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RESEARCH ARTICLE

Isomorphisms of Cayley graphs of surface groups

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Abstract. A combinatorial proof is given for the fact that the Cayley graph of the fundamental group Γ_g of the closed, orientable surface of genus $g \geq 2$ with respect to the usual generating set is isomorphic to the Cayley graph of a certain Coxeter group generated by $4g$ elements.

1. Introduction

The fundamental group of the closed, orientable surface of genus $g \geq 2$ in its usual presentation is

$$\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle. \qquad (1)$$

It is an open question, whether the spectral radius of the simple random walk on Γ_g with respect to the symmetric generating set

$$V_g = \{a_1, a_1^{-1}, b_1, b_1^{-1}, \ldots, a_g, a_g^{-1}, b_g, b_g^{-1}\} \qquad (2)$$

is an algebraic number. Bounds on this spectral radius have been obtained by several authors; see [2], [1], [7] and [3].

For a group G with symmetric generating set S, let $\mathcal{G}(G, S)$ denote the resulting Cayley graph. The spectral radius of the random walk mentioned above depends only on the Cayley graph of $\mathcal{G}(\Gamma_g, V_g)$. It well known to experts that there is a graph isomorphism from the Cayley

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graph of this surface group onto the Cayley graph $\mathcal{G}(G_{4g}, W_{4g})$, where G_{4g} is the Coxeter group

$$G_{4g} = \langle s_1, \dots, s_{4g} \mid s_i^2 = 1,\ (s_i s_{i+1})^{2g} = 1,\ 1 \le i \le 4g \rangle, \tag{3}$$

with all subscripts of s taken modulo $4g$, and where

$$W_{4g} = \{s_1, \dots, s_{4g}\}. \tag{4}$$

This result is of interest in part because it opens new avenues for techniques of free probability theory (see [6]) to be applied to the random walks on the surface groups. See the appendix, where a geomtric proof (that was kindly shown to us by J.G. Ratcliffe) is given. In this paper we provide an elementary combinatorial proof of this graph isomorphism. This proof utilizes some techniques involving free monoids that may be of further interest.

We now summarize the contents of this paper. In §2, we show that certain bijections between free monoids induce isomorphisms of Cayley graphs. In §3, we prove the graph isomorphism

$$\mathcal{G}(\Gamma_g, V_g) \cong \mathcal{G}(G_{4g}, W_{4g}) \tag{5}$$

in the case $g = 2$. The proof of this special case is simpler than and motivates our proof of the general case. Parts of the Caylay graphs $\mathcal{G}(\Gamma_2, V_2)$ and $\mathcal{G}(G_8, W_8)$ are drawn in Figures 1 and 2, and these drawings motivate our construction of an isomorphism. In §4, we construct an isomorphism (5) for general $g \ge 2$.

2. Certain maps yielding isomorphisms of Cayley graphs

For $i \in \{1, 2\}$, let G_i be a group with symmetric generating set $S_i \subseteq G_i$. Let S_i^* denote the free monoid on S_i. We will let $|w|$ denote the length of a word $w \in S_i^*$. Take $R_i \subseteq S_i^* \times S_i^*$ and let $C_i \subseteq S_i^* \times S_i^*$ denote the congruence generated by R_i, namely, the translation–invariant equivalence relation generated by R_i; suppose that G_i is the quotient of S_i^* by C_i, i.e. that $G_i = \mathrm{Mon}\langle S_i \mid R_i \rangle$ is a presentation of G_i as a monoid. (See, for example, Chapter 1 of [5] for basic facts about monoid presentations.)

The following lemma follows directly from the definition of a congruence.

Lemma 2.1. *Let $\psi : S_1^* \to S_2^*$ and suppose that whenever $(u, v) \in R_1$ and $w, z \in S_1^*$, we have $(\psi(wuz), \psi(wvz)) \in C_2$. Then $(\psi \times \psi)(C_1) \subseteq C_2$.*

Proposition 2.2. *Suppose $\psi : S_1^* \to S_2^*$ is a bijection such that*

(i) if $w \in S_1^*$ and $x \in S_1$, then there is $y \in S_2$ such that $\psi(wx) = \psi(w)y$,

(ii) $(\psi \times \psi)(C_1) = C_2$.

Then, considering the quotients $G_i = S_i^*/C_i$, ψ descends to a bijection from G_1 onto G_2 that implements a graph isomorphism from the Cayley graph $\mathcal{G}(G_1, S_1)$ onto the Cayley graph $\mathcal{G}(G_2, S_2)$.

Proof. Let $S_i^*(n)$ denote the set of words belonging to S_i having length n. Taking $w = \emptyset$ to be the empty word in S_1^*, from (i) we get $\psi(S_1) \subseteq S_2^*(m+1)$, where m is the length of $\psi(\emptyset)$. Moreover, using induction on n, we see $\psi(S_1^*(n)) \subseteq S_2^*(m+n)$ for all $n \in \mathbf{N}$. Using that ψ is a bijection, we conclude that $m = 0$ and

$$\psi(S_1^*(n)) = S_2^*(n).$$

Let $\tilde{w} \in S_2^*$ and $y \in S_2$. Let $n = |\tilde{w}|$ and let $w = \psi^{-1}(\tilde{w})$. Since $|\psi^{-1}(\tilde{w}y)| = n+1$, we have $\psi^{-1}(\tilde{w}y) = w'x$ for some $w' \in S_1^*(n)$. By (i), $\tilde{w}y = \psi(w'x) = \psi(w')\tilde{y}$ for some $\tilde{y} \in S_2$. But we must have $\tilde{w} = \psi(w')$ and $\tilde{y} = y$. To summarize, we have shown the analogue of (i) for ψ^{-1}, namely:

$$\forall \tilde{w} \in S_2^* \; \forall y \in S_2 \; \exists x \in S_1 \text{ such that } \psi^{-1}(\tilde{w}y) = \psi^{-1}(\tilde{w})x. \tag{6}$$

Let $\overline{\psi} : G_1 \to G_2$ denote the map of equivalence classes induced by ψ. Clearly, $\overline{\psi}$ is a bijection. Consider any edge of $\mathcal{G}(G_1, S_1)$; its endpoints are g and gx for some $g \in G_1$ and $x \in S_1$. Let $w \in S_1^*$ be a representative of g. By (i), there is $y \in S_2$ such that $\psi(wx) = \psi(w)y$. Therefore, $\overline{\psi}(gx) = \overline{\psi}(g)y$. So $\overline{\psi}(gx)$ and $\overline{\psi}(g)$ are the endpoints of an edge of $\mathcal{G}(G_2, S_2)$. Arguing similarly, but using (6) instead of (i), we see that the endpoints of any edge of $\mathcal{G}(G_2, S_2)$ get mapped by ψ^{-1} to the endpoints of an edge of $\mathcal{G}(G_1, S_1)$. Thus, $\overline{\psi}$ implements an isomorphism of Cayley graphs. □

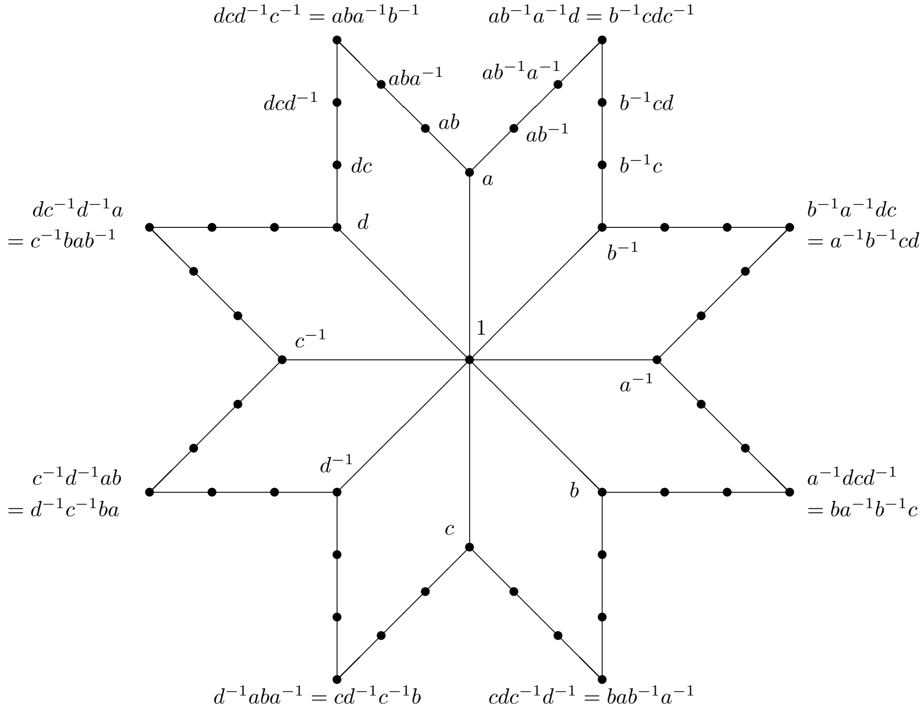
3. The genus 2 case

Consider the fundumental group of the closed orientable surface of genus $g = 2$ in its usual presentation:

$$\Gamma_2 = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$$

and take the symmetric generating set $V = \{a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}\}$. Part of the Cayley graph $\mathcal{G}(\Gamma_2, V)$ around the identity element is drawn

Figure 1: Part of the Cayley graph $\mathcal{G}(\Gamma_2, V)$



in Figure 1. (We have chosen to draw Cayley graphs with respect to multiplication on the right.) Consider also the Coxeter group

$$G_8 = \langle s_1, \dots, s_8 \mid s_i^2 = (s_i s_{i+1})^4 = 1, \ (1 \le i \le 8) \rangle,$$

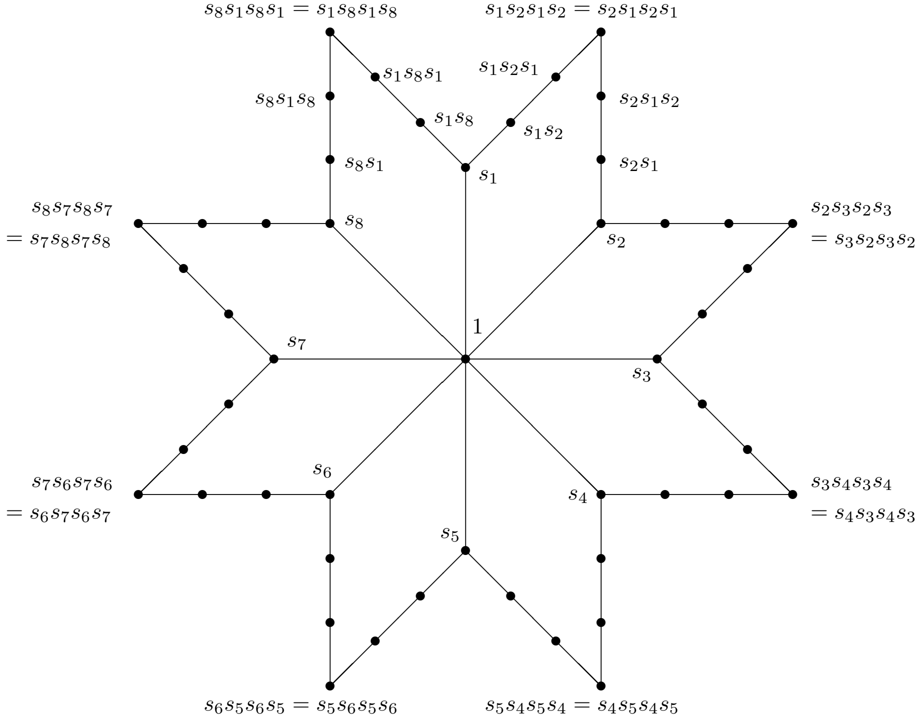
where the subscript in s_{i+1} is to be taken modulo 8. Take the symmetric generating set $W = \{s_1, \dots, s_8\}$. Part of the Cayley graph $\mathcal{G}(G_8, W)$ around the identity element is drawn in Figure 2.

Theorem 3.1. *The Cayley graphs $\mathcal{G}(\Gamma_2, V)$ and $\mathcal{G}(G_8, W)$ are isomorphic.*

Proof. Let V^* and W^* be the free monoids on generating sets V and W, respectively. Note that we continue to use the notation a^{-1}, etc., for elements of V, even though in the monoid V^* they are not invertible. Thus, for example, $ab^{-1}bc$ and ac are distinct elements of V^*. We will keep this notation because we will use the order–two permutation $x \mapsto x^{-1}$ of V. Monoid presentations of the groups Γ_2 and G_8 are

$$\Gamma_2 = \mathrm{Mon}\langle V \mid R_\Gamma \rangle, \qquad G_8 = \mathrm{Mon}\langle W \mid R_G \rangle,$$

Figure 2: Part of the Cayley graph $\mathcal{G}(G_8, W)$



where

$$
\begin{aligned}
R_\Gamma &= \{(aba^{-1}b^{-1}, dcd^{-1}c^{-1})\} \cup \{(xx^{-1}, \emptyset) \mid x \in V\} \subseteq V^* \times V^* \\
R_G &= \{(s_is_i, \emptyset) \mid 1 \le i \le 8\} \cup \{(s_is_{i+1}s_is_{i+1}, s_{i+1}s_is_{i+1}s_i) \mid 1 \le i \le 8\} \subseteq \\
&\subseteq W^* \times W^*,
\end{aligned}
$$

where $\emptyset$ denotes the empty word, i.e. the identity element of the monoid V^* or W^* and where the subscript in s_{i+1} should be taken modulo 8. We will construct a bijection $\psi : W^* \to V^*$ and use Proposition 2.2 to show that ψ implements an isomorphism of Cayley graphs. Let $C_\Gamma \subseteq V^* \times V^*$ and $C_G \subseteq W^* \times W^*$ be the congruence relations generated by R_Γ and, respectively, R_G.

An inspection of the drawings in Figures 1 and 2 suggests an obvious relation (of being "neighbors") on the generating sets V and W, respectively. Namely, two generators are neighbors if there is an octagon in the Cayley graph that contains both of them. These relations of V and, respectively, W are encoded in the octagons of Figures 3 and 4. As suggested by these figures, let us define the bijection $\eta : \{1, \dots, 8\} \to V$

Figure 3: A relation on the generators of Γ_2

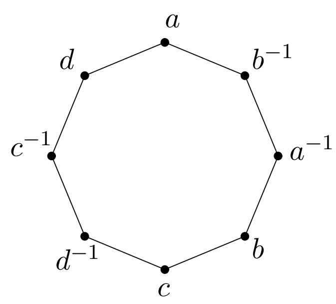
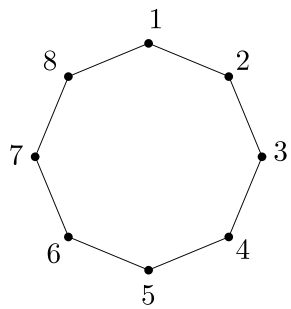


Figure 4: A relation on the generators of G_8



by

$$\eta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ a & b^{-1} & a^{-1} & b & c & d^{-1} & c^{-1} & d \end{pmatrix}.$$

and set $\psi(s_i) = \eta(i)$. Suppose we try to define $\psi(s_1 s_j) = \psi(s_1)\gamma(j) = a\gamma(j)$ for some bijection $\gamma : \{1, \ldots, 8\} \to V$. Inspecting Figures 1 and 2, we see that we need

$$\gamma(1) = a^{-1}, \qquad \gamma(2) = b^{-1}, \qquad \gamma(8) = b. \tag{7}$$

But we also want γ to send neighbors in Figure 4 to neighbors in Figure 3. The values (7) are, therefore, sufficient to determine γ. We have $\gamma = \eta \circ \tau$, where

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 8 & 7 & 6 & 5 & 4 \end{pmatrix}$$

is the permutation arising as the reflection of the octagon in Figure 4 through the axis containing vertices 2 and 6. Similarly, we are led to define $\psi(s_2 s_j) = b^{-1}\eta(\sigma(j))$, where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 2 & 1 & 8 & 7 & 6 \end{pmatrix}$$

is the permutation arising from the reflection of the octagon through the axis containing vertices 3 and 7. Exploring further, we are led to the recursive definition of ψ described below.

Let $\rho = \tau\sigma$. Then ρ is the rotation of the octagon through angle $\pi/2$. Let H be the group generated by σ and τ. Then H is the dihedral group of order 8, and the sets $\{1,3,5,7\}$ and $\{2,4,6,8\}$ are both preserved by all elements of H. Consider the map $W^* \to H$ denoted $w \mapsto h_w$ and defined recursively by $h_\emptyset = \mathrm{id}$ and

$$h_{ws_k} = \begin{cases} \tau h_w, & k \text{ odd} \\ \sigma h_w, & k \text{ even,} \end{cases}$$

for all $w \in W^*$. Define $\psi : W^* \to V^*$ by $\psi(\emptyset) = \emptyset$ and

$$\psi(ws_k) = \psi(w)\eta(h_w(k))$$

for all $w \in W^*$. It is clear that ψ is a bijection from W^* onto V^* that preserves word length and that condition (i) of Proposition 2.2 is satisfied (with $S_1 = W$ and $S_2 = V$). We also observe the following.

Claim 3.2. If $w, w' \in W^*$ and if $h_w = h_{w'}$, then for every $z \in W^*$ there is $\tilde{z} \in V^*$ such that $\psi(wz) = \psi(w)\tilde{z}$ and $\psi(w'z) = \psi(w')\tilde{z}$. Moreover, this map $z \mapsto \tilde{z}$ is a bijection from W^* onto V^* that preserves word length.

Suppose $i \in \{1,3,5,7\}$ and $w \in W^*$. Then $h_{ws_i} = \tau h_w$,

$$\psi(ws_is_i) = \psi(w)\eta(h_w(i))\eta(\tau h_w(i))$$

and $h_{ws_is_i} = \tau\tau h_w = h_w$. But $h_w(i)$ is odd and for all $k \in \{1,3,5,7\}$ we have $\eta(\tau(k)) = \eta(k)^{-1}$. Thus, using Claim 3.2, for every $z \in W^*$ there is $\tilde{z} \in V^*$ such that

$$\psi(wz) = \psi(w)\tilde{z}, \qquad \psi(ws_is_iz) = \psi(w)xx^{-1}\tilde{z}, \tag{8}$$

where $x = \eta(h_w(i)) \in V$, and we have $(\psi(ws_is_iz), \psi(wz)) \in C_\Gamma$.

Similarly, if $i \in \{2,4,6,8\}$ and $w \in W^*$ then

$$\psi(ws_is_i) = \psi(w)\eta(h_w(i))\eta(\sigma h_w(i))$$

and $h_{ws_is_i} = \sigma\sigma h_w = h_w$. But $h_w(i)$ is even, and for all $k \in \{2,4,6,8\}$ we have $\eta(\sigma(k)) = \eta(k)^{-1}$. Thus, using Claim 3.2, for every $z \in W^*$ there is $\tilde{z} \in V^*$ such that

$$\psi(wz) = \psi(w)\tilde{z}, \qquad \psi(ws_is_iz) = \psi(w)xx^{-1}\tilde{z}, \tag{9}$$

where $x = \eta(h_w(i)) \in V$, and again we have $(\psi(ws_is_iz), \psi(wz)) \in C_\Gamma$.

Let $i \in \{1, 3, 5, 7\}$ and $w \in W^*$. Then

$$\psi(ws_is_{i\pm1}s_is_{i\pm1}) = \psi(w)t,$$

where

$$t = \eta(h_w(i))\eta(\tau h_w(i \pm 1))\eta(\sigma\tau h_w(i))\eta(\tau\sigma\tau h_w(i \pm 1)) \tag{10}$$

and

$$h_{ws_is_{i\pm1}s_is_{i\pm1}} = \sigma\tau\sigma\tau h_w = \rho^2 h_w.$$

On the other hand,

$$\psi(ws_{i\pm1}s_is_{i\pm1}s_i) = \psi(w)u,$$

where

$$u = \eta(h_w(i \pm 1))\eta(\sigma h_w(i))\eta(\tau\sigma h_w(i \pm 1))\eta(\sigma\tau\sigma h_w(i)) \tag{11}$$

and

$$h_{ws_{i\pm1}s_is_{i\pm1}s_i} = \tau\sigma\tau\sigma h_w = \rho^{-2} h_w = \rho^2 h_w.$$

Invoking Claim 3.2, for every $z \in W^*$ there is $\tilde{z} \in V^*$ such that

$$\psi(ws_is_{i\pm1}s_is_{i\pm1}z) = \psi(w)t\tilde{z}, \qquad \psi(ws_{i\pm1}s_is_{i\pm1}s_iz) = \psi(w)u\tilde{z},$$

where t and u are the words given in (10) and (11) that are determined by the values of $h_w(i)$ and $h_w(i\pm1)$. But $h_w(i) \in \{1, 3, 5, 7\}$ and $h_w(i\pm1)$ is a neighbor of $h_w(i)$ on the octagon in Figure 4. The values of t and u are easily computed in the eight possible cases and these are displayed in Table 1. We always get that (t, u) belongs to C_Γ. Therefore, for every $w, z \in W^*$,

$$\big(\psi(ws_is_{i\pm1}s_is_{i\pm1}z), \psi(ws_{i\pm1}s_is_{i\pm1}s_iz)\big) = (\psi(w)t\tilde{z}, \psi(w)u\tilde{z}) \in C_\Gamma. \tag{12}$$

By Lemma 2.1, we conclude

$$(\psi \times \psi)(C_G) \subseteq C_\Gamma. \tag{13}$$

In order to show the reverse inclusion in (13), we argue backwards. Let $\tilde{w} \in V^*$ and $x \in V$. There is $w \in W^*$ such that $\psi(w) = \tilde{w}$. Choose $i \in \{1, \ldots, 8\}$ such that $\eta(h_w(i)) = x$. When we invoked Claim 3.2 to find (8) and (9), we found that for every $z \in W^*$ there is $\tilde{z} \in V^*$ such that

$$\psi(ws_is_iz) = \tilde{w}xx^{-1}\tilde{z}, \qquad \psi(wz) = \tilde{w}\tilde{z},$$

and that the map $z \mapsto \tilde{z}$ is a bijection from W^* onto V^*. Hence, for all $\tilde{w}, \tilde{z} \in V^*$ and $x \in V$, there is $z \in W^*$ such that

$$(\psi^{-1}(\tilde{w}xx^{-1}\tilde{z}), \psi^{-1}(\tilde{w}\tilde{z})) = (ws_is_iz, wz) \in C_G.$$

It remains to consider the relation $(aba^{-1}b^{-1}, dcd^{-1}c^{-1}) \in R_\Gamma$. Given $\tilde{w} \in V^*$, let $w = \psi^{-1}(\tilde{w})$. Since h_w is a symmetry of the octagon in Figure 4 that maps odds to odds, we may choose $i \in \{1, 3, 5, 7\}$ and a sign $\pm$ such that $h_w(i) = 1$ and $h_w(i \pm 1) = 8$. When we invoked Claim 3.2 above to conclude (12), for every $\tilde{z} \in V^*$ we found $z \in W^*$ such that

$$\begin{aligned}(\psi^{-1}(\tilde{w}aba^{-1}b^{-1}\tilde{z}), \psi^{-1}(\tilde{w}dcd^{-1}c^{-1}\tilde{z})) &= \\ &= (ws_is_{i\pm1}s_is_{i\pm1}z, ws_{i\pm1}s_is_{i\pm1}s_iz) \in C_G.\end{aligned}$$

Applying Lemma 2.1, we conclude

$$(\psi^{-1} \times \psi^{-1})(C_\Gamma) \subseteq C_G.$$

All the conditions needed to apply Proposition 2.2 have been proved and we conclude that ψ induces an isomorphism of the Cayley graphs. □

4. The general case of genus $g \geq 2$

Let g be an integer, $g \geq 2$. Consider the fundamental group Γ_g of the closed, orientable surface of genus g in its usual presentation (1). Let $V = V_g$ be the symmetric generating set (2). Consider the Coxeter group $G = G_{4g}$ given at (3) and let $W = W_{4g}$ be as in (4).

Theorem 4.1. *The Cayley graphs $\mathcal{G}(\Gamma_g, V)$ and $\mathcal{G}(G, W)$ are isomorphic.*

Table 1: Values of t and u in the different cases.

$h_w(i)$	$h_w(i \pm 1)$	t	u
1	8	$aba^{-1}b^{-1}$	$dcd^{-1}c^{-1}$
1	2	$ab^{-1}a^{-1}d$	$b^{-1}cdc^{-1}$
3	2	$a^{-1}b^{-1}cd$	$b^{-1}a^{-1}dc$
3	4	$a^{-1}dcd^{-1}$	$ba^{-1}b^{-1}c$
5	4	$cdc^{-1}d^{-1}$	$bab^{-1}a^{-1}$
5	6	$cd^{-1}c^{-1}b$	$d^{-1}aba^{-1}$
7	6	$c^{-1}d^{-1}ab$	$d^{-1}c^{-1}ba$
7	8	$c^{-1}bab^{-1}$	$dc^{-1}d^{-1}a.$

Proof. Let V^* and W^* denote the free monoids on V and W, respectively. We will construct a bijection $\psi : W^* \to V^*$ for which we can invoke Proposition 2.2. This construction is analogous to the one in the proof of the special case, Theorem 3.1, but more complicated.

In V^*, a_j^{-1} and b_j^{-1} should be understood as symbols only and not as multiplicative inverses. Thus, for example, $a_1b_1b_1^{-1}a_2$ and a_1a_2 are distinct elements of V^*. We chose not to introduce extra symbols to replace a_j^{-1} and b_j^{-1}, because we will use the notation $x \mapsto x^{-1}$ (here taking inverses in Γ_g) for the obvious permutation of V. When $k \in \mathbf{Z}$ we will take a_k to mean a_j where $j \in \{1,\dots,g\}$ and $j \equiv k \pmod g$ and similarly for a_j^{-1}, b_j and b_j^{-1}. Similarly, for elements s_j of W, the subscript j is always taken modulo $4g$.

For future use, it will be convenient to have names for certain elements of V^* that correspond to taking the first half u of a cyclic permutation of the word

$$a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$$

and the inverse v of the second half. Let $n \in \mathbf{Z}$. If g is even, then set

$$u_1(n) = (a_nb_na_n^{-1}b_n^{-1})(a_{n+1}b_{n+1}a_{n+1}^{-1}b_{n+1}^{-1})\cdots(a_{n+\frac{g}{2}-1}b_{n+\frac{g}{2}-1}a_{n+\frac{g}{2}-1}^{-1}b_{n+\frac{g}{2}-1}^{-1})$$
$$u_2(n) = (b_na_n^{-1}b_n^{-1}a_{n+1})(b_{n+1}a_{n+1}^{-1}b_{n+1}^{-1}a_{n+2})\cdots(b_{n+\frac{g}{2}-1}a_{n+\frac{g}{2}-1}^{-1}b_{n+\frac{g}{2}-1}^{-1}a_{n+\frac{g}{2}})$$
$$u_3(n) = (a_n^{-1}b_n^{-1}a_{n+1}b_{n+1})(a_{n+1}^{-1}b_{n+1}^{-1}a_{n+2}b_{n+2})\cdots(a_{n+\frac{g}{2}-1}^{-1}b_{n+\frac{g}{2}-1}^{-1}a_{n+\frac{g}{2}}b_{n+\frac{g}{2}})$$
$$u_4(n) = (b_n^{-1}a_{n+1}b_{n+1}a_{n+1}^{-1})(b_{n+1}^{-1}a_{n+2}b_{n+2}a_{n+2}^{-1})\cdots(b_{n+\frac{g}{2}-1}^{-1}a_{n+\frac{g}{2}}b_{n+\frac{g}{2}}a_{n+\frac{g}{2}}^{-1})$$
$$v_1(n) = (b_{n-1}a_{n-1}b_{n-1}^{-1}a_{n-1}^{-1})(b_{n-2}a_{n-2}b_{n-2}^{-1}a_{n-2}^{-1})\cdots(b_{n-\frac{g}{2}}a_{n-\frac{g}{2}}b_{n-\frac{g}{2}}^{-1}a_{n-\frac{g}{2}}^{-1})$$
$$v_2(n) = (a_n^{-1}b_{n-1}a_{n-1}b_{n-1}^{-1})(a_{n-1}^{-1}b_{n-2}a_{n-2}b_{n-2}^{-1})\cdots(a_{n-\frac{g}{2}+1}^{-1}b_{n-\frac{g}{2}}a_{n-\frac{g}{2}}b_{n-\frac{g}{2}}^{-1})$$
$$v_3(n) = (b_n^{-1}a_n^{-1}b_{n-1}a_{n-1})(b_{n-1}^{-1}a_{n-1}^{-1}b_{n-2}a_{n-2})\cdots(b_{n-\frac{g}{2}+1}^{-1}a_{n-\frac{g}{2}+1}^{-1}b_{n-\frac{g}{2}}a_{n-\frac{g}{2}})$$
$$v_4(n) = (a_nb_n^{-1}a_n^{-1}b_{n-1})(a_{n-1}b_{n-1}^{-1}a_{n-1}^{-1}b_{n-2})\cdots(a_{n-\frac{g}{2}+1}b_{n-\frac{g}{2}+1}^{-1}a_{n-\frac{g}{2}+1}^{-1}b_{n-\frac{g}{2}}),$$

while if g is odd, then set

$$\begin{aligned} u_1(n) = {} & (a_nb_na_n^{-1}b_n^{-1})(a_{n+1}b_{n+1}a_{n+1}^{-1}b_{n+1}^{-1})\cdots \\ & \cdots(a_{n+\frac{g-1}{2}-1}b_{n+\frac{g-1}{2}-1}a_{n+\frac{g-1}{2}-1}^{-1}b_{n+\frac{g-1}{2}-1}^{-1})a_{n+\frac{g-1}{2}}b_{n+\frac{g-1}{2}} \\ u_2(n) = {} & (b_na_n^{-1}b_n^{-1}a_{n+1})(b_{n+1}a_{n+1}^{-1}b_{n+1}^{-1}a_{n+2})\cdots \\ & \cdots(b_{n+\frac{g-1}{2}-1}a_{n+\frac{g-1}{2}-1}^{-1}b_{n+\frac{g-1}{2}-1}^{-1}a_{n+\frac{g-1}{2}})b_{n+\frac{g-1}{2}}a_{n+\frac{g-1}{2}}^{-1} \\ u_3(n) = {} & (a_n^{-1}b_n^{-1}a_{n+1}b_{n+1})(a_{n+1}^{-1}b_{n+1}^{-1}a_{n+2}b_{n+2})\cdots \\ & \cdots(a_{n+\frac{g-1}{2}-1}^{-1}b_{n+\frac{g-1}{2}-1}^{-1}a_{n+\frac{g-1}{2}}b_{n+\frac{g-1}{2}})a_{n+\frac{g-1}{2}}^{-1}b_{n+\frac{g-1}{2}}^{-1} \end{aligned}$$

$$\begin{aligned}
u_4(n) = {} & (b_n^{-1}a_{n+1}b_{n+1}a_{n+1}^{-1})(b_{n+1}^{-1}a_{n+2}b_{n+2}a_{n+2}^{-1})\cdots \\
& \cdots(b^{-1}_{n+\frac{g-1}{2}-1}a_{n+\frac{g-1}{2}}b_{n+\frac{g-1}{2}}a^{-1}_{n+\frac{g-1}{2}})b^{-1}_{n+\frac{g-1}{2}}a_{n+\frac{g-1}{2}+1} \\
v_1(n) = {} & (b_{n-1}a_{n-1}b_{n-1}^{-1}a_{n-1}^{-1})(b_{n-2}a_{n-2}b_{n-2}^{-1}a_{n-2}^{-1})\cdots \\
& \cdots(b_{n-\frac{g-1}{2}}a_{n-\frac{g-1}{2}}b^{-1}_{n-\frac{g-1}{2}}a^{-1}_{n-\frac{g-1}{2}})b_{n-\frac{g-1}{2}-1}a_{n-\frac{g-1}{2}-1} \\
v_2(n) = {} & (a_n^{-1}b_{n-1}a_{n-1}b_{n-1}^{-1})(a_{n-1}^{-1}b_{n-2}a_{n-2}b_{n-2}^{-1})\cdots \\
& \cdots(a^{-1}_{n-\frac{g-1}{2}+1}b_{n-\frac{g-1}{2}}a_{n-\frac{g-1}{2}}b^{-1}_{n-\frac{g-1}{2}})a^{-1}_{n-\frac{g-1}{2}}b_{n-\frac{g-1}{2}-1} \\
v_3(n) = {} & (b_n^{-1}a_n^{-1}b_{n-1}a_{n-1})(b_{n-1}^{-1}a_{n-1}^{-1}b_{n-2}a_{n-2})\cdots \\
& \cdots(b^{-1}_{n-\frac{g-1}{2}+1}a^{-1}_{n-\frac{g-1}{2}+1}b_{n-\frac{g-1}{2}}a_{n-\frac{g-1}{2}})b^{-1}_{n-\frac{g-1}{2}}a^{-1}_{n-\frac{g-1}{2}} \\
v_4(n) = {} & (a_nb_n^{-1}a_n^{-1}b_{n-1})(a_{n-1}b_{n-1}^{-1}a_{n-1}^{-1}b_{n-2})\cdots \\
& \cdots(a_{n-\frac{g-1}{2}+1}b^{-1}_{n-\frac{g-1}{2}+1}a^{-1}_{n-\frac{g-1}{2}+1}b_{n-\frac{g-1}{2}})a_{n-\frac{g-1}{2}}b^{-1}_{n-\frac{g-1}{2}}.
\end{aligned}$$

Let

$$R_\Gamma = \{(u_1(1), v_1(1))\} \cup \bigcup_{i=1}^{g}\{(a_ia_i^{-1},\emptyset),(a_i^{-1}a_i,\emptyset),(b_ib_i^{-1},\emptyset),(b_i^{-1}b_i,\emptyset)\} \subseteq V^* \times V^*$$

$$R_G = \{(s_is_i,\emptyset) \mid 1 \le i \le 4g\} \cup \{((s_is_{i+1})^g,(s_{i+1}s_i)^g) \mid 1 \le i \le 4g\} \subseteq W^* \times W^*.$$

Then

$$\Gamma_g = \mathrm{Mon}\langle V \mid R_\Gamma\rangle, \qquad G_{4g} = \mathrm{Mon}\langle W \mid R_G\rangle$$

are monoid presentations of the groups. Let $C_\Gamma \subseteq V^* \times V^*$ and $C_G \subseteq W^* \times W^*$ be the congruences generated by R_Γ and R_G, respectively. We have

$$(u_j(n), v_j(n)) \in C_\Gamma, \qquad (n \in \mathbf{Z},\ 1 \le j \le 4). \tag{14}$$

Consider the regular $4g$–gon P with vertices labeled $1, 2, \dots, 4g$ going clockwise and for $p \in \{1, \dots, 4g\}$ let τ_p be the reflection of P through the axis that contains the vertex p. Note that τ_p depends only on p modulo $2g$. Let ρ be the clockwise rotation of P through angle $\frac{2\pi k}{2g}$. For $n \in \mathbf{Z}$, we will let τ_n denote the reflection τ_k where $k \in \{1, \dots 4g\}$ and $k \equiv n \pmod g$. We have

$$\tau_{p_1}\tau_{p_2} = \rho^{p_2-p_1}, \qquad (p_1, p_2 \in \mathbf{Z}).$$

Let H denote the group of symmetries of P generated by $\{\tau_p \mid 1 \le p \le 4g\}$. We will identify elements of H with the corresponding permutations of $\{1, \dots, 4g\}$ according to how they move the vertices. Note that every element of H preserves the set $\{1, 3, \dots, 4g-1\}$ of odd numbers. Also, H is isomorphic to the dihedral group of order $4g$.

Consider the bijection $\eta : \{1, 2, \dots, 4g\} \to V$ given by

$$\eta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots & 4g-3 & 4g-2 & 4g-1 & 4g \\ a_1 & b_1^{-1} & a_1^{-1} & b_1 & a_2 & b_2^{-1} & a_2^{-1} & b_2 & \dots & a_g & b_g^{-1} & a_g^{-1} & b_g \end{pmatrix}.$$

Thus,

$$\eta(4k-3) = a_k, \quad \eta(4k-2) = b_k^{-1}, \quad \eta(4k-1) = a_k^{-1}, \quad \eta(4k) = b_k$$

for all $k \in \{1, \dots, g\}$. Consider the map $\sigma : \{1, \dots, 4g\} \to H$ defined by

$$\begin{aligned} \sigma(4k-3) = \sigma(4k-1) &= \tau_{4k-2} \\ \sigma(4k-2) = \sigma(4k) &= \tau_{4k-1}, \qquad (1 \le k \le g). \end{aligned}$$

Thus, we have

$$\eta(\sigma(i)i) = \eta(i)^{-1} \tag{15}$$

$$\sigma(\sigma(i)i) = \sigma(i) \tag{16}$$

for all $i \in \{1, \dots, 4g\}$. Define recursively a map $W^* \to H$, denoted $w \mapsto h_w$, by

$$\begin{aligned} h_\emptyset &= \mathrm{id} \\ h_{ws_i} &= \sigma(h_w(i))h_w \qquad (w \in W^*,\ s_i \in W). \end{aligned}$$

Define recursively a map $\psi : W^* \to V^*$ by

$$\begin{aligned} \psi(\emptyset) &= \emptyset \\ \psi(ws_i) &= \psi(w)\eta(h_w(i)) \qquad (w \in W^*,\ s_i \in W). \end{aligned}$$

It is clear that ψ is a bijection from W^* onto V^* that preserves word length and that condition (i) of Proposition 2.2 is satisfied (with $S_1 = W$ and $S_2 = V$).

Claim 4.2. Suppose $w, w' \in W^*$ and suppose $h_w = h_{w'}$. Then for every $z \in W^*$ we have $h_{wz} = h_{w'z}$ and there is a unique $\tilde{z} \in V^*$ such that

$$\psi(wz) = \psi(w)\tilde{z} \tag{17}$$

$$\psi(w'z) = \psi(w')\tilde{z}. \tag{18}$$

Moreover, the map $z \mapsto \tilde{z}$ is a bijection from W^* onto V^*.

Proof. Uniqueness of $\tilde{z}$ is clear. If $z = \emptyset$ then (17) and (18) hold with $\tilde{z} = \emptyset$. Take $z = s_i$. Then $h_{wz} = \sigma(h_w(i))h_w = \sigma(h_{w'}(i))h_{w'} = h_{w'z}$. Moreover,

$$\psi(wz) = \psi(w)\eta(h_w(i))$$
$$\psi(w'z) = \psi(w')\eta(h_{w'}(i)),$$

so (17) and (18) hold with $\tilde{z} = \eta(h_w(i))$, and the map $z \mapsto \tilde{z}$ is a bijection from W onto V. Now one easily shows by induction on n that if $z \in W^*$ with $|z| = n$, then $\tilde{z}$ exists such that (17) and (18) hold and the map $z \mapsto \tilde{z}$ is a bijection from the set of words in W^* of length n onto the set of words in V^* of length n. This will finish the proof of Claim 4.2. □

Claim 4.3. Let $w \in W^*$ and let $i \in \{1, \dots, 4g\}$. Then

$$h_{ws_is_i} = h_w$$

and

$$\psi(ws_is_i) = \psi(w)xx^{-1},$$

where $x = \eta(h_w(i))$.

Proof. We have $\psi(ws_i) = \psi(w)x$ and $h_{ws_i} = \sigma(h_w(i))h_w$. Thus,

$$\psi(ws_is_i) = \psi(w)x\eta(h_{ws_i}(i)) = \psi(w)x\eta(\sigma(h_w(i))h_w(i)) = \psi(w)xx^{-1},$$

where we have used (15) to obtain the last equality. Also,

$$h_{ws_is_i} = \sigma(h_{ws_i}(i))h_{ws_i} = \sigma(\sigma(h_w(i))h_w(i))\sigma(h_w(i))h_w = h_w,$$

where we have used (16) and the fact that the each $\sigma(n)$ is a reflection. This finishes the proof of Claim 4.3 □

Claim 4.4. Let $w \in W^*$, let $i \in \{1, 3, 5, \dots, 4g-1\}$ and choose a sign $\pm$. Then

$$h_{w(s_is_{i\pm1})^g} = h_{w(s_{i\pm1}s_i)^g} \tag{19}$$

and there is $(\tilde{u}, \tilde{v}) \in C_\Gamma \subseteq V^* \times V^*$ such that

$$\psi(w(s_is_{i\pm1})^g) = \psi(w)\tilde{u}, \qquad \psi(w(s_{i\pm1}s_i)^g) = \psi(w)\tilde{v}. \tag{20}$$

Proof. Let $k = h_w(i)$ and $\ell = h_w(i \pm 1)$. Then k is odd, ℓ is even, and $|k - \ell| = 1$. let $p_1 = k$ and $p_2 = \sigma(p_1)\ell$. The possible values of k, ℓ, p_1 and p_2 are displayed in Table 2, where n is an integer. We assign to a pair (p, p') a name as indicated in Table 3, and we have included in Table 2 the names of the pairs (p_1, p_2). We find

$$\psi(ws_i) = \psi(w)\eta(p_1) \qquad h_{ws_i} = \sigma(p_1)h_w$$
$$\psi(ws_is_{i\pm1}) = \psi(w)\eta(p_1)\eta(p_2) \qquad h_{ws_is_{i\pm1}} = \sigma(p_2)\sigma(p_1)h_w.$$

Defining recursively

$$p_{j+2} = \sigma(p_{j+1})\sigma(p_j)p_j, \qquad (j \geq 1), \tag{21}$$

we find

$$\psi(w(s_is_{i\pm1})^g) = \psi(w)\eta(p_1)\eta(p_2)\cdots\eta(p_{2g-1})\eta(p_{2g}) \tag{22}$$
$$h_{w(s_is_{i\pm1})^g} = \sigma(p_{2g})\sigma(p_{2g-1})\cdots\sigma(p_2)\sigma(p_1)h_w. \tag{23}$$

In the various cases, a step of the recursion $(p_j, p_{j+1}) \mapsto (p_{j+2}, p_{j+3})$ determined by (21) for j odd is described in Table 4. There, "(in)" refers to the is the name of the case (p_j, p_{j+1}) and "(out)" refers to the name of the case corresponding to (p_{j+2}, p_{j+3}).

Similarly, letting $q_1 = \ell$ and $q_2 = \sigma(q_1)k$ and making the recursive definition

$$q_{j+2} = \sigma(q_{j+1})\sigma(q_j)q_j, \qquad (j \geq 1), \tag{24}$$

we find

$$\psi(w(s_{i\pm1}s_i)^g) = \psi(w)\eta(q_1)\eta(q_2)\cdots\eta(q_{2g-1})\eta(q_{2g}) \tag{25}$$
$$h_{w(s_{i\pm1}s_i)^g} = \sigma(q_{2g})\sigma(q_{2g-1})\cdots\sigma(q_2)\sigma(q_1)h_w. \tag{26}$$

The starting cases are given in Table 5, the names of pairs (q, q') are assigned according to Table 6 and the names of (q_1, q_2) are included in Table 5. The recursive step $(q_j, q_{j+1}) \mapsto (q_{j+2}, q_{j+3})$ is described in Table 7.

Suppose we start with $k = 4n-3$, $\ell = 4n-4$. When we recursively calculate $p_1, p_2, \ldots, p_{2g-1}, p_{2g}$, we run through g blocks of two whose names are

$$\begin{cases} \mathrm{A}_n, \mathrm{C}_n, \mathrm{A}_{n+1}, \mathrm{C}_{n+1}, \ldots, \mathrm{A}_{n+\frac{g}{2}-1}, \mathrm{C}_{n+\frac{g}{2}-1}, & g \text{ even} \\ \mathrm{A}_n, \mathrm{C}_n, \mathrm{A}_{n+1}, \mathrm{C}_{n+1}, \ldots, \mathrm{A}_{n+\frac{g-1}{2}-1}, \mathrm{C}_{n+\frac{g-1}{2}-1}, \mathrm{A}_{n+\frac{g-1}{2}}, & g \text{ odd.} \end{cases}$$

Table 2: Possible values of k, ℓ, p_1 and p_2.

k	ℓ	p_1	$\sigma(p_1)$	p_2	$\sigma(p_2)$	Name
$4n-3$	$4n-4$	$4n-3$	τ_{4n-2}	$4n$	τ_{4n-1}	A_n
$4n-3$	$4n-2$	$4n-3$	τ_{4n-2}	$4n-2$	τ_{4n-1}	B_n
$4n-1$	$4n-2$	$4n-1$	τ_{4n-2}	$4n-2$	τ_{4n-1}	C_n
$4n-1$	$4n$	$4n-1$	τ_{4n-2}	$4n-4$	τ_{4n-5}	D_n

We find

$$(p_1, p_2, \ldots, p_{2g-1}, p_{2g}) =$$

$$= \begin{cases} (4n-3,\, 4n,\, 4n-1,\, 4n-2,\, 4n+1,\, 4n+4,\, 4n+3,\, 4n+2,\, \ldots \\ \quad \ldots,\, 4n+2g-7,\, 4n+2g-4,\, 4n+2g-5,\, 2n+2g-6), & g \text{ even} \\ (4n-3,\, 4n,\, 4n-1,\, 4n-2,\, 4n+1,\, 4n+4,\, 4n+3,\, 4n+2,\, \ldots \\ \quad \ldots,\, 4n+2g-9,\, 4n+2g-6,\, 4n+2g-7,\, 4n+2g-8, \\ \qquad\qquad 4n+2g-5,\, 4n+2g-2), & g \text{ odd} \end{cases}$$

and

$$\eta(p_1)\eta(p_2)\cdots\eta(p_{2g-1})\eta(p_{2g}) = u_1(n). \tag{27}$$

Furthermore, using the fourth column in Table 4, we immediately get

$$\sigma(p_{2g})\sigma(p_{2g-1})\cdots\sigma(p_2)\sigma(p_1) = \rho^g. \tag{28}$$

On the other hand, recursively calculating $q_1, q_2, \ldots, q_{2g-1}, q_{2g}$, we have g blocks of two whose names are

$$\begin{cases} \mathrm{E}_n, \mathrm{G}_{n-1}, \mathrm{E}_{n-1}, \mathrm{G}_{n-2}, \ldots, \mathrm{E}_{n-\frac{g}{2}+1}, \mathrm{G}_{n-\frac{g}{2}}, & g \text{ even} \\ \mathrm{E}_n, \mathrm{G}_{n-1}, \mathrm{E}_{n-1}, \mathrm{G}_{n-2}, \ldots, \mathrm{E}_{n-\frac{g-1}{2}+1}, \mathrm{G}_{n-\frac{g-1}{2}}, \mathrm{E}_{n-\frac{g-1}{2}}, & g \text{ odd.} \end{cases}$$

This yields

$$\eta(q_1)\eta(q_2)\cdots\eta(q_{2g-1})\eta(q_{2g}) = v_1(n). \tag{29}$$

Using the fourth column in Table 7, we get

$$\sigma(q_{2g})\sigma(q_{2g-1})\cdots\sigma(q_2)\sigma(q_1) = (\rho^{-1})^g = \rho^{-g}. \tag{30}$$

From (22) and (27) we have $\psi(w(s_i s_{i\pm1})^g) = \psi(w)u_1(n)$. From (25) and (29) we have $\psi(w(s_{i\pm1}s_i)^g) = \psi(w)v_1(n)$. By (23) and (28), $h_{w(s_i s_{i\pm1})^g} = \rho^g h_w$. By (26) and (30), $h_{w(s_{i\pm1}s_i)^g} = \rho^{-g} h_w = \rho^g h_w$. By (14), $(u_1(n), v_1(n)) \in C_\Gamma$. We have proved the claim in the case $k = 4n-3$ and $\ell = 4n-4$.

Table 3: Names of (p, p').

p	p'	Name
$4n-3$	$4n$	A_n
$4n-3$	$4n-2$	B_n
$4n-1$	$4n-2$	C_n
$4n-1$	$4n-4$	D_n

In a similar manner, if $k = 4n-3$ and $\ell = 4n-2$, then we find

$$\eta(p_1)\eta(p_2)\cdots\eta(p_{2g-1})\eta(p_{2g}) = v_4(n),$$
$$\eta(q_1)\eta(q_2)\cdots\eta(q_{2g-1})\eta(q_{2g}) = u_4(n)$$

and

$$\sigma(p_{2g})\sigma(p_{2g-1})\cdots\sigma(p_2)\sigma(p_1) = \begin{cases} (\rho\rho^{-3})^{\frac{g}{2}} = \rho^{-g}, & g \text{ even,} \\ (\rho\rho^{-3})^{\frac{g-1}{2}}\rho = \rho^{-g+2}, & g \text{ odd,} \end{cases}$$
$$\sigma(q_{2g})\sigma(q_{2g-1})\cdots\sigma(q_2)\sigma(q_1) = \begin{cases} (\rho^{3}\rho^{-1})^{\frac{g}{2}} = \rho^{g}, & g \text{ even,} \\ (\rho^{3}\rho^{-1})^{\frac{g-1}{2}}\rho^{3} = \rho^{g+2}, & g \text{ odd.} \end{cases}$$

If $k = 4n-1$ and $\ell = 4n-2$, then we find

$$\eta(p_1)\eta(p_2)\cdots\eta(p_{2g-1})\eta(p_{2g}) = u_3(n),$$
$$\eta(q_1)\eta(q_2)\cdots\eta(q_{2g-1})\eta(q_{2g}) = v_3(n)$$

and

$$\sigma(p_{2g})\sigma(p_{2g-1})\cdots\sigma(p_2)\sigma(p_1) = (\rho)^g = \rho^g$$
$$\sigma(q_{2g})\sigma(q_{2g-1})\cdots\sigma(q_2)\sigma(q_1) = (\rho^{-1})^g = \rho^{-g}.$$

Finally, if $k = 4n-1$ and $\ell = 4n$, then we find

$$\eta(p_1)\eta(p_2)\cdots\eta(p_{2g-1})\eta(p_{2g}) = v_2(n),$$
$$\eta(q_1)\eta(q_2)\cdots\eta(q_{2g-1})\eta(q_{2g}) = u_2(n)$$

and

$$\sigma(p_{2g})\sigma(p_{2g-1})\cdots\sigma(p_2)\sigma(p_1) = \begin{cases} (\rho^{-3}\rho)^{\frac{g}{2}} = \rho^{-g}, & g \text{ even,} \\ (\rho^{-3}\rho)^{\frac{g-1}{2}}\rho^{-3} = \rho^{-g-2}, & g \text{ odd,} \end{cases}$$
$$\sigma(q_{2g})\sigma(q_{2g-1})\cdots\sigma(q_2)\sigma(q_1) = \begin{cases} (\rho^{-1}\rho^{3})^{\frac{g}{2}} = \rho^{g}, & g \text{ even,} \\ (\rho^{-1}\rho^{3})^{\frac{g-1}{2}}\rho^{-1} = \rho^{g-2}, & g \text{ odd.} \end{cases}$$

Table 4: The recursion $(p_j, p_{j+1}) \mapsto (p_{j+2}, p_{j+3})$.

(in)	p_j	p_{j+1}	$\sigma(p_{j+1})\sigma(p_j)$	p_{j+2}	$\sigma(p_{j+2})$	p_{j+3}	(out)
A_n	$4n-3$	$4n$	ρ	$4n-1$	τ_{4n-2}	$4n-2$	C_n
B_n	$4n-3$	$4n-2$	ρ	$4n-1$	τ_{4n-2}	$4n-4$	D_n
C_n	$4n-1$	$4n-2$	ρ	$4n+1$	τ_{4n+2}	$4n+4$	A_{n+1}
D_n	$4n-1$	$4n-4$	ρ^{-3}	$4n-7$	τ_{4n-6}	$4n-6$	B_{n-1}

Using (14), (22), (23), (25) and (26) in all these cases, the proof of Claim 4.4 is finished. □

Using Claims 4.2, 4.3 and 4.4 and using Lemma 2.1, we have

$$(\psi \times \psi)(C_G) \subseteq C_\Gamma. \tag{31}$$

In order to show the reverse inclusion in (31), we make use of some of the calculations done in the proof of Claim 4.4, but reverse the argument. Let $\tilde{w} \in V^*$ and $x \in V$. There is $w \in W^*$ such that $\psi(w) = \tilde{w}$. Choose $i \in \{1, \ldots, 4g\}$ such that $\eta(h_w(i)) = x$. When we proved Claim 4.3, we found that for every $z \in W^*$ there is $\tilde{z} \in V^*$ such that

$$\psi(ws_is_iz) = \tilde{w}xx^{-1}\tilde{z}, \qquad \psi(wz) = \tilde{w}\tilde{z},$$

and the map $z \mapsto \tilde{z}$ is a bijection from W^* onto V^*. Hence, for all $\tilde{w}, \tilde{z} \in V^*$ and $x \in V$, there is $z \in W^*$ such that

$$(\psi^{-1}(\tilde{w}xx^{-1}\tilde{z}), \psi^{-1}(\tilde{w}\tilde{z})) = (ws_is_iz, wz) \in C_G.$$

It remains to treat the relation $(u_1(1), v_1(1)) \in R_\Gamma$. Given $\tilde{w} \in V^*$, let $w = \psi^{-1}(\tilde{w})$. We may choose $i \in \{1, 3, \ldots, 4g-1\}$ and a sign $\pm$ such that $h_w(i) = 1$ and $h_w(i \pm 1) = 4g$. In the proof of Claim 4.4 for every $\tilde{z} \in V^*$ we found $z \in W^*$ such that

$$(\psi^{-1}(\tilde{w}u_1(1)\tilde{z}), \psi^{-1}(\tilde{w}v_1(1)\tilde{z})) = (w(s_is_{i\pm1})^g z, w(s_{i\pm1}s_i)^g z) \in C_G.$$

Applying Lemma 2.1, we conclude

$$(\psi^{-1} \times \psi^{-1})(C_\Gamma) \subseteq C_G.$$

We now apply Proposition 2.2 to conclude that ψ induces an isomorphism of the Cayley graphs. □

Table 5: Possible values of k, ℓ, q_1 and q_2.

k	ℓ	q_1	$\sigma(q_1)$	q_2	$\sigma(q_2)$	Name
$4n-3$	$4n-4$	$4n-4$	τ_{4n-5}	$4n-7$	τ_{4n-6}	E_n
$4n-3$	$4n-2$	$4n-2$	τ_{4n-1}	$4n+1$	τ_{4n+2}	F_n
$4n-1$	$4n-2$	$4n-2$	τ_{4n-1}	$4n-1$	τ_{4n-2}	G_n
$4n-1$	$4n$	$4n$	τ_{4n-1}	$4n-1$	τ_{4n-2}	H_n

5. Appendix. A geometric proof

In this section we present a classical, geometric proof of Theorem 4.1, that was shown to us by J.G. Ratcliffe. We take $g = n$ to avoid confusion with the notation for side–pairing maps.

Geometric proof of Theorem 4.1. As in Example 4 on p. 382 of [4], there is a regular hyperbolic $4n$–gon P with dihedral angle $\pi/2n$. We position P in the conformal disk model of the hyperbolic plane as in Figure 9.2.3 in [4] and we label the edges as in this figure (with a slight modification) in positive order

$$S_1, T_1', S_1', T_1, \ldots, S_n, T_n', S_n', T_n.$$

Let

$$g_{S_i}, g_{S_i'}, g_{T_i}, g_{T_i'} \qquad (1 \le i \le n) \tag{32}$$

be the side–pairing maps. By Poincaré's fundamental polyhedron theorem (Theorem 11.2.2 in [4]; see also Theorem 6.7.7), the side–pairing maps generate a discrete group Γ with fundamental polygon P and and Γ has the presentation with generators (32) and relations

$$\left(g_{S_i} g_{S_i'}\right)_{1\le i\le n}, \quad \left(g_{T_i} g_{T_i'}\right)_{1\le i\le n}, \quad g_{S_1} g_{T_1} g_{S_1'} g_{T_1'} \cdots g_{S_n} g_{T_n} g_{S_n'} g_{T_n'}.$$

This means that $\{gP \mid g \in \Gamma\}$ is an exact tessellation of the hyperbolic plane. Moreover, if S is a side of P, then $S = P \cap g_S P$. Hence, $gS = gP \cap g g_S P$. This implies that the dual graph of the tessellation is the undirected Cayley graph of the presentations.

Now P is a Coxeter polygon and so reflecting in the sides of P generates a Coxeter group G with $4n$ generators. Again, by Poincaré's fundamental polyhedron theorem (see Theorems 7.1.3 and 7.1.4 in [4]), G is a discrete group with fundamental polygon P. The tessellation $\{gP \mid g \in G\}$ is the same tessellation as before and the undirected Cayley

Table 6: Names of (q, q').

q	q'	Name
$4n-4$	$4n-7$	E_n
$4n-2$	$4n+1$	F_n
$4n-2$	$4n-1$	G_n
$4n$	$4n-1$	H_n

Table 7: The recursion $(q_j, q_{j+1}) \mapsto (q_{j+2}, q_{j+3})$.

(in)	q_j	q_{j+1}	$\sigma(q_{j+1})\sigma(q_j)$	q_{j+2}	$\sigma(q_{j+2})$	q_{j+3}	(out)
E_n	$4n-4$	$4n-7$	ρ^{-1}	$4n-6$	τ_{4n-5}	$4n-5$	G_{n-1}
F_n	$4n-2$	$4n+1$	ρ^3	$4n+4$	τ_{4n+3}	$4n+3$	H_{n+1}
G_n	$4n-2$	$4n-1$	ρ^{-1}	$4n-4$	τ_{4n-5}	$4n-7$	E_n
H_n	$4n$	$4n-1$	ρ^{-1}	$4n-2$	τ_{4n-1}	$4n+1$	F_n

graph of the Coxeter presentation of G is the dual graph of the tessellation. Therefore, Γ and G have isomorphic Cayley graphs with respect to the above generators. □

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