots \\ 0 & 0 & & 1 & 0 & 0 & \cdots & -1 \\ -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

where the sub diagonal, whose elements are equal to -1 has number n_1, i.e. the element of first line of X^i, which is equal to -1, is in the column number $n_1 + 1$.

Matrix B looks like

$$B = \begin{pmatrix} L_2 & -L_1 & 0 & \cdots & 0 & 0 \\ L_3 & 0 & -L_1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ L_{q-1} & 0 & 0 & \cdots & -L_1 & 0 \\ L_q & 0 & 0 & \cdots & 0 & -L_1 \end{pmatrix}$$

where L_i means the same as in matrix X''.

Consider block matrix $\begin{pmatrix} X \\ X'' \end{pmatrix}$.

Note that widthes of blocks of matrices A, B, and X'' are equal, and its easy to see, that if we will subtract the first column of each block of the width $l_1 - 1,\ l_2, \dots l_q$, from all other it's columns, then each block of matrices B and X'' will have number l_i at the first place and zeros at the others. Denote them by $\widetilde{L}_i$, and matrices which appear after transformation of B and X'' by $\widetilde{B}$ and $\widetilde{X}''$.

Let's find the rang of matrix $\begin{pmatrix} \widetilde{B} \\ \widetilde{X}'' \end{pmatrix}$. It's obvious that it is not less then $q-1$, because its first $q-1$ lines are linear independent.

Let's consider the first $q-1$ lines and arbitrary other line, which contains elements l_i and l_j, obtaining

$$\det \begin{pmatrix} l_2 & -l_1 & 0 & \cdots & 0 & 0 \\ l_3 & 0 & -l_1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ l_{q-1} & 0 & 0 & \cdots & -l_1 & 0 \\ l_q & 0 & 0 & \cdots & 0 & -l_1 \\ 0 & l_i & \cdots & -l_j & \cdots & 0 \end{pmatrix} =$$

$= (-l_1)^{q-3} \cdot \det \begin{pmatrix} l_j & -l_1 & 0 \\ l_i & 0 & -l_1 \\ 0 & l_i & -l_j \end{pmatrix} = (-l_1)^{q-3} \cdot (l_1 l_i l_j - l_1 l_i l_j) = 0.$ And, the rang of matrix $\begin{pmatrix} \widetilde{B} \\ \widetilde{X}'' \end{pmatrix}$ is equal to $q-1$.

That is why the rang of matrix $\begin{pmatrix} X \\ X'' \end{pmatrix}$ is equal to one of matrix A minus $q-1$, And we may generalize our calculations of dimension of Kirichenko space.

Thus, if $q > 2$, then the dimension of Kirichenko space may be written out as $\left[\frac{l_1}{2}\right] + \sum\limits_{s=2}^{q}(l_s, l_1) + \sum\limits_{r=2}^{q}\left[\frac{l_r-2}{2}\right] + \sum\limits_{r=2}^{q}\sum\limits_{s=r+1}^{q}(l_s, l_r) - q + 1 = \sum\limits_{r=1}^{q}\left[\frac{l_r}{2}\right] + \sum\limits_{r=1}^{q}\sum\limits_{s=r+1}^{q}(l_s, l_r) - 2q + 2.$

If $q = 2$, then matrix X'' will be absent, whence the dimension of Kirichenko space will be $\left[\frac{l_1}{2}\right] + \left[\frac{l_2-2}{2}\right] + (l_2, l_1) - 1 = \left[\frac{l_1}{2}\right] + \left[\frac{l_2}{2}\right] + (l_2, l_1) - 2$

If $q = 1$, then all count relations will be represented by equality (13), and the dimension of Kirichenko space will be equal to $\left[\frac{n_1}{2}\right]$.

Thus, we have proved the main theorem

Main Theorem. Let σ be a permutation, which is composition of cycles of lengthes $l_1, \ldots, l_q$. Then the dimension of Kirichenko space is equal to

$2 - 2q + \sum\limits_{r=1}^{q}\left[\frac{l_r}{2}\right] + \frac{1}{2}\sum\limits_{r\neq s}(l_s, l_r)$, where (a, b) denotes the greatest common divisor of numbers a and b.

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