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Isolated and nilpotent subsemigroups in the variants of $\mathcal{IS}_n$

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Abstract. All isolated, completely isolated, and nilpotent subsemigroups in the semigroup $\mathcal{IS}_n$ of all injective partial transformations of an n-element set, considered as a semigroup with a sandwich multiplication are described.

Introduction and main definitions

In the monograph [5] Ljapin proposed some constructions for the semigroups, a certain modification of which is the following. Let S be a semigroup. For a fixed $a \in S$ define an operation $*_a$ via $x *_a y = xay$, $x, y \in S$. Obviously, this multiplication $*_a$ is associative, therefore the set S with respect to this operation is the semigroup which is called *the semigroup* S *with a sandwich operation* or the *variant* of S and is denoted by $(S, *_a)$.

The variants of the classical transformation semigroups, $\mathcal{IS}_n$, $\mathcal{T}_n$, $\mathcal{PT}_n$, are interesting examples of this construction. In [7], [8], and [9], criteria of isomorphisms of these variants are detailed, and some of their properties are described.

Recall, a subsemigroup T of S is called *isolated*, provided that $x^k \in T$ for some $k \in \mathbb{N}$ implies $x \in S$ for all $x, y \in S$; T is called *completely isolated*, provided that $xy \in T$ implies $x \in T$ or $y \in T$ for all $x, y \in S$. Note that every completely isolated subsemigroup is isolated, while the contrary does not hold true. Isolated and completely isolated subsemigroups of the variants of $\mathcal{T}_n$ are described in [6].

Key words and phrases: *Inverse symmetric semigroup, variants of a semigroup, sandwich semigroup, isolated subsemigroups, nilpotent subsemigroups.*

A semigroup S with a zero 0 is called *nilpotent of the nilpotency degree* $k \geq 1$, provided that $x_1x_2\cdots x_k = 0$ for all $x_1, x_2, \ldots, x_k \in S$ and there exist $y_1, y_2, \ldots, y_{k-1} \in S$ such that $y_1y_2\cdots y_{k-1} \neq 0$. Denote $n(S) = k$. It is known[1], that a finite semigroup S is nilpotent if and only if each element of S is nilpotent, that is, for each $x \in S$ there exist $k \in \mathbb{N}$ such that $x^k = 0$ and $x^{k-1} \neq 0$.

In this paper we describe completely isolated, isolated and nilpotent subsemigroups of the variants of $\mathcal{IS}_n$, the inverse symmetric semigroup of all partial injective transformations of the set $N = \{1, 2, \ldots, n\}$.

In particular, in section 1 we study isolated and completely isolated subsemigroups, and we produce the full description of nilpotent subsemigroups of the variants of $\mathcal{IS}_n$ with respect to a natural zero (a nowhere defined map) in section 2. In the proofs we use the technique presented in [4].

For an arbitrary $\beta \in \mathcal{IS}_n$ denote $\mathrm{dom}(\beta)$ and $\mathrm{im}(\beta)$ the domain and image of β, respectively. The value $|\,\mathrm{dom}(\beta)| = |\,\mathrm{im}(\beta)|$ is called the range of β and is denoted by $\mathrm{rank}(\beta)$.

It is proved in [8] that semigroups $(\mathcal{IS}_n, *_\alpha)$ and $(\mathcal{IS}_n, *_\beta)$ are isomorphic if and only if $\pi\alpha = \beta\tau$ for some permutations $\pi, \tau \in \mathcal{S}_n$. Note if α is a permutation then $(\mathcal{IS}_n, *_\alpha)$ is isomorphic to $\mathcal{IS}_n$, and the subsemigroup construction of the latter is studied in details in, for example,[2], [3]. Therefore, without loss of generality, we may assume that the sandwich element α is a non-identity idempotent in $\mathcal{IS}_n$.

Now let us fix an idempotent $\alpha \in \mathcal{IS}_n$. Set $(\mathcal{IS}_n, *) = (\mathcal{IS}_n, *_\alpha)$, $\mathrm{dom}(\alpha) = A$, $\mathrm{rank}(\alpha) = l$, $l < n$, and let α, as an element of $\mathcal{IS}_n$, be defined on A identically.

Fix some $z \in N \setminus A$ and for arbitrary pairwise different elements $x_1, \ldots, x_k \in A$ define the following partial permutations:

$$(x_1, \ldots, x_k)(x) = \begin{cases} x_{i+1}, & \text{if } x = x_i, \quad i < k, \\ x_1, & \text{if } x = x_k, \\ x, & \text{if } x \in N \setminus \{x_1, \ldots, x_k\} \end{cases}$$

and

$$[x_1, \ldots, x_k](x) = \begin{cases} x_{i+1}, & \text{if } x = x_i, \quad i < k, \\ z, & \text{if } x = x_k, \\ x, & \text{if } x \in N \setminus \{x_1, \ldots, x_k, z\}. \end{cases}$$

For any $\beta \in (\mathcal{IS}_n, *)$ we denote $\beta^{*s} = \underbrace{\beta * \beta * \cdots * \beta}_{s}$.

Proposition 1 ([8]). *An element $\varepsilon \in \mathcal{IS}_n$ is an idempotent in $(\mathcal{IS}_n, *)$ if and only if ε is an idempotent in $\mathcal{IS}_n$ and* $\mathrm{dom}(\varepsilon) \subset A$.

Remark 2. *For each $x \in A$ denote ε_x the idempotent of $\mathcal{IS}_n$ such that* $\operatorname{dom}(\varepsilon_x) = A \setminus \{x\}$*. Then any idempotent $\varepsilon \neq \alpha$ of $(\mathcal{IS}_n, *)$ may be factorized as*

$$\varepsilon = \prod_{x \in A \setminus \operatorname{dom}(\varepsilon)} \varepsilon_x = \prod^{*}_{x \in A \setminus \operatorname{dom}(\varepsilon)} \varepsilon_x.$$

1. Isolated and completely isolated subsemigroups in $(\mathcal{IS}_n, *)$

Let S be a semigroup. For the idempotent e in S define

$$\sqrt{e} = \{x \in S : x^m = e \text{ for some } m > 0\}.$$

Proposition 3 ([6])**.** *If $\sqrt{e}$ is a subsemigroup of S then $\sqrt{e}$ is a minimal with respect to inclusion isolated subsemigroup containing e.*

Denote $\mathcal{C}_A$ the set of those elements from $\mathcal{IS}_n$, which are one-to-one maps on A and are arbitrarily defined on $N \setminus A$. That is,

$$\mathcal{C}_A = \{\beta \in \mathcal{IS}_n \mid \beta(A) = A\}.$$

Theorem 4. *The only completely isolated subsemigroups of $(\mathcal{IS}_n, *)$ are $\mathcal{C}_A$, $(\mathcal{IS}_n, *) \setminus \mathcal{C}_A$ and $(\mathcal{IS}_n, *)$.*

Proof. It is clear that $\mathcal{C}_A$ is a subsemigroup of $(\mathcal{IS}_n, *)$. Let $\beta * \gamma \in \mathcal{C}_A$ for some β, γ from $(\mathcal{IS}_n, *)$. Then $\operatorname{dom}(\beta * \gamma) \supset A$, therefore $\beta(x) \in A$ for all $x \in A$. This implies $\beta \in \mathcal{C}_A$. Hence $\mathcal{C}_A$ is a completely isolated subsemigroup. $(\mathcal{IS}_n, *) \setminus \mathcal{C}_A$ is also completely isolated as a complement to the completely isolated subsemigroup. Obviously, $(\mathcal{IS}_n, *)$ is completely isolated.

Conversely, let $T \subset (\mathcal{IS}_n, *)$ be a completely isolated subsemigroup. Assume that $T \cap \mathcal{C}_A \neq \varnothing$, $\beta \in \mathcal{C}_A \cap T$. Then $\beta^{*s} = \alpha$ for some $s \in \mathbb{N}$ hence $\alpha \in T$. However, for any γ from $\mathcal{C}_A$ we have $\gamma^{*t} = \alpha$ for some $t \in \mathbb{N}$, hence $\gamma \in T$. Therefore

$$T \cap \mathcal{C}_A = \varnothing \quad \text{or} \quad \mathcal{C}_A \subset T. \tag{1}$$

Let $T \cap ((\mathcal{IS}_n, *) \setminus \mathcal{C}_A) \neq \varnothing$ and β belong to the intersection. Then T contains an idempotent ε as a power of β and $\operatorname{rank}(\varepsilon) < l$. From remark 2 it follows that since T is completely isolated $\varepsilon_x \in T$ for some $x \in A$. But for any $y \in A$ we have $\varepsilon_x = (x, y) * \varepsilon_y * (x, y)$. So if $\mathcal{C}_A \subset T$ then T contains ε_y, as it is a semigroup. Otherwise T contains ε_y as it is completely isolated. Consequently, T contains all idempotents of ranks $< l$, and since some power of each element $\gamma \in (\mathcal{IS}_n, *) \setminus \mathcal{C}_A$ equals such idempotent, we get

$$((\mathcal{IS}_n, *) \setminus \mathcal{C}_A) \subset T \quad \text{or} \quad ((\mathcal{IS}_n, *) \setminus \mathcal{C}_A) \cap T = \varnothing. \tag{2}$$

Now the statement of the theorem follows from (1) and (2). □

For every $x \in A$ denote

$$\mathcal{G}(x) = \{\lambda \in \mathcal{IS}_n \ : \lambda(x) \notin A \text{ and } \lambda(y) \in A \setminus \{x\} \text{ for all } y \in A \setminus \{x\}\}.$$

Observe, if $|A| = 1$ then $\mathcal{G}(x) = (\mathcal{IS}_n, *) \setminus \mathcal{C}_A$, $x \in A$.

Lemma 5. $\mathcal{G}(x) = \sqrt{\varepsilon_x}$.

Proof. It is clear that $\mathcal{G}(x)$ is a subsemigroup of $(\mathcal{IS}_n, *)$ and $\mathcal{G}(x) \subset \sqrt{\varepsilon_x}$. Further $\varepsilon_x \in \mathcal{G}(x)$ by definition of $\mathcal{G}(x)$.

Let $\lambda^{*k} \in \mathcal{G}(x)$ for some $k \in \mathbb{N}$. If $\lambda(x) \in A$ then whereas λ^{*k} is not one-to-one map on A there exists $y \in A \setminus \{x\}$ such that $\lambda(y) \in N \setminus A$. Then $\lambda^{*k}(y)$ is not defined for $k \geq 2$, which contradicts the definition of $\mathcal{G}(x)$. Now let $\lambda(x) \in N \setminus A$. Clearly, for $\lambda^{*k} \in \mathcal{G}(x)$ it is necessary $\lambda(y) \in A \setminus \{x\}$ for all $y \in A \setminus \{x\}$. Hence, $\lambda \in \mathcal{G}(x)$ and $\mathcal{G}(x)$ is isolated subsemigroup. Finally, proposition 3 completes the proof. □

Theorem 6. *(i) If* $\operatorname{rank}(\alpha) \geq 2$ *then the only isolated subsemigroups of* $(\mathcal{IS}_n, *)$ *are* $(\mathcal{IS}_n, *)$, $\mathcal{C}_A$, $(\mathcal{IS}_n, *) \setminus \mathcal{C}_A$ *and* $\mathcal{G}(x)$, $x \in A$.

(ii) If $\operatorname{rank}(\alpha) = 1$ *then the only isolated subsemigroups of* $(\mathcal{IS}_n, *)$ *are* $\mathcal{C}_A$, $(\mathcal{IS}_n, *) \setminus \mathcal{C}_A$, $(\mathcal{IS}_n, *)$, *in particular all of them are completely isolated.*

Proof. By theorem 4 and lemma 5 all listed subsemigroups are isolated.

Let T be isolated subsemigroup of $(\mathcal{IS}_n, *)$. As in the proof of theorem 4 it can be shown that (1) holds.

Assume that $\mathcal{C}_A \subset T$. If $T \neq \mathcal{C}_A$ then there exists $\beta \in T \setminus \mathcal{C}_A$ and T contains some idempotent ε of rank $< l$. Denote $A \setminus \operatorname{im}(\varepsilon) = \{a_1, \ldots, a_k\}$ and consider elements $\lambda = [a_1, \ldots, a_k]$ and $\mu = [a_k, \ldots, a_1]$. From $\lambda^{*k} = \mu^{*k} = \varepsilon$ it follows that $\lambda, \mu \in T$ and $\lambda * \mu = \varepsilon_{a_k}$ belongs to T. However as $\mathcal{C}_A \subset T$, T contains all ε_x, $x \in A$, and hence it contains all idempotents of ranks $\leq l - 1$. Therefore in this case $T = (\mathcal{IS}_n, *)$.

Now assume that $\mathcal{C}_A \cap T = \varnothing$ and T contains an element β and some idempotent ε of rank $< l$ as a power of β. Consider the cases.

1) The case of $l = 1$.
   Obviously, the power of any element from $(\mathcal{IS}_n, *) \setminus \mathcal{C}_A$ is a nowhere defined map, so $T = (\mathcal{IS}_n, *) \setminus \mathcal{C}_A$.

2) The case of $l \geq 2$.

   If $\operatorname{rank}(\varepsilon) \leq l - 2$ then by the analogous arguments produced above it can be shown that T contains at least two different idempotents ε_x

and ε_y, $x, y \in A$, $x \neq y$. We show that T contains all idempotents of rank $l-1$, and hence, $T = (\mathcal{IS}_n, *) \setminus \mathcal{C}_A$. Indeed, let $z \in A$ and $z \neq x, y$. Consider element $\mu = (x, z) * [y]$. $\mu^{*2} = \varepsilon_y$, hence $\mu \in T$. Then $(\varepsilon_x * \mu)^{*2} = [x] * [y] * [z] \in T$. Let $\sigma = [x, y, z]$, $\tau = [z, y, x]$. We have $\sigma^{*3} = \tau^{*3} = [x]*[y]*[z]$, hence $\sigma, \tau \in T$ and finally $\sigma * \tau = \varepsilon_z \in T$.

Now let $\mathrm{rank}(\varepsilon) = l - 1$. If ε is not the only idempotent in T then by the above reasoning $T = (\mathcal{IS}_n, *) \setminus \mathcal{C}_A$. If ε is the only idempotent in T then $T = \mathcal{G}(x)$.

The theorem is proved. □

2. Nilpotent subsemigroups in $(\mathcal{IS}_n, *)$

For every positive integer k denote Nil_k the set of all nilpotent subsemigroups of $(\mathcal{IS}_n, *)$ of nilpotency degree $\leq k$. The set Nil_k is partially ordered with respect to inclusions in a natural way. Set $M = \overline{A}^{(1)} \cup A \cup \overline{A}^{(2)}$, where $\overline{A}^{(1)} = N \setminus A$, and $\overline{A}^{(2)}$ is a disjoint copy of $\overline{A}^{(1)}$. For every $x \in \overline{A}^{(1)}$ denote x' the corresponding element from $\overline{A}^{(2)}$.

Denote $\mathrm{Ord}_k(M)$ the ordered set of all strict partial orders, Λ, on M which satisfy the following two conditions:

(1) the cardinalities of chains of Λ are bounded by k,

(2) $\overline{A}^{(1)} \subseteq \min(\Lambda)$, $\overline{A}^{(2)} \subseteq \max(\Lambda)$, where $\min(\Lambda)$ and $\max(\Lambda)$ mean the sets of all minimal and maximal elements of order Λ respectively.

If $k \leq m$ then we have natural inclusions $\mathrm{Nil}_k \hookrightarrow \mathrm{Nil}_m$ and $\mathrm{Ord}_k(M) \hookrightarrow \mathrm{Ord}_m(M)$, which preserve the partial order. Therefore we can consider the ordered sets

$$\mathrm{Nil} = \bigcup_k \mathrm{Nil}_k \quad \text{and} \quad \mathrm{Ord}(M) = \bigcup_k \mathrm{Ord}_k(M).$$

For every partial order $\Lambda \in \mathrm{Ord}(M)$ consider the set

$$\mathrm{Mon}(\Lambda) = \{\beta \in \mathcal{IS}_n : \beta(x) \neq x \text{ and } (x, \beta(x)) \in \Lambda \text{ for all } x \in \mathrm{dom}(\beta)\} \tag{3}$$

and for every subsemigroup $S \in \mathrm{Nil}$ the relation

$$\begin{aligned}\Lambda_S = {} & \{(x, y) : x \in A \text{ and there exists } \beta \in S \text{ such that } \beta(x) = y\} \\ & \cup \{(x, y) : y \in A \text{ and there exists } \beta \in S \text{ such that } \beta(x) = y\} \\ & \cup \{(x, y) : x \in \overline{A}^{(1)}, y \in \overline{A}^{(2)} \text{ and there exists } \beta \in S \text{ such that } \beta(x) = y\}.\end{aligned} \tag{4}$$

Let $\beta \in \mathcal{IS}_n$ and $\operatorname{rank}(\beta) = k$, $k \leq n$. Write β as

$$\beta = \begin{pmatrix} x_{11} & \dots & x_{1i_1} & x_{21} & \dots & x_{2i_2} & x_{31} & \dots & x_{3i_3} & x_{41} & \dots & x_{4i_4} \\ y_{11} & \dots & y_{1i_1} & y_{21} & \dots & y_{2i_2} & y_{31} & \dots & y_{3i_3} & y_{41} & \dots & y_{4i_4} \end{pmatrix}, \quad (5)$$

that is, $\operatorname{dom}(\beta) = \{x_{11}, \dots, x_{4i_4}\}$ and $\beta(x_{ij}) = y_{ij}$ for all i, j, moreover

$$\begin{array}{ll} \{x_{11}, \dots, x_{1i_1}\} \subset A, & \{y_{11}, \dots, y_{1i_1}\} \subset A, \\ \{x_{21}, \dots, x_{2i_2}\} \subset A, & \{y_{21}, \dots, y_{2i_2}\} \subset N \setminus A, \\ \{x_{31}, \dots, x_{3i_3}\} \subset N \setminus A, & \{y_{31}, \dots, y_{3i_3}\} \subset A, \\ \{x_{41}, \dots, x_{4i_4}\} \subset N \setminus A, & \{y_{41}, \dots, y_{4i_4}\} \subset N \setminus A. \end{array}$$

Consider a map $f : (\mathcal{IS}_n, *) \to \mathcal{IS}(M)$ defined in the following way:

if β is given by (5) then $\operatorname{dom}(f(\beta)) = \{x_{11}, \dots, x_{1i_1}, x_{21}, \dots, x_{4i_4}\}$ and

$$f(\beta) = \begin{pmatrix} x_{11} & \dots & x_{1i_1} & x_{21} & \dots & x_{2i_2} & x_{31} & \dots & x_{3i_3} & x_{41} & \dots & x_{4i_4} \\ y_{11} & \dots & y_{1i_1} & y'_{21} & \dots & y'_{2i_2} & y_{31} & \dots & y_{3i_3} & y'_{41} & \dots & y'_{4i_4} \end{pmatrix}.$$

Proposition 7. *The above map $f : (\mathcal{IS}_n, *) \to \mathcal{IS}(M)$ is a monomorphism, besides,*

$$\operatorname{Im}(f) = \{\gamma \in \mathcal{IS}(M) : \operatorname{dom}(\gamma) \subset \overline{A}^{(1)} \cup A, \operatorname{im}(\gamma) \subset A \cup \overline{A}^{(2)}\}.$$

Proof. Clear from the definition. □

Hence, every nilpotent subsemigroup of $(\mathcal{IS}_n, *)$ is mapped by f to the corresponding nilpotent subsemigroup of $\mathcal{IS}(M)$. This allows one to apply the results from [4] for $\mathcal{IS}(M)$ to the semigroup $(\mathcal{IS}_n, *)$. In particular, by a word for word repetition of corresponding proofs from [4] one may prove propositions 8-10:

Proposition 8. *(i) For each $k \geq 1$ the map $\Lambda \mapsto \operatorname{Mon}(\Lambda)$ is a homomorphism from the poset $\operatorname{Ord}_k(M)$ to the poset Nil_k.*

(ii) For every $k \geq 1$ the map $S \mapsto \Lambda_S$ is a homomorphism from the poset Nil_k to the poset $\operatorname{Ord}_k(M)$.

Proposition 9. *Let $n(S) = k$. Then $\Lambda_S \in \operatorname{Ord}_k(M) \setminus \operatorname{Ord}_{k-1}(M)$.*

Proposition 10. *Let $S \in \operatorname{Nil}$ and $\Lambda \in \operatorname{Ord}(M)$. Then*

(i) $\operatorname{Mon}(\Lambda_S) \supset S$, $\Lambda_{\operatorname{Mon}(\Lambda)} \subset \Lambda$;

(ii) $\operatorname{Mon}(\Lambda_{\operatorname{Mon}(\Lambda)}) = \operatorname{Mon}(\Lambda)$;

(iii) $\Lambda_{\mathrm{Mon}(\Lambda_S)} = \Lambda_S$.

From proposition 9 we derive

Corollary. *The nilpotency degree, $n(S)$, of any nilpotent subsemigroup $S \subset (\mathcal{IS}_n, *)$ does not exceed* $\mathrm{rank}(\alpha) + 2$*, moreover the equality holds true if and only if* $|N \setminus \mathrm{im}(\alpha)| \geq 1$ *and* $\min(\Lambda_S) = \overline{A}^{(1)}$, $\max(\Lambda_S) = \overline{A}^{(2)}$*. All maximal nilpotent subsemigroups are isomorphic.*

By an *ordered A-partition* of M into k non-empty blocks we mean the partition $M = M_1 \cup M_2 \cup \cdots \cup M_k$ where $M_1 \supset \overline{A}^{(1)}$, $M_k \supset \overline{A}^{(2)}$, and the order of the blocks is also taken into account. With every ordered A-partition of $M = M_1 \cup M_2 \cup \cdots \cup M_k$ we associate the set

$$\mathrm{ord}(M_1, \ldots, M_k) = \bigcup_{1 \leq i < j \leq k} M_i \times M_j \subset M \times M. \tag{6}$$

Lemma 11. *Let $k \leq |M|$ be fixed. Then*

(i) for every ordered A-partition $M = M_1 \cup M_2 \cup \cdots \cup M_k$ the set $\mathrm{ord}(M_1, \ldots, M_k)$ *is a maximal element in* $\mathrm{Ord}_k(M)$*;*

(ii) different ordered A-partitions of M correspond to different elements in $\mathrm{Ord}_k(M)$*;*

(iii) each maximal element in $\mathrm{Ord}_k(M)$ *has the form* $\mathrm{ord}(M_1, \ldots, M_k)$ *for some ordered A-partition of $M = M_1 \cup M_2 \cup \cdots \cup M_k$.*

Proof. Statements (i) and (ii) are proved analogously to lemma 7 from [4].

(iii) Let the order $\Lambda \in \mathrm{Ord}_k(M)$ be fixed. Denote by M_1 the set of all minimal elements of the relation Λ. By definition $\overline{A}^{(1)} \subset M_1$, therefore $M_1 \neq \varnothing$. For every increasing (with respect to Λ) chain $x_1 < \cdots < x_m$ of elements in $M \setminus M_1$ there exists $x_0 \in M$ such that $x_0 < x_1$. Hence, the cardinality of every increasing chain in $M \setminus M_1$ is bounded by $k-1$. Denote by M_2 the set of all minimal elements in $M \setminus M_1$, by M_3 the set of all minimal elements in $M \setminus (M_1 \cup M_2)$ and so on. In k steps we get the partition $M = M_1 \cup M_2 \cup \cdots \cup M_k$, for which $\overline{A}^{(1)} \subset M_1$. Observe that the elements of $\overline{A}^{(2)}$ are maximal by the definition of Λ. Hence set

$$\widetilde{M_i} = M_i \setminus \overline{A}^{(2)}, \quad 1 \leq i \leq k-1,$$
$$\widetilde{M_k} = M_k \cup \overline{A}^{(2)}$$

and get the new partition $M = \widetilde{M_1} \cup \widetilde{M_2} \cup \cdots \cup \widetilde{M_k}$. Clearly, $\Lambda \subset \mathrm{ord}(\widetilde{M_1}, \ldots, \widetilde{M_k})$ and from the maximality of Λ we have $\Lambda = \mathrm{ord}(\widetilde{M_1}, \ldots, \widetilde{M_k})$. The lemma is proved. □

For every ordered A-partition of $M = M_1 \cup M_2 \cup \cdots \cup M_k$ denote

$$T(M_1, \ldots, M_k) = \{\beta \in \mathcal{IS}_n : x \in M_i \text{ and } \beta(x) \in M_j \text{ imply } i < j \text{ for all } x \in \mathrm{dom}(\beta)\}. \quad (7)$$

Lemma 12.

$$T(M_1, \ldots, M_k) = \mathrm{Mon}(\mathrm{ord}(M_1, \ldots, M_k)).$$

Proof. The assertion follows from definitions (3), (4), (6) and (7). □

Theorem 13. *(i) For every ordered A-partition of $M = M_1 \cup M_2 \cup \cdots \cup M_k$ the semigroup $T(M_1, \ldots, M_k)$ is maximal in* Nil_k.

(ii) Different ordered A-partitions of M correspond to different subsemigroups in Nil_k.

(iii) Every maximal subsemigroup in Nil_k *has the form $T(M_1, \ldots, M_k)$ for some ordered A-partition of $M = M_1 \cup M_2 \cup \cdots \cup M_k$.*

Proof. From lemma 12 it follows that the set $T(M_1, \ldots, M_k)$ is a subsemigroup in Nil_k. We show that

$$\Lambda_{T(M_1,\ldots,M_k)} = \mathrm{ord}(M_1, \ldots, M_k). \quad (8)$$

From proposition 10(i) we have $\Lambda_{T(M_1,\ldots,M_k)} \subset \mathrm{ord}(M_1, \ldots, M_k)$.
We prove the contrary inclusion. Let $(x, y) \in \mathrm{ord}(M_1, \ldots, M_k)$. Take $\beta \in \mathcal{IS}_n$ such that $\mathrm{rank}(\beta) = 1$, $\mathrm{dom}(\beta) = \{x\}$, $\mathrm{im}(\beta) = \{y\}$. By definition $\beta \in \mathrm{Mon}(\mathrm{ord}(M_1, \ldots, M_k))$, and using (4) we have $(x, y) \in \Lambda_{\mathrm{Mon}(\mathrm{ord}(M_1,\ldots,M_k))}$. Hence $\Lambda_{T(M_1,\ldots,M_k)} \supset \mathrm{ord}(M_1, \ldots, M_k)$ and the equality (8) is proved.

Now let $S \in \mathrm{Nil}_k$ be such that $S \supset T(M_1, \ldots, M_k)$. According to lemma 11 the order $\mathrm{ord}(M_1, \ldots, M_k)$ is a maximal element in $\mathrm{Ord}_k(M)$, therefore by proposition 8(ii) we get $\Lambda_S = \mathrm{ord}(M_1, \ldots, M_k)$. Finally from proposition 10(i), lemma 12, and equality (8) it follows

$$S \subset \mathrm{Mon}(\Lambda_S) = \mathrm{Mon}(\mathrm{ord}(M_1, \ldots, M_k)) = T(M_1, \ldots, M_k)$$

and the statement (i) is proved.

Statements (ii) and (iii) follow from proposition 8, lemma 11 and lemma 12. □

Let S be maximal nilpotent subsemigroup of $(\mathcal{IS}_n, *)$ of nilpotency degree k and let $M = M_1 \cup M_2 \cup \cdots \cup M_k$ be the ordered A-partition of $M = \overline{A}^{(1)} \cup A \cup \overline{A}^{(2)}$, which corresponds to the partial order Λ_S. We call the set $(|M_1|, |M_2|, \ldots, |M_k|) \in \mathbb{N}^k$ *the type of nilpotent subsemigroup* S and denote it by $\mathrm{type}(S)$. Set $(|M_1|, \ldots, |M_k|)^{\#} = (|M_k|, \ldots, |M_1|)$.

Proposition 14. *Let T_1 and T_2 be two maximal nilpotent subsemigroups of $(\mathcal{IS}_n, *)$ of nilpotency degree k. Then*

(i) if $k = 2$ then T_1 and T_2 are isomorphic if and only if $\mathrm{type}(T_1) = \mathrm{type}(T_2)$ *or* $\mathrm{type}(T_1) = \mathrm{type}(T_2)^{\#}$.

(ii) if $k > 2$ then T_1 and T_2 are isomorphic if and only if $\mathrm{type}(T_1) = \mathrm{type}(T_2)$. *$T_1$ and T_2 are anti-isomorphic if and only if* $\mathrm{type}(T_1) = \mathrm{type}(T_2)^{\#}$.

The proof follows from proposition 7 and corresponding statements about nilpotent subsemigroups of $\mathcal{IS}(M)$ from [3] (lemma 14.4 and theorem 14.1).

□

References

[1] *The algebraic theory of automata, languages and semigroups* (under the editorship of M.A.Arbiba), M.:Statistics, 1975. (Russian)

[2] O. Ganyushkin, T. Kormysheva, *Isolated and nilpotent subsemigroups of a finite inverse symmetric semigroup*, Dopov. Akad. Nauk Ukrainy, N.**9**, 1993, pp. 5-9. (Ukrainian)

[3] A. Ganyushkin, V. Mazorchuk, *The full finite inverse symmetric semigroup* $\mathcal{IS}_n$, Preprint 2001:37, Göteborg University, Göteborg, Sweden, 2001.

[4] A. Ganyushkin, V. Mazorchuk, *On the structure of* $\mathcal{IO}_n$, Semigroup Forum, Vol.**66**, (2003), pp.455-483.

[5] E. Ljapin, *Semigroups*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

[6] V. Mazorchuk, G. Tsyaputa, *Isolated subsemigroups in the variants of* $\mathcal{T}_n$, Preprint 2005:09, Uppsala University, Uppsala, Sweden, 2005.

[7] J. Symons, *On a generalization of the transformation semigroup*, J. Austral. Math. Soc. **19**, 1975, pp. 47–61.

[8] G. Tsyaputa, *Transformation semigroups with the deformed multiplication*, Bulletin of the University of Kiev, Series: Physics and Mathematics, N.**4**, 2003, pp. 82–88.

[9] G. Tsyaputa, *Deformed multiplication in the semigroup* $\mathcal{PT}_n$, Bulletin of the University of Kiev, Series: Mechanics and Mathematics, N.**11-12**, 2004, pp. 35–38.

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