Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 1. **(2006).** pp. 67 – 80

Classification of inverse semigroups generated by two-state partially defined invertible automata over the two-symbol alphabet

Janusz Konrad Slupik

Communicated by V. I. Sushchansky

Abstract. The classification of inverse semigroups generated by two-state partially defined invertible automata over a two-symbol alphabet is investigated. Two presentations of such semigroups are given. The structures of these semigroups are analyzed.

Introduction

The problem of classification of (semi)groups generated by Mealy automata have been actively investigated for the last few years. It seems natural to study (semi)groups defined by finite automata with small parameters (numbers of states and symbols in the alphabet). R. I. Grigorchuk proved that there exist 6 pairwise nonisomorphic groups generated by two-state invertible Mealy automata over a two-symbol alphabet (finite groups: the trivial group, the cyclic group of order two and the Klein four-group; infinite groups: the infinite cyclic group, the infinite dihedral group and the lamplighter group, see [2]). I. I. Reznykov and V. I. Sushchansky proved that there are 29 pairwise nonisomorphic semigroups and six aforementioned groups generated by fully defined two-state automata over a two-symbol alphabet (see [3]).

In paper [4], we started to investigate inverse semigroups generated by two-state partially defined invertible automata over a two-symbol al-

2000 Mathematics Subject Classification: *20M18; 20M35, 68Q35.*

Key words and phrases: *inverse semigroup, Mealy automata,partially defined Mealy automata, automaton transformations.*

phabet. Namely, we proved that such semigroups are finite. The aim of this paper is to give a full classification of these inverse semigroups.

Let us denote the set of all partially defined invertible Mealy automata with m states over some n-symbol alphabet by $IPMA(m, n)$.

The computer counting was applied to investigate isomorphisms of inverse semigroups generated by automata from $IPMA(2, 2)$. The calculation was made in Institute of Mathematics of Silesian University of Technology. The following results were obtained.

Theorem 1. *Let S be an inverse semigroup defined by some automaton from $IMPA(2, 2)$. Then $|S|$ is a number from the set*

$$M = [1, 16] \cup [18, 20] \cup [23, 26] \cup$$

$$\cup \{28, 30, 32, 36, 39, 43, 52, 59, 89, 149, 357, 388\} \ .$$

For every number $m \in M$ there exists an automaton $A \in IPMA(2, 2)$ such that $|S_A| = m$.

Theorem 2. *There exist 59 pairwise nonisomorphic inverse semigroups generated by automata from $IPMA(2, 2)$.*

The complete list of these 59 inverse semigroups is given in the last section of the paper. We give two descriptions of such semigroups: as subsemigroups of symmetric inverse semigroups, and as presentation with generators and determining relations. We present some structural aspects of such semigroups, which are expressed in theorem 3.

1. Preliminaries

The standard terminology and notations from the theory of inverse semigroups (see [5, 6, 7]) is used.

Let X be a nonempty finite set (an alphabet) and let X^* be the free monoid over X. Its subsemigroup consisting of all nonempty words is denoted by X^+. If $\omega = x_1 x_2 ... x_n \in X^*$, then $|\omega| = n$ is the length of the word ω. The length of the empty word is equal to zero. We denote the domain of any partial transformation f by the symbol $\mathrm{dom}(f)$ and the image of f we denote by $\mathrm{im}(f)$.

Let $A = \langle X, Q, \pi, \lambda \rangle$ be a Mealy automaton, where X is the alphabet of input and output symbols ($|X| < \infty$), Q is the set of internal states (not necesserly finite), $\pi : X \times Q \to Q$ and $\lambda : X \times Q \to X$ are its transition and output functions respectively.

Definition 1. *The automaton $A = \langle X, Q, \pi, \lambda\rangle$ is called a partially defined automaton if its transition function π or output function λ are partially defined, i.e.*

$$\mathrm{dom}(\pi) \cap \mathrm{dom}(\lambda) \neq X \times Q \ .$$

Hence, the set of patially defined automata and the set of fully defined automata are disjoint.

Each state $q \in Q$ of a Mealy automaton determines a certain partial transformation $\lambda_q : X \to X$ defined by the formula $\lambda_q(x) = \lambda(x, q)$.

The automaton A with a fixed (initial) state $q \in Q$ is called an initial automaton and is denoted by (A, q).

Definition 2. *An automaton A is called invertible if λ_q is a partial permutation on X for every $q \in Q$.*

Let $A = \langle X, Q, \pi, \lambda\rangle$ be an invertible partially defined automaton. Let $A' = \langle X, Q, \pi', \lambda'\rangle$ be a partially defined automaton such that the following conditions hold

i) $\lambda'_q(x) = \lambda_q^{-1}(x)$ for every $q \in Q$,

ii) if $(\lambda_q^{-1}(x), q) \in \mathrm{dom}(\pi)$, then $\pi'(x, q) = \pi(\lambda_q^{-1}(x), q)$ otherwise $(x, q) \notin \mathrm{dom}(\pi')$.

Definition 3. *The automaton A' is called an inverse automaton to A.*

The functions π and λ can be naturally extended to mappings $\bar\pi$, $\bar\lambda$ of subsets of the $X^* \times Q$ into the sets Q and X^* by the following equalities:

$$\bar\pi(e, q) = q, \quad \bar\pi(\omega x, q) = \pi(x, \bar\pi(\omega, q)),$$

$$\bar\lambda(e, q) = e, \quad \bar\lambda(\omega x, q) = \bar\lambda(\omega, q)\lambda(x, \bar\pi(\omega, q)),$$

where e is the empty word, $q \in Q$, $\omega \in X^*$, $x \in X$. Moreover, if $(\omega, q) \notin \mathrm{dom}\ \bar\lambda$ then $(\omega x, q) \notin \mathrm{dom}\ \bar\lambda$.

Now, let us introduce the partial transformation $f_{A,q} : X^* \to X^*$ determined by the partially defined automaton A at the state $q \in Q$ in the following way

$$f_{A,q}(u) = \bar\lambda(u, q),$$

where $u \in X^*$.

Definition 4. *A partial transformation $f : X^* \to X^*$ is called an automaton transformation if there exists a finite automaton $A = \langle X, Q, \pi, \lambda\rangle$ and a state $q \in Q$ such that $f = f_{A,q}$.*

The partially defined Mealy automaton A with the set of states Q defines the set of partial transformations $T_A = \{f_{A,q}\ ;\ q \in Q\}$ on the set X^*.

Definition 5. *The semigroup $S_A = \langle T_A \rangle$ is called the semigroup of partial transformations which is defined by the automaton A with the set of internal states Q.*

If A is an invertible partially defined automaton, then for every $q \in Q$ the transformation $f_{A,q}$ is a partial permutation over X^*. In this case we denote by S_A the inverse semigroup generated by such automaton, i.e.

$$S_A = \left\langle f_{A,q}, f_{A,q}^{-1}\ ;\ q \in Q \right\rangle .$$

Now, let us adduce very useful properties of automaton transformations.

Lemma 1. *Let $f : X^* \to X^*$ be a partial transformation. The automaton transformation f holds*
a) f preserves the lengths of words, i.e. if $\omega \in \mathrm{dom}(f)$ then $|f(\omega)| = |\omega|$,
b) f preserves the beginnings of words, i.e. if $f(u) = u'$ then $f(uv) = u'v'$ for $v \in X^*$ such that $uv \in \mathrm{dom}(f)$,
c) if $u \notin \mathrm{dom}(f)$, then $ux \notin \mathrm{dom}(f)$ for every $x \in X$.

Let $A_1 = \langle X, Q_1, \pi_1, \lambda_1 \rangle$ and $A_2 = \langle X, Q_2, \pi_2, \lambda_2 \rangle$ be two automata.

Let us construct a new automaton $B = A_1 * A_2$ with the set of states $Q_1 \times Q_2$ and the transition function π and the output function λ defined by the equalities, respectively

$$\pi(x, (q_1, q_2)) = \big(\pi_1(x, q_1), \pi_2(\lambda_1(x, q_1), q_2)\big) ,$$

$$\lambda(x, (q_1, q_2)) = \lambda_2\big(\lambda_1(x, q_1), q_2\big) .$$

The automaton B is called the composition of automata A_1 and A_2. If we consider initial automata (A_1, q_1^0) and (A_2, q_2^0), then B is the automaton with the initial state (q_1^0, q_2^0).

It is clear that, if $\omega \in X^*$ is a word such that $\omega \in \mathrm{dom}\big(f_{(A_1,q_1^0)}\big)$ and $f_{(A_1,q_1^0)}(\omega) \in \mathrm{dom}\big(f_{(A_2,q_2^0)}\big)$, then

$$f_{(A_1 * A_2,(q_1^0,q_2^0))}(\omega) = f_{(A_2,q_2^0)}\big(f_{(A_1,q_1^0)}(\omega)\big) .$$

Hence, the superposition of partial automaton transformations is a partial automaton transformation.

Definition 6. *Two initial automata* (A_1, q_1) *and* (A_2, q_2) *are called equivalent if they define the same partial transformations, i.e.* $f_{(A_1,q_1)} = f_{(A_2,q_2)}$.

There exist well known algorithms that determine whether or not two given finite automata are equivalent. One of the basis facts in the theory of finite automata is the existence of a unique automaton in the class of equivalent automata that has the minimal number of states, so called the minimal automaton. The procedure of determination of minimal automaton for (A, q) is called the reduction of (A, q). Let us denote the result of this procedure by $\mathrm{red}\big((A, q)\big)$.

In the sequel of this section we recall the essential definitions.

Let S be an inverse semigroup. We will denote by $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ Green's relations on S. Let $E(S)$ denote the set of all idempotents of a semigroup S.

Definition 7. *An inverse semigroup* S *is called a Clifford semigroup if for every* $s \in S$ $ss^{-1} = s^{-1}s$.

It is equivalent that every $\mathcal{H}$-class of the inverse semigroup S is a group.

Definition 8. *An inverse semigroup* S *is said to be* E*-reflexive if for every* $s, t \in S$

$$st \in E(S) \;\Leftrightarrow\; ts \in E(S)\ .$$

Definition 9. *An inverse semigroup* S *(without zero) having a single* D*-class is said to be bisimple. An inverse semigroup with zero* S *is called 0-bisimple if it contains two* $\mathcal{D}$*-classes.*

2. Algorithms

The set of all two-state partially defined Mealy automata over the two-symbol alphabet has 6305 elements. The set $IPMA(2, 2)$ of all partially defined invertible Mealy automata with 2 states over 2-symbol alphabet has 3905 elements. We introduce an equivalence relation on the set $IPMA(2, 2)$ by setting two automata to be equivalent if they can be obtained from each other by permutations of the states or letters of the alphabet.

Let $A \in IPMA(2, 2)$ and λ, π are its output and transition functions respectively. If $(x, q) \notin \mathrm{dom}(\lambda)$ for certain $x \in X$ and $q \in Q$, then $f_{A,q}$ does not depend on value $\pi(x, q)$. Hence, we can add to the equivalence relation pairs of automata which differ in this way.

Straightforward calculation shows that $IPMA(2,2)$ divides into 209 classes of equivalence. It is enough to investigate representative automata from each class.

Of course, with an enormous machine calculations of this nature, one inevitably asks how far the result can be trusted, and whether it is likely that a small logical or other error in the code could have resulted in an incorrect final result. This is extremely unlikely in this case, for the following reason. All of the calculations have been made by two completely different versions of the program and yielded the same results.

The algorithm 1 was used to investigate the orders of inverse semigroups generated by automata from $IPMA(2,2)$.

Algorithm 1.
1. Input: Given an automaton A from $IPMA(2,2)$.
2. Let L be a list of elements of S_A. Now, let $L = \emptyset$. Setting $M = \{(A, q_0), (A, q_1), (A^{-1}, q_0), (A^{-1}, q_1)\}$.
3. Reduction of elements from M and adding different one to L.
4. $n := |L|$
5. For every automaton $(B, s) \in L$ and $(C, q) \in M$, the following superpositions are callculated $(D, s') = (B, s) * (C, q)$.
Next:
    5.1. The reduction of the automaton (D, s'),
    5.2. If $\{\text{red}\big((D, s')\big)\} \cap L = \emptyset$, then $\text{red}\big((D, s')\big)$ is added to L.
6. After step 5 if $|L| = n$, then the algorithm stops and returns: the list L and the number n. Otherwise go to 4.
7. Ouput: $|S_A|$ and the list L of different elements of S_A, (each element of S_A is represented by some minimal initial automaton from L).

Algorithm 2.
1. Input: Given the list L of different elements of S_A. (L contains inital automata)
2. The Cayley table is made according to multiplication defined by the rule: for each (A, q) and (B, q') from L,
$$(A, q) \circ (B, q') = \text{red}\left((A, q) * (B, q')\right)$$
3. Ouput: the Cayley table T_A.

Algorithm 3.
1. Input: Given Cayley tables T_A and T_B.
2. For every element from S_A and S_B the frequency of appearance in the table is counted. Making the lists of all frequences $L(S_A)$ and $L(S_B)$.
3. Ordering the lists $L(S_A)$, $L(S_B)$ and replacing elements in T_A and T_B according to the lists.
4. If $L(S_A)$ has different elements than $L(S_B)$, then the semigroups are not isomorphic and the algorithm stops.
Otherwise:
4.1. Permuting elements from $L(S_A)$ which have the same frequency.
4.2. Enumerating the table T_A according to such permutation.
4.3. Compering the tables: T_A and T_B. If the tables are the same, then the semigroups are isomorphic and the algorithm stops.
5. If for every possible permutation described in point 4.1. the Cayley tables are different, then the semigroups are not isomorphic. The algorithm stops.
6. Ouput: the answer 'isomorphic' or 'nonisomorphic'.

Algorithm 4.
1. Input: Given the Cayley table T_A.
2. According to the Wagner-Preston theorem, partial permutations are made.
3. Output: two partial permutations.

Algorithm 5.
1. Input: Given the Cayley table T_A.
2. Let n be a dimension of the table T_A.
For arbitrary element $s \in S_A$, we denote by $w(s)$ element s rewritten as a product of generators.
3. We construct n^2 relations by the rule:
if $a \circ b = c$ where a, b, c are elements of T_A,
then we have a relation $w(a)w(b) = w(c)$.
4. We apply preliminary reduction according to patterns:
- relations $uu^{-1}u = u$ are deleted,
- if $u = v$, then relation $u^{-1} = v^{-1}$ is deleted,
- if $uu = u$, then relation $u = u^{-1}$ is deleted,
- if $u = v$, then relations $aub = avb$ are deleted,

where u, v, a, b are words in the alphabet consisted of generators.
5. Output: the set of relations.

The isomorphism test (i.e. whether or not two given inverse semigroups defined by Cayley tables are isomorphic), is complex. The analysis of n element semigroup gives rise to checking $n!$ permutations of rows and columns of Cayley tables. In the case of inverse semigroups generated by automata from $IPMA(2,2)$, the algorithm 3 was proposed to solve that problem.

3. Results of calculations

Our investigations can be briefly described in the following way. First, for every automaton $A \in IPMA(2,2)$ a list of elements in S_A is made by the algorithm 1. Next for each inverse semigroup S_A the Cayley table is made by the algorithm 2. Then, we use the algorithm 3, which generates a list of non isomorphic inverse semigroup. Next, the algorithm 4 is used and the charts presentations are made. The reduction of the lengths of partial permutations is made manually. Finally, we use the algorithm 5, which made lists of relations. The presentations with generator sets and relations are made by manual reduction from obtained lists of relations.

We use the following symbols: n - serial number, o - order of a semigroup, z - zero, i - the identity, e - number of idempotents in a semigroup.

n	o	z	i	e
1	1	0	1	1
2	2	_	1	1
3	2	0	1	2
4	3	0	1	2
5	3	0	_	3
6	3	_	1	2
7	4	_	1	2
8	4	_	1	2
9	5	0	_	3
10	6	0	_	4
11	6	0	1	4
12	6	_	_	3
13	7	0	_	4
14	7	0	1	4
15	7	_	1	4
16	8	0	_	5
17	8	_	_	4
18	8	_	1	4
19	9	0	_	5
20	9	0	_	5

n	o	z	i	e
21	10	0	_	6
22	10	0	_	6
23	10	0	_	6
24	10	0	_	6
25	10	_	_	4
26	11	0	_	6
27	11	0	_	6
28	11	0	1	4
29	12	_	_	6
30	12	_	1	4
31	13	0	_	7
32	14	0	_	6
33	14	_	_	7
34	15	0	_	8
35	16	_	_	8
36	18	0	_	8
37	19	0	_	10
38	19	0	_	9
39	19	_	_	6
40	20	_	_	10

n	o	z	i	e
41	23	0	_	11
42	23	0	_	12
43	23	_	_	8
44	24	_	_	12
45	25	_	_	9
46	26	0	_	10
47	28	0	_	11
48	30	0	_	12
49	32	_	_	10
50	36	0	_	13
51	36	_	_	12
52	39	0	_	17
53	43	0	_	17
54	52	0	_	17
55	59	_	_	17
56	89	0	_	27
57	149	_	_	27
58	357	0	_	66
59	388	_	_	66

Our notation is slightly different from that of [6]. We will write every cycle of the length one and we will omit paths of the length one. This convention gives us shorter chart decompositions of generators.

According to the above numeration of inverse semigroups, the following presentations are written:

n	*charts*	
1	$f = 0$	$g = 0$
2	$f = (1,2)$	$g = (1,2)$
3	$f = (1)$	$g = 0$
4	$f = (1,2)$	$g = 0$
5	$f = (1)$	$g = (2)$
6	$f = (1)(2)(3)$	$g = (2,3)$
7	$f = (1,3)(2)(4)$	$g = (2,4)$
8	$f = (1,3)(2,4)$	$g = (2,4)$
9	$f = (2,1]$	$g = 0$
10	$f = (1)(2)$	$g = (3,2]$
11	$f = (1)(2)(3)$	$g = (3,2]$
12	$f = (4,1](2,3)$	$g = (2,3)$
13	$f = (3,1](2,4)$	$g = 0$
14	$f = (1,3)(2,4)$	$g = (2)$
15	$f = (1)(2)(3)(4)$	$g = (3,1](2,4)$
16	$f = (1,3)(4,5)$	$g = (2)(5)$
17	$f = (5,1](2,4)(3,6)$	$g = (2,4)$
18	$f = (1,3)(2,4)(5,6)$	$g = (2)(5)(6)$
19	$f = (3,1](4,2]$	$g = (2)$
20	$f = (1,3)(4,6)$	$g = (2,5)(6)$
21	$f = (4,1](2)(3)$	$g = (5,2]$
22	$f = (3,1](2)(4)$	$g = (4,2]$
23	$f = (2)(4)$	$g = (3,1](4,2]$
24	$f = (1)(2)(4)$	$g = (3,1](2,4]$
25	$f = (5,1](2,4)(3,6,8,7)$	$g = (2)(4)$
26	$f = (5,1](2,4)(3,6)$	$g = (2)$
27	$f = (1,3)(4,6)$	$g = (5,2](4,6]$
28	$f = (1,3)(2,4)(5,6)$	$g = (2,5)$
29	$f = (5,1](2,4)(3,7)(6,8)$	$g = (4,2](6,8)$
30	$f = (1,3)(2,4)(5,6)(7,8)$	$g = (2,5)(7)(8)$
31	$f = (4,1](6,3]$	$g = (5,2](3,6]$
32	$f = (5,1](2,4](6,7]$	$g = (8,1](3,4](7)$
33	$f = (5,1](3,7)(8,4]$	$g = (6,2](3,7)(4,8]$

34	$f = (5,1](6,2](3,7)(4,8)$	$g = (6,2](4,8]$
35	$f = (5,1](6,2](3,8)(4,9)(7,10)$	$g = (6,2](4,9](7,10)$
36	$f = (1,3)(4,6)(7,10)(8,11)$	$g = (2,5)(6,12)(9,11)(10)$
37	$f = (6,1](3,8)(4,9)(10,5]$	$g = (7,2](4,9](10,5]$
38	$f = (3,1](6,2](4)(5)(7)(8)$	$g = (7,2](8,5]$
39	$f = (1,4](3,10)(6)(8)$	$g = (2,4](12,3](6,8)(9,10]$
40	$f = (6,1](3,4](5,10)(7,9)$	$g = (8,2](4,3](10,5](7,9)$
41	$f = (7,1](8,2](3)(4)(5)(6)$	$g = (10,2](9,5](6,3]$
42	$f = (1,4](2,5](6,3)$	$g = (4,1](2,5](3,6]$
43	$f = (6,4)(11,7)(3,10)(9,1)$	$g = (2,5)(8,6)(10)(1,11)(3)$
44	$f = (12,1](6,4](3,5](11,7)(9,10)$	$g = (8,2](4,6](3,5](11,7)(9,10]$
45	$f = (1)(2)(3,4)(5)(6)(8,9](7)$	$g = (1,2)(3,6](4,5](8,7]$
46	$f = (7,2](3,5](4)(1,8](9,10](6)$	$g = (9,2](3,8](6,4]$
47	$f = (7,2](13,4)(6,5)(8,1](17,10](9,11]$	$g = (12,2](3,11](1,10)(6)$
48	$f = (1)(2)(3,4](5,6]$	$g = (1,2](3,5](4,6]$
49	$f = (8,2](6,5)(9)(1,3](4,10)(7)$	$g = (1,2](10,5](7,9)(4,6]$
50	$f = (1,2)(3,4,5,6]$	$g = (1)$
51	$f = (6,5](1)(4)(10,9)(3,2](7,8)$	$g = (5,2](1,4)(8,9](6,3](7,10]$
52	$f = (1,4)(6,3](7,5](9,10]$	$g = (1)(2,3](8,5)(10)$
53	$f = (2,4](5)(11,1](10)(8)$	$g = (7,4](5,8](9,10](6,1](3,2]$
54	$f = (1,2)(4,5,3,6]$	$g = (1)(3)$
55	$f = (5,3,1,2](7,9,8,4)(6,10)$	$g = (1)(4)(6)(9)(10)$

56	$f = (1)(2)(3)(4)(5,6](7)(8)(9,10](11,13]$ $(12,14](15,16](20,18](19,21]$ $g = (1,2](3,5](4,6](7,11](8,12](9,14](15,17](20,21]$
57	$f = (1)(2)(3,4)(5)(6)(7,9](8,10](11,12)(13)(14)(15,20](16,19]$ $(17,22](23,25](24,26](27,28)(30)(18,21](29,31]$ $g = (1,2)(3,5](4,6](7,12](8,11](9,13](10,14](15,25](16,26]$ $(19,28](20,27](22,30](21,31]$
58	$f = (1,2)(4,5,3,6](12,7,14](9,11,8,13]$ $(16,29](25,15,30](20,23,18](19,24,17,27]$ $g = (1)(3,4)(8,9,7,10](17,19,16,21](20,15,22]$
59	$f = (1,2)(3,5,4,6)(13,8,11,9](14,7,12,10](16,15,17]$ $(19,18,21,20](25,24,26](32,31,33,30](37,38,27](39,29]$ $g = (1)(2)(3,4)(7,9,8,10)(22,15,20,18](23,24,27](28,29,30,31]$

Now, we have the following presentations with generators and relations.

n	*presentation*
1	$\langle f, g; f = g = 0\rangle$
2	$\langle f, g; f = g, f^2 = 0\rangle$
3	$\langle f, g; f = 1, g = 0\rangle$
4	$\langle f, g; f^2 = 1, g = 0\rangle$
5	$\langle f, g; f = f^2, g = g^2, fg = 0\rangle$
6	$\langle f, g; f = 1, g = g^3\rangle$
7	$\langle f, g; f^2 = 1, g = g^{-1}, fg = g\rangle$
8	$\langle f, g; f^2 = 1, g = g^{-1}, gf = fg = g^2\rangle$
9	$\langle f, g; f^2 = 0, g = 0\rangle$
10	$\langle f, g; f^2 = f, gf = 0, fg = g\rangle$
11	$\langle f, g; f = 1, g^2 = 0\rangle$
12	$\langle f, g; f^2 = f^4, g = f^3\rangle$
13	$\langle f, g; f^2 = f^4, g = 0\rangle$
14	$\langle f, g; f^2 = 1, g^2 = g, gfg = 0\rangle$
15	$\langle f, g; f = 1, g^2 = g^4\rangle$
16	$\langle f, g; f = f^{-1}, g^2 = g, gfg = 0, gf^2 = f^2g\rangle$
17	$\langle f, g; f^2 = f^4, g = g^{-1}, g = gf^2, gf = fg = g^2\rangle$
18	$\langle f, g; f^2 = 1, g = g^2, gfgf = fgfg\rangle$
19	$\langle f, g; f^2 = 0 = fg, g = g^2, ff^{-1}g = g\rangle$
20	$\langle f, g; f = f^{-1}, g = g^{-1}, gfg = 0, gf^2 = f^2g, fg = fg^2\rangle$
21	$\langle f, g; f^2 = f^3, g = fg = f^{-1}g, fg^{-1} = gf = 0\rangle$
22	$\langle f, g; f^2 = f^3, g^2 = 0, fg = gf = g = f^{-1}g = gf^{-1}\rangle$
23	$\langle f, g; f = f^2, g^2 = 0, fg = gf\rangle$
24	$\langle f, g; f = f^2, g^2 = 0, fg = g\rangle$
25	$\langle f, g; f^2 = f^{-2}, g = g^2, gf = fg = f^{-1}g, gf^2 = g\rangle$
26	$\langle f, g; f^2 = f^4, g = g^2 = gf^2, gfg = 0, fg = f^{-1}g\rangle$
27	$\langle f, g; f = f^{-1}, g^2 = 0, gf = fg^{-1}, fg = g^{-1}f, gf^2 = gfg\rangle$
28	$\langle f, g; f^2 = 1, g = g^{-1}, gfg = 0\rangle$
29	$\langle f, g; f^2 = f^4, g^2 = g^4, gf = gf^{-1} = gg^{-1}, fg = f^{-1}g = g^{-1}g\rangle$
30	$\langle f, g; f^2 = 1, g = g^{-1}, fgfg = gfgf, gfg = g^2fg = gfg^2\rangle$
31	$\langle f, g; f^2 = g^2 = 0 = fg^{-1} = f^{-1}g, (gf)^2 = gf, (fg)^2 = fg\rangle$
32	$\langle f, g; f^2 = 0 = fg = fg^{-1}, g^2 = g^3, gg^{-1} = ff^{-1}, gf = g^{-1}gf\rangle$
33	$\langle f, g; f^2 = f^4 = g^2 = fg^{-1} = g^{-1}f, gff^{-1} = gfg,$ $gf^{-1}f = gf^2 = f^2g, fg = g^{-1}f^{-1}\rangle$
34	$\langle f, g; f^2 = f^4, g^2 = 0, g = gf^{-1}f = ff^{-1}g, gf^{-1} = gg^{-1},$ $gff^{-1} = f^{-1}fg = gf^2 = gfg = f^2g\rangle$
35	$\langle f, g; f^2 = f^4, g^2 = g^4, g = gf^{-1}f = ff^{-1}g, gf^{-1} = gg^{-1}$ $g^{-1}g = f^{-1}g, gfg = gff^{-1} = f^{-1}fg\rangle$

36	$\langle f,g; f=f^{-1}, g=g^{-1}, gfg=0, f^2g^2=g^2f^2, fgf^2=fgf^2g\rangle$
37	$\langle f,g; f^2=f^4, g^2=0, gff^{-1}=gf^2=gfg=f^2g=f^{-1}fg,$ $ff^{-1}g=gf^{-1}g=gf^{-1}f\rangle$
38	$\langle f,g; f^2=f^3, g^2=0, g=gf=gf^{-1}, fg=f^2g=gg^{-1}fg\rangle$
39	$\langle f,g; f^2=f^4, g^2=g^4, gg^{-1}=ff^{-1}, gfg=gfgf=fg^2,$ $gf=gf^2=gf^{-1}=fgf=f^{-1}g^{-1}\rangle$
40	$\langle f,g; f^2=f^4, g^2=g^4, gfg=gff^{-1}=f^{-1}fg,$ $gf^2=gf^{-1}f=ff^{-1}g=gf^{-1}g, g^3=fg^2=g^2f\rangle$
41	$\langle f,g; f^2=f^3, g^2=0=gfg, gf=gf^{-1}=gf^2=fgf,$ $g=ff^{-1}g, fg^{-1}=fg^{-1}f, fg^{-1}g=g^{-1}gf,$ $fgg^{-1}=gg^{-1}f^{-1}=fgg^{-1}f\rangle$
42	$\langle f,g; f^2=f^4, g^2=0, gf^{-1}g=gf^{-1}f=ff^{-1}g,$ $gfg=f^{-1}fg=gff^{-1}\rangle$
43	$\langle f,g; f=f^{-1}, g=g^{-1}, g^2f^2=f^2g^2, fgfg=gfgf,$ $gfg^2=gfg=f^2gfg, gf^2gf^2=f^2gf^2g\rangle$
44	$\langle f,g; f^2=f^4, g^2=g^4, gf^2=f^2g, gfg=gff^{-1}=f^{-1}fg,$ $gf^{-1}g=gg^{-1}f=gf^{-1}f=ff^{-1}g, g^3=g^2f=fg^2\rangle$
45	$\langle f,g; f^2=f^4, g^2=g^4, g=gf=gf^{-1}=ff^{-1}g, f^{-1}fg=f^2g,$ $fgg^{-1}=gg^{-1}f^{-1}, g^{-1}f^{-1}g=g^{-1}fg\rangle$
46	$\langle f,g; f^2=f^3, g^2=0=gf^{-1}g, gf=fg=gff^{-1}=f^{-1}fg,$ $g=gf^{-1}f=ff^{-1}g, gg^{-1}f^{-1}=fgg^{-1}, g^{-1}gf=f^{-1}g^{-1}g\rangle$
47	$\langle f,g; f^2=f^4, g^2=g^4, gfg=0=gf^{-1}g, g=ff^{-1}g,$ $f^{-1}fg=gf^{-1}f=gf^2=f^2g\rangle$
48	$\langle f,g; f^2=f^3, g^2=0, gf=fg, gf^{-1}=f^{-1}g\rangle$
49	$\langle f,g; f^2=f^4, g^2=g^4, gf=fg, g=gf^{-1}f=ff^{-1}g\rangle$
50	$\langle f,g; f^4=f^6, g=g^2=gf^2=f^2g=gff^{-1}=gf^{-1}f, gfg=0\rangle$
51	$\langle f,g; f^2=f^4, g^2=g^4, gf=fg, gf^{-1}=f^{-1}g, gf^2=f^2g\rangle$
52	$\langle f,g; f^2=f^4, g^2=g^4, gfg=gf^{-1}g=0, fg^{-1}=fg,$ $gf^2=f^2g=gf^{-1}f=f^{-1}fg\rangle$
53	$\langle f,g; f^2=f^3, g^2=0=gfg, g^{-1}gf=fg^{-1}g,$ $gf=gf^{-1}=gf^2=fgf\rangle$
54	$\langle f,g; f^4=f^6, g=g^2=gff^{-1}=gf^{-1}f, gfg=0,$ $f^2g=gf^{-2}=gf^4, gf^3=gf^{-3}=gf^5, gf=gf^2f^{-1}\rangle$
55	$\langle f,g; f^4=f^8, g=g^2=gff^{-1}=f^{-1}fg=gf^2f^{-2},$ $gfg=gf^{-1}g, gf^4=f^4g=f^2gf^2=gf^{-2}f^2, gfgf=fgfg\rangle$
56	$\langle f,g; f^2=f^3, g^2=0=gfg=gf^{-1}g, g=gf^{-1}f,$ $fg=fgf=fgf^{-1}, f^{-1}gf=gg^{-1}f^{-1}gf\rangle$
57	$\langle f,g; f^2=f^4, g^2=g^4=g^2f=gfg=gf^{-1}g=gf^2g=fg^2,$ $fgf^{-1}=fgf, g=gf^{-1}f, gf=gf^3, f^2g=f^2gf,$ $gfg^{-1}=gf^{-1}g^{-1}, g^{-1}gf=f^{-1}g^{-1}g, fgg^{-1}f=f^2gg^{-1}\rangle$

58*	$\langle f,g; f^4=f^6, g^4=g^6, gfg=0=gf^{-1}g=gfg^{-1}=g^{-1}fg,$ $gf^3g=0=gf^{-3}g, gf^3=g^{-1}f^3=gf^5=f^2g^2f,$ $g=gf^{-1}f, gf^4=gf^4g=f^4g, (f^{-1}gf)^2=f^{-1}gf=f^2f^{-1}gf,$ $(g^{-1}f^2)^2=g^{-1}f^2, g^{-1}g^2=f^{-1}fg, f^{-2}fg=f^{-1}g,$ $(gf^2)^2=gf^2, (f^2g)^2=f^2g, ...\rangle$
59*	$\langle f,g; f^4=f^8, g^4=g^8, (gf^2)^2=gf^2, (f^2g)^2=f^2g,$ $gfg=gf^{-1}g=gfg^{-1}=g^{-1}fg=gf^3g=gf^{-3}g,$ $gfg=gfg^2=g^{-1}f^{-1}g^{-1}, g^{-1}f^2=f^{-2}g, f^{-1}fg=g^{-1}g^2,$ $gfgf=fgfg, gf^4=f^4g=f^4g^{-1}, gf^3=g^{-1}f^3,$ $g=gf^{-1}f, f^{-1}gf=f^{-1}g^{-1}f, gf^5=ff^{-1}gf=f^2g^2f,$ $g^{-1}f^{-1}f^2=g^{-1}f, (f^{-1}gf)^2=f^2f^{-1}gf, ...\rangle$

(* - these two presentations may have not enough relations)

According to the above numeration of semigroups, we can formulate the following theorem:

Theorem 3. *Let S be an inverse semigroup generated by an automaton from $IPMA(2,2)$.*

a) S is a Clifford semigroup if, and only if S have a number from 1÷8,

b) S is bisimple if, and only if S have a number 1 or 2,

c) S is 0-bisimple if, and only if S have a number 3, 4 or 9,

d) S is E-reflexive if, and only if S have a number 1÷8, 12, 13, 15, 17,18, 25, 29, 30, 33, 35, 39, 40, 44, 45, 49, 51, 55, 57, 59.

References

[1] C.K. Gupta, V.I. Sushchansky, *Semigroups of Automatic Transformations, in: Topics in Infinite Groups*, Quaderni di Matematica, **8** Aracne, Napoli 2001, 205-218.

[2] R.I. Grigorchuk, V.V. Nekrashevych, V.I. Sushchansky, *Automata, Dynamical Systems, and Groups*, Proceedings of the Steklov Institute of Mathematics, **231** 2000, 134-214.

[3] I.I. Reznykov, V.I. Sushchansky, *Growth functions of semigroups defined by Mealy automata with two states over two-letter alphabet*, Report of Ukrainian NAN 2002, N2, 23-27.

[4] J.K. Slupik, V.I. Sushchansky, *Inverse semigroups generated by two-state partially defined automata*, Contr. Gen. Alg. 16, Klagenfurt 2005, 261-274.

[5] M.V. Lawson, *Inverse semigroups: The Theory of Partial Symmetries*, World Scientific, Singapore 1998.

[6] S.L. Lipskomb, *Symmetric inverse semigroups*, Mathematical Surves and monographs, 46, AMS, Providence, RI, 1996, 166 p.

[7] M. Petrich, *Inverse semigroups*, John Wiley & Sons Inc., New York 1984.

Contact information

J. K. Slupik
Institute of Mathematics, Silesian University of Technology, ul. Kaszubska 23, Gliwice, Poland
E-Mail: Janusz.Slupik@polsl.pl
URL: 157.158.16.184/js/