Algebra and Discrete Mathematics
Number 1. **(2006).** pp. 81 – 88

RESEARCH ARTICLE

Uncountably many non-isomorphic nilpotent real n-Lie algebras

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Communicated by V. M. Futorny

Abstract. There are an uncountable number of non-isomorphic nilpotent real Lie algebras for every dimension greater than or equal to 7. We extend an old technique, which applies to Lie algebras of dimension greater than or equal to 10, to find corresponding results for n-Lie algebras. In particular, for $n \geq 6$, there are an uncountable number of non-isomorphic nilpotent real n-Lie algebras of dimension $n + 4$.

Classifying nilpotent real Lie algebras has been an often studied subject since Engel. In 1962, Chao [1] proved that there are uncountably many such Lie algebras of dimension 10 and greater that are non-isomorphic. We shall prove an n-Lie algebra analogue of this theorem.

Before we proceed we recall the identities of n-Lie algebras as introduced by Fillipov [2]. An n-Lie algebra, is an algebra equiped with an n-linear, skew-symmetric bracket with the identity

$$[[x_1, x_2, \ldots, x_n]y_2 \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_2, \ldots, y_n], x_{i+1}, \ldots, x_n]$$

which we call the n-Jacobi identity. For further resuts, see [2], [3] and [4].

Theorem 1. *There are uncountably many non-isomorphic n-Lie algebras of dimensions d and nilpotent of length 2 when*
1) $n = 2$ *and* $d = 10$.
2) $n = 3$ *and* $d = 10$.

2000 Mathematics Subject Classification: *17A42.*

Key words and phrases: *n-Lie algebras, nilpotent, algebraically independent, transcendence degree.*

3) $n = 4$ *and* $d = 9$.
4) $n = 5$ *and* $d = 10$.
5) $n \geq 6$ *and* $d = n + 4$.

Definition 2. *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$. An n-Lie algebra A over R is said to be an $\mathbb{F}$-algebra if its structure constants with respect to some basis of A lie in $\mathbb{F}$.*

Let $\mathbb{F}$ be a subfield of $\mathbb{R}$ and $C^k_{i_1,i_2,\ldots,i_n}$ be real numbers in $\mathbb{F}$ such that $\sigma(C^k_{i_1,i_2,\ldots,i_n}) = C^k_{i_{\sigma(1)},i_{\sigma(2)},\ldots,i_{\sigma(n)}} = sgn\sigma(C^k_{i_1,i_2,\ldots,i_n})$ for all $\sigma \in S_n$, the symmetric group. Let A be an n-Lie algebra over $\mathbb{F}$ with a basis $(x_1, x_2, \ldots, x_\ell, y_1, y_2, \ldots, y_m)$ where $\ell \geq n$ and multiplication given by $[x_{i_1}, x_{i_2}, \ldots, x_{i_n}] = \sum_{k=1}^{n} C^k_{i_1,i_2,\ldots,i_n} y_k$ and all other products 0. Note that this fits the anti-symmetric condition as,

$$\begin{aligned}
\sigma([x_{i_1}, x_{i_2}, \ldots, x_{i_n}]) &= [x_{i_{\sigma(1)}}, x_{i_{\sigma(2)}}, \ldots, x_{i_{\sigma(n)}}] \\
&= \sigma(\sum_{k=1}^{n} C^k_{i_1,i_2,\ldots,i_n} y_k) \\
&= \sum_{k=1}^{n} \sigma(C^k_{i_1,i_2,\ldots,i_n}) y_k \\
&= \sum_{k=1}^{n} sgn\sigma(C^k_{i_1,i_2,\ldots,i_n}) y_k \\
&= sgn\sigma\left(\sum_{k=1}^{n} C^k_{i_1,i_2,\ldots,i_n} y_k\right) \\
&= sgn\sigma\big([x_{i_1}, x_{i_2}, \ldots, x_{i_n}]\big).
\end{aligned}$$

Lemma 3. *If the numbers $C^k_{i_1,i_2,\ldots,i_n}$, for $1 \leq i_1 < i_2 < \ldots < i_n \leq \ell$, and $1 \leq k \leq m$ are algebraically independent over $\mathbb{F}$ and if $\binom{\ell}{n} m > m^2 + \ell^2$, then A is not an $\mathbb{F}$-algebra.*

Proof of Lemma 3

We want to show that $A^2 =< y_1, \ldots, y_m >= Z(A)$. First we show that $A^2 =< y_1, \ldots, y_m >$. Since $\binom{\ell}{n} > m$ we can pick m distinct sets of n integers between 1 and ℓ. Each such set determines a set of vectors from $x_1, \ldots, x_\ell$ and we label these sets S_j, $j = 1, \ldots, m$. Let z_j be the product of the elements in S_j where the indices are arranged in increasing order in the product. As a result $z_j = \sum_{k=1}^{m} C^k_j y_k$ for $j = 1, \ldots, m$ where $C^k_j = C^k_{j_1,j_2,\ldots,j_n}$ if $x_{j_1}, x_{j_2}, \ldots, x_{j_n} \in S_j$. The polynomial $det(x_{ij})$ for $i, j = 1, \ldots, m$ has integer coefficients and thus lies in $\mathbb{F}[x_{11}, \ldots, x_{kj}, \ldots, x_{mm}]$. If $det(C^k_j) = 0$, then the C^k_j's are not algebraically independent, which is a contradiction. Therefore (C_j^k) is a non-singular matrix which generates $< y_1, \ldots, y_m >$ and hence $A^2 =< y_1, \ldots, y_m >$.

Now we show that $Z(A) =< y_1, \ldots, y_m >$. Since the only non-zero products in A are products of the x_i's it is clear that $< y_1, \ldots, y_m >\subset Z(A)$.

Let $z = \sum_{j=1}^{\ell} a_j x_j + \sum_{j=1}^{m} b_k y_k \in Z(A)$ and let $R_\pi = [\_, x_{i_2}, x_{i_3}, \ldots, x_{i_n}]$ then,

$$\begin{aligned} 0 &= zR_\pi \\ &= \left(\sum_{j=1}^{\ell} a_j x_j\right) R_\pi + \left(\sum_{k=1}^{m} b_k y_k\right) R_\pi \\ &= \sum_{j=1}^{\ell} a_j (x_j R_\pi) + 0 \\ &= \sum_{j=1}^{\ell}\sum_{k=1}^{m} a_j C_{j\pi}^k y_k \end{aligned}$$

where $C_{j\pi}^k = C_{ji_2i_3\ldots i_n}^k$.

By virtue of the linear independence of the y_k's we obtain $\sum_{j=1}^{\ell} a_i C_{j\pi}^k = 0$. For each $1 \leq t \leq \ell$ choose $\pi_t = t_2, \ldots, t_n$ such that $t_r \neq t_s$ for $r \neq s$ and $t \neq t_2, \ldots, t_n$. Then $\sum_{j=1}^{\ell} a_i C_{j\pi_t}^k = 0$. We observe that $C_{t\pi_t}^k \neq 0$ and $C_{t\pi_t}^k$ is in the algebraically independent set. Repeating this process for each t, $1 \leq t \leq \ell$ gives us a system of ℓ equations and ℓ unknowns. The coefficient matrix C has non-zero elements on the diagonal and hence are algebraically independent. Considering $det(x_{ij})$ as in the last paragraph, gives us a polynomial in ℓ^2 variables with coefficients ± 1. If C is singular then the elements of C satisfy $det(x_{ij})$. The non-zero elements of C satisfy a polynomial obtained from $det(x_{ij})$ by deleting terms if necessary, from any elements of C that are 0. The resulting polynomial is non-zero because of the non-zero diagonal of C. This non-zero polynomial is satisfied by a set of algebraically independent elements. This is a contradiction and hence C is non-singular. As a result $a_1 = a_2 = \ldots = a_\ell = 0$ and $z = \sum_{j=1}^{\ell} a_j x_j + \sum_{j=1}^{m} b_k y_k = \sum_{j=1}^{m} b_k y_k \in< y_1, \ldots, y_m >$. Thus $Z(A) =< y_1, \ldots, y_m >= A^2$.

Now we prove lemma 3. Suppose, to the contrary, that A satisfies the conditions of the lemma. Namely, A is an $\mathbb{F}$-algebra with basis $(z_1, \ldots, z_\ell, z_{\ell+1}, \ldots, z_{\ell+m})$ and structure constants $D_{i_1,i_2,\ldots,i_n}^k$'s for $1 \leq i_1, i_2, \ldots, i_n \leq \ell$ and $1 \leq k \leq \ell + m$. We can assume without loss of generality that $(z_1, \ldots, z_\ell)$ form a basis for C a compliment of A^2. We can write $z_{\ell+i} = v_i + t_i$ for all i where $v_i \in C$ and $t_i \in A^2$ for $i = 1, \ldots, m$.

We observe

$$[z_{i_1}, z_{i_2}, \ldots, z_{i_n}] = \\ = \sum_{r=1}^{\ell} D^r_{i_1,i_2,\ldots,i_n} z_r + \sum_{s=\ell+1}^{\ell+m} D^s_{i_1,i_2,\ldots,i_n} v_{s-\ell} + \sum_{s=\ell+1}^{\ell+m} D^s_{i_1,i_2,\ldots,i_n} t_{s-\ell}$$

Since $[z_{i_1}, z_{i_2}, \ldots, z_{i_n}] \in A^2$ we see the first two summands must be 0 and we obtain

$$\begin{aligned}[z_{i_1}, z_{i_2}, \ldots, z_{i_n}] &= \sum_{s=\ell+1}^{\ell+m} D^s_{i_1,i_2,\ldots,i_n} t_{s-\ell} \\ &= \sum_{u=1}^{m} D^{u+\ell}_{i_1,i_2,\ldots,i_n} t_u.\end{aligned}$$

As a result $(z_1, \ldots, z_\ell, t_{\ell+1}, \ldots, t_{\ell+m})$ is a new basis for A whose structure coefficients are a subset of the structure coefficients for the old basis. We observe that $(x_1, \ldots, x_\ell)$ is a basis for C' a compliment of A^2. Now let s_i be such that $s_i - z_i \in A^2$ and $s_i \in C'$ for $1 \leq i \leq \ell$. We obtain yet another basis $(s_1, \ldots, s_\ell, t_1, \ldots, t_m)$ which has the same structure coefficients as $(z_1, \ldots, z_\ell, t_{\ell+1}, \ldots, t_{\ell+m})$.

Indeed,

$$\begin{aligned}[s_{i_1}, s_{i_2}, \ldots, s_{i_n}] &= [z_{i_1} + A^2, z_{i_2} + A^2, \ldots, s_{i_n} + A^2] \\ &= [z_{i_1} + Z(A), z_{i_2} + Z(A), \ldots, z_{i_n} + Z(A)] \\ &= [z_{i_1}, z_{i_2}, \ldots, z_{i_n}].\end{aligned}$$

Since $(s_1, \ldots, s_\ell)$ and $(x_1, \ldots, x_\ell)$ both form a basis for C' there exists a non-singular matrix, $B = (b_{ip})$ such that $s_i = \sum_{p=1}^{\ell} b_{ip} x_p$ for all $1 \leq i \leq \ell$. Likewise there exists a non-singular $G = (g_{ur})$ such that $t_u = \sum_{r=1}^{m} g_{ur} y_r$ for all $1 \leq u \leq m$. Substituting into

$$[z_{i_1}, \ldots, z_{i_n}] = [s_{i_1}, \ldots, s_{i_n}] = \sum_{u=1}^{m} D^{\ell+u}_{i_1,\ldots,i_n} t_u$$

we observe for all $1 \leq i_1, i_2, \ldots, i_n \leq \ell$

$$\begin{aligned}
[s_{i_1}, s_{i_2}, \ldots, s_{i_n}] &= \left[\sum_{p_1=1}^{\ell} b_{i_1p_1}x_{p_1}, \sum_{p_2=1}^{\ell} b_{i_2p_2}x_{p_2}, \ldots, \sum_{p_n=1}^{\ell} b_{i_np_n}x_{p_n}\right] \\
&= \sum_{p_1=1}^{\ell}\sum_{p_2=1}^{\ell}, \ldots, \sum_{p_n=1}^{\ell}\Big(b_{i_1p_1}b_{i_2p_2}, \ldots, b_{i_np_n}[x_{p_1}, x_{p_2}, \ldots, x_{p_n}]\Big) \\
&= \sum_{p_1=1}^{\ell}\sum_{p_2=1}^{\ell}, \ldots, \sum_{p_n=1}^{\ell}\Big(b_{i_1p_1}b_{i_2p_2}, \ldots, b_{i_np_n}\sum_{r=1}^{m} C^r_{p_1,p_2,\ldots,p_n}y_r\Big) \\
&= \sum_{u=1}^{m} D^{\ell+u}_{i_1,i_2,\ldots,i_n}t_u \\
&= \sum_{u=1}^{m}\sum_{r=1}^{m} D^{\ell+u}_{i_1,i_2,\ldots,i_n}g_{ur}y_r.
\end{aligned}$$

This implies that for fixed $i_1, i_2, \ldots, i_n$ and r we obtain

$$\sum_{p_1=1}^{\ell}\sum_{p_2=1}^{\ell}\ldots, \sum_{p_n=1}^{\ell} b_{i_1p_1}b_{i_2p_2}, \ldots, b_{i_np_n}C^r_{p_1,p_2,\ldots,p_n} = \sum_{u=1}^{m} D^{\ell+u}_{i_1,i_2,\ldots,i_n}g_{ur}.$$

We claim that this in turn implies that,

$$C^r_{p_1,p_2,\ldots,p_n} = \sum_{p_1=1}^{\ell}\sum_{p_2=1}^{\ell}\ldots, \sum_{p_n=1}^{\ell}\sum_{u=1}^{m} D^{\ell+u}_{i_1,i_2,\ldots,i_n}g_{ur}\bar{b}_{p_1i_1}\bar{b}_{p_2i_2}, \ldots, \bar{b}_{p_ni_n}$$

where $B^{-1} = [\bar{b}_{ip}]$.
We show the t^{th} step. Suppose for
$1 \leq p_1, p_2 \ldots p_{t-1} \leq \ell$ and $1 \leq i_t, \ i_{t+1} \ \ldots \ i_n \leq \ell$ and r fixed that

$$\begin{aligned}
&\sum_{p_t=1}^{\ell}\sum_{p_{t+1}=1}^{\ell}, \ldots, \sum_{p_n=1}^{\ell} b_{i_tp_t}b_{i_{t+1}p_{t+1}}, \ldots, b_{i_np_n}C^r_{p_1,p_2,\ldots,p_n} \\
= &\sum_{i_1=1}^{\ell}\sum_{i_2=1}^{\ell}\ldots, \sum_{i_{t-1}=1}^{\ell} \bar{b}_{p_1i_1}\bar{b}_{p_2i_2}\ldots\bar{b}_{p_{t-1}i_{t-1}}\sum_{u=1}^{m} D^r_{i_1i_2\ldots i_n}g_{ur}.
\end{aligned}$$

Let

$$A_{p_t} = \sum_{p_{t+1}=1}^{\ell}\sum_{p_{t+2}=1}^{\ell}\ldots, \sum_{p_n=1}^{\ell} b_{i_{t+1}p_{t+1}}b_{i_{t+2}p_{t+2}}, \ldots, b_{i_np_n}C^r_{p_1,p_2,\ldots,p_n}$$

for $p_t = 1, \ldots, \ell$ and

$$E_{i_t} = \sum_{i_1=1}^{\ell} \sum_{i_2=1}^{\ell} \ldots, \sum_{i_{t-1}=1}^{\ell} \overline{b}_{p_1 i_1} \overline{b}_{p_2 i_2} \ldots \overline{b}_{p_{t-1} i_{t-1}} D^r_{i_1 i_2 \ldots i_n} g_{ur}$$

for $i_t = 1, 2, \ldots, \ell$.

This implies that

$$b_{i_t 1} A_1 + b_{i_t 2} A_2 + \ldots + b_{i_t \ell} A_\ell = E_{i_t}$$

or

$$\begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1\ell} \\ b_{21} & b_{22} & \ldots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{\ell 1} & b_{\ell 2} & \ldots & b_{\ell\ell} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_\ell \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_\ell . \end{bmatrix}$$

So

$$A_{p_t} = \overline{b}_{p_t 1} E_1 + \overline{b}_{p_t 2} E_2 + \ldots + \overline{b}_{p_t n} E_n$$

and

$$A_{p_t} = \sum_{i_1=1}^{\ell} \sum_{i_2=1}^{\ell} \ldots, \sum_{i_t=1}^{\ell} \overline{b}_{p_1 i_1} \overline{b}_{p_2 i_2} \ldots \overline{b}_{p_t i_t} D^r_{i_1 i_2 \ldots i_n}.$$

Finally,

$$\sum_{p_{t+1}=1}^{\ell} \sum_{p_{t+2}=1}^{\ell} \ldots, \sum_{p_n=1}^{\ell} b_{i_{t+1} p_{t+1}} b_{i_{t+2} p_{t+2}}, \ldots, b_{i_n p_n} C^r_{p_1, p_2, \ldots, p_n}$$
$$= \sum_{i_1=1}^{\ell} \sum_{i_2=1}^{\ell} \ldots, \sum_{i_t=1}^{\ell} \overline{b}_{p_1 i_1} \overline{b}_{p_2 i_2} \ldots \overline{b}_{p_t i_t} D^r_{i_1 i_2 \ldots i_n}.$$

This proves the claim.

The claim implies that $C^r_{p_1, p_2, \ldots, p_n} \in \mathbb{E} = \mathbb{F}(b_{ip}, g_{ur})$. But the degree of transcendence of $\mathbb{E}$ over $\mathbb{F}$ is at most $\ell^2 + m^2$ which is less than, $\binom{\ell}{n} m$, the number of $C^r_{p_1, p_2, \ldots, p_n}$'s. This a contradiction and hence A is not an $\mathbb{F}$-algebra, proving the lemma.

Proof of Theorem 1

It is known that there exists a set S of uncountably many real numbers that are algebraically independent over $\mathbb{Q}$. We can divide S into uncountably many disjoint subsets $\{C^k_{i_1,i_2,\ldots,i_n}\}_\alpha$ of size $\binom{\ell}{n}m$ where α distinguishes subsets. Define the n-Lie algebra A_α with basis $(x_1, x_2, \ldots, x_\ell, y_1, y_2, \ldots, y_m)$ and multiplication given by

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_n}] = \sum_{k=1}^{n} C^k_{i_1,i_2,\ldots,i_n} y_k$$

and all other products 0 where $C^k_{i_1,i_2,\ldots,i_n} \in \{C^k_{i_1,i_2,\ldots,i_n}\}_\alpha$ for all $1 \leq i_1, i_2, \ldots, i_n \leq \ell$ and $1 \leq k \leq m$. For $\alpha \neq \beta$ we claim that A_α and A_β are non-isomorphic. Indeed, since the $(C^k_{i_1,i_2,\ldots,i_n})_\alpha$'s are algebraically independent over $\mathbb{Q}[\{C^k_{i_1,i_2,\ldots,i_n}\}_\beta]$, if we apply lemma 3 to A_α, we see that it is not a $\mathbb{Q}[(C^k_{i_1,i_2,\ldots,i_n})_\beta]$-algebra. Hence A_α and A_β are non-isomorphic as claimed.

To prove the theorem it remains to find for each given n and d in 1-4, an m and k where $d = n + m + k$ such that $f(k, m, n) = \binom{n+k}{n}m - (n + k)^2 - m^2 > 0$. We do this case by case.

1) When $k = m = 4$, we obtain $f(4, 4, n) = 1/6n^4 + 5/3n^3 + 29/6n^2 + 1/3n - 28$. The only positive root is approximately $n = 1.807126451$. Hence $f(4, 4, n) > 0$ if $n \geq 2$. Setting $n = 2$ gives $d = n + m + k = 10$. This coincides with Chao's result.
2) When $k + m = 3 + 4 = 7$, we obtain $f(3, 4, n) = 2/3n^3 + 3n^2 + 4/3n - 21$. The only positive root is approximately $n = 2.046172397$. Hence $f(3, 4, n) > 0$ if $n \geq 3$. Hence $f(3, 4, n) > 0$ if $n \geq 3$. Setting $n = 3$ gives $d = n + m + k = 10$.
3) When $k = 3, m = 2$, we obtain $f(3, 2, n) = 1/3n^3 + n^2 - 7/3n - 11$. The only positive root is $n = 3$. Hence $f(3, 2, n) > 0$ if $n \geq 4$. Setting $n = 4$ gives $d = 9$ and setting $n = 5$ gives $d = 10$.
4) When $k = 3, m = 1$, we obtain $f(3, 1, n) = 1/6n^3 - 25/6n - 9$. The only positive root is approximately $n = 5.850622760$. Hence $f(3, 1, n) > 0$ if $n \geq 6$. Thus $d = n + m + k = n + 4$.

Note that if $n \geq 6$, then $d - n$ cannot be less than 4. That is to say, we have found the minimal $k + m$ such that $f(k, m, n) > 0$. If we set $k + m < 4$, we get no solutions. Indeed, if $k = 0$ or $m = 0$, we obtain $f(k, m, n) = m - m^2 - (n)^2 \leq 0$ and $f(k, m, n) = -(n + k)^2 \leq 0$. For $m + k = 1 + 1 = 2$ we obtain $f(k, m, n) = -n - n^2 - 1$ which has no real roots. For $m + k = 2 + 1 = 3$ and $m + k = 1 + 2 = 3$ we obtain $f(k, m, n) = -1/2n^2 - 5/2n - 4$ and $f(k, m, n) = -3 - n^2$ neither of which have real roots and are always negative.

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