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On quantales of preradical Bland filters and differential preradical filters

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Dedicated to Professor V. V. Kirichenko
on the occasion of his 65th birthday

Abstract. We prove that the set of all Bland preradical filters over an arbitrary differential ring form a quantale with respect to meets where the role of multiplication is played by the usual Gabriel pro-duct of filters. A subset of a differential pretorsion theory is a subquantale of this quantale.

Introduction

Rings considered in calculus are often equipped with additive maps that have properties of the operation of derivation. They include the ring of infinite differentiable real functions, the ring of all integer functions, the field of meromorphic functions and many others. That is why the study of rings equipped with derivations and, more generally, modules equipped with derivations is an important part of differential algebra. Recently Bland [2] has initiated the investigation of a new type of hereditary tor-sion theories over such rings, which he called differential torsion theories. The investigation continued in [16], where such torsion theories were called Bland torsion theories. Another type of differential torsion theories was considered by O. L. Horbachuk, M. Ya. Komarnytskyi in [9]. The purpose of the present paper is to reveal the interrelation between these types of torsion theories and to investigate the lattices of

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both Bland pretorsion theories and differential pretorsion theories from the point of view of quantale theory. In fact, we are dealing with more general functors than the functors of a torsion part, namely with the kernel functors, the general theory of which is developed in [6].

The concept of quantale [15] goes back to 1920's, when W. Krull, followed by R. P. Dilworth and M.Ward, considered a lattice of ideals equipped with multiplication. The term 'quantale' itself (short for 'quantum locale') was suggested by C.J. Mulvey. A quantale is a complete lattice L satisfying the law $a \cdot (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \cdot b_i)$ for all $a, b_i \in L$, where I is an index set. Note that, in this definition, we can substitute the meets by the joins, so that we obtain the dual quantale with respect to meets.

It is well known that the set of all ideals of any ring is a complete lattice on which an operation of multiplication of ideals is defined. It satisfies an infinite distributive law, so the lattice of all ideals of the ring is a quantale. This quantale is intensively exploited in the paper [3]. By analogy, the set of all preradical filters of left ideals of an arbitrary ring is a complete lattice, moreover, an operation of Gabriel multiplication of preradical filters is defined. Hence, the lattice of preradical filters is, in fact, a quantale with respect to meets, see [5].

We wish to discover, which sets of differential pretorsion theories in the sense of Bland and pretorsion theories in the sense O.Horbachuk and M.Komarnytskyi over differential rings form a quantale.

Preliminaries

Throughout the paper, all rings are assumed to be associative with nonzero identity. All modules are unital left modules. The word 'ideal', without any further comments, signifies a two-sided ideal. $R - Mod$ denotes the category of left R-modules and module homomorphisms. Also, if I is a left ideal of the ring R and $S \subseteq R$, then the set $(I : S) = \{r \in R | rS \subseteq I\}$ is a left ideal of R. In particular, when $S = \{a\}$, where $a \in R$, $(I : a)$ denotes the left ideal of R given by $\{r \in R | ra \in I\}$. In the paper a standard ring-theoretic terminology will be used, following [8], [11].

Now we provide the basic definitions and facts on differential algebra, for more details, see [10].

An *ordinary differential ring* (or a *δ-ring* for short) is a pair (R, δ), where R is a ring and $\delta : R \to R$ is a map, called a *derivation* on R, satisfying the following conditions:

1) $\delta(r + s) = \delta(r) + \delta(s)$,
2) $\delta(rs) = \delta(r)s + r\delta(s)$,

where in both equalities r and s are arbitrary elements of the ring R. If $\Delta = \{\delta_i, \quad i = 1, ..., n\}$ is the set of pairwise commutative derivations on R, then the ring R together with the set Δ is said to be *a partial differential ring* (*a Δ-ring* for short).

A *left partial differential R-module* (a *D-module* for short) over a Δ-ring (R, Δ) is a pair (M, D), where M is an R -module and D is a set of pairwise commutative maps $d_i : M \to M$, $i = 1, 2, ..., n$, called *derivations on the module M* consistent with $\delta_i, \quad i = 1, ..., n$ (Δ-*derivation*s on M for short), each of which satisfies the following conditions for module derivations:

1) $d_i(m + n) = d_i(m) + d_i(n)$ and

2) $d_i(rm) = rd_i(m) + \delta_i(r)m$

for any $m, n \in M$, $r \in R$.

Let (R, Δ) be a Δ-ring and let (M, D) be a left D-module over (R, Δ). A left ideal I of the ring R is called a Δ-*ideal* (short for *differential ideal*) if $\delta(I) \subset I$ for each $\delta \in \Delta$. Similarly, a submodule N of the module M is a *D-submodule* (short for *differential submodule*) if $d(N) \subset N$ for each $d \in D$.

A differential module M is called *differentially simple* if it has no differential submodules other than zero and itself (though it may have nontrivial non-differential submodules). Sometimes it may be useful to consider so called *simple differential modules*, i. e. simple modules, in the ordinary sense, equipped with some derivations. A differential submodule N of a differential module M is called a *maximal differential submodule* if there is no proper differential submodule between N and M. It is equivalent for the d-factor module M/N to be differentially simple, and the latter is equivalent for the module M/N to be simple when considered as a left module over the ring D_R of linear differential operators of the differential ring R. The intersection of all maximal differential submodules of the left D-module M is a *differential Jackobson radical* of M, denoted by $J_d(M)$. If M has no maximal differential submodules, then assume that $J_d(M) = M$. In case when M is a left regular differential module (i.e., $M = {}_RR$), we obtain the notion of a differential Jackobson radical of a differential ring R. It is clear that $J(R) \subseteq J_d(R)$, where $J_d(R)$ is the intersection of those left ideals which are maximal among the differential ideals of R. The differential Jacobson radical is vastly examined in many publications (see, e.g. [7]).

A *D-homomorphism* of D-modules is a usual homomorphism of modules which commutes with the operation of taking a derivative element with respect to each derivation, and an *exact sequence of D-modules* is *D-exact* if the linking maps are d-homomorphisms. The class of all left D-modules is, in fact, a category, where the role of morphisms is played by D-homomorphisms. This category is called the category of differential modules and denoted by $R - Dmod$. The set of all differential homomorphisms from the D-module M to the D-module N is denoted by $Dhom_R(M, N)$.

Given a differential ring R together with the set of pairwise commutative derivations $\Delta = \{\delta_1, \delta_2, ..., \delta_n\}$ and a is an arbitrary element of R, we denote by $a^{(\infty)}$ the set of all elements obtained from a by applying a finite number of differential operators $\delta_1, \delta_2, ..., \delta_n$, not necessarily distinct. Symbolically it is designated by $a^{(\infty)} =$

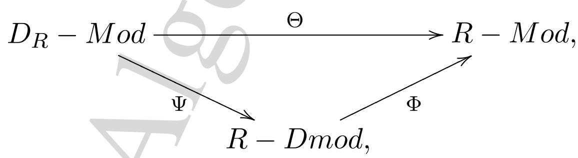
$$= \left\{ \left(\delta_{k_1}^{i_{k_1}} \delta_{k_2}^{i_{k_2}} ... \delta_{k_s}^{i_{k_s}} \right) (a) \,|k_1, ..., k_s = 1, ..., n; \quad i_{k_1}, ..., i_{k_s} \in \mathbb{N} \cup \{0\} \right\}.$$

Evidently, for any left differential ideal I and for any element $a \in R$, the ideal $\left(I : a^{(\infty)}\right)$ is differential. It is easy to check that

$$\left(\left(I : a^{(\infty)} \right) : b^{(\infty)} \right) = \left(I : (ab)^{(\infty)} \right)$$

for every $a, b \in R$.

Since the study of differential modules over the differential ring R is equivalent to the study of usual modules over the ring of linear differential operators D_R with the coefficients of R, we must distinguish, which coefficient ring is considered in the context. The following functorial diagram will be helpful



where Θ is an inclusion functor, Ψ is the identity functor and Φ is a forgetful functor.

1. Differential preradical filters and preradical Bland filters

Let (R, Δ) be a differential ring, $\Delta = \{\delta_1, \delta_2, ..., \delta_n\}$, $\delta_i \delta_j = \delta_j \delta_i$, $i, j = 1, ..., n$.

A nonempty collection $\mathcal{F}$ of left differential ideals of the differential ring R is said to be a *differential preradical filter* of R (see [9]) if the following conditions hold:

DF1. If $I \in \mathcal{F}$ and $I \subseteq J$, where J is a left differential ideal of R, then $J \in \mathcal{F}$;

DF2. If $I \in \mathcal{F}$ and $I \in \mathcal{F}$, then $I \bigcap J \in \mathcal{F}$;

DF3. If $I \in \mathcal{F}$, then $(I : a^{(\infty)}) \in \mathcal{F}$ for each $a \in R$.

If a differential preradical filter $\mathcal{F}$ satisfies an extra condition

DF4. If $I \subseteq J$ with $J \in \mathcal{F}$ and $(I : a^{(\infty)}) \in \mathcal{F}$ for all $a \in J$, then $I \in \mathcal{F}$,

then the filter $\mathcal{F}$ is called a *differential radical filter*. Further we will call such filters *HK-filters* (*differential radical filters in the sense of Horbachuk-Komarnytskyi*).

When the derivatives δ_i , $i = 1, ..., n$ are trivial, we obtain the definition of the usual preradical and Gabriel filter.

A preradical filter $\mathcal{E}$ of left ideals of the differential ring (R, Δ) is called a *preradical Bland filter* [2] if the following condition holds:

For every $I \in \mathcal{E}$, there exists a $J \in \mathcal{E}$ such that $\delta_i(J) \subseteq I$ for each $i = 1, \ldots, n$.

A Gabriel filter of left ideals of the differential ring R is called a *radical Bland filter* if it is the Bland filter as a preradical filter.

The set of all Bland preradical filters will be denoted by $Fil_B(R)$.

Bland remarked that all Gabriel filters over a commutative ring are differential; in our terms, it means that all Gabriel filters over such a ring are radical Bland filters [2]. In that paper the examples confirming the importance of radical Bland filters are given. We now prove some propositions which make the Bland's remark more accurate and show that there are plenty of Bland filters. In fact, it is much more difficult to give an example of the Gabriel filter of a differential ring which is not a radical Bland filter. Remind that a Gabriel filter which has a basis consisting of two-sided ideals is called *symmetric* (or *bounded*) (see [5]).

Proposition 1. *Any symmetric preradical filter (Gabriel filter) of an arbitrary differential ring is a preradical (radical) Bland filter. If a preradical filter (Gabriel filter) of left ideals of a differential ring R, not necessarily symmetric, has a basis consisting of left differential ideals, then it is a preradical (radical) Bland filter.*

Proof. The proof of the part of the proposition concerning preradical filters is not difficult. Thus, we prove it for Gabriel filters.

Let $\mathcal{E}$ be a symmetric Gabriel filter and let I be a left ideal of the ring R which belong to the filter $\mathcal{E}$. Then there exists a two-sided ideal $K \in \mathcal{E}$ such that $K \subseteq I$. Since $\mathcal{E}$ is multiplicatively closed, $K^2 \in \mathcal{E}$ and $\delta_i(K^2) \subseteq I$ for every $i = 1, ..., n$. Hence, $\mathcal{E}$ is a Bland filter.

Assume now that the Gabriel filter $\mathcal{E}$ has a basis B consisting of left differential ideals. We again consider an arbitrary left ideal I of $\mathcal{E}$. Choose $K \in B$ such that $K \subseteq I$. Since K is differential, then $\delta_i(K) \subseteq I$, and we are done. $\square$

The following Proposition shows that each HK-filter generates some Bland filter, necessarily unique, and that the inverse correspondence is not one-to-one.

Proposition 2. *Every differential HK-filter $\mathcal{F}$ of the Δ-ring R is a basis for some Gabriel filter $\mathcal{E}_{\mathcal{F}}$ of the ring R, which obviously is a radical Bland filter called a radical Bland filter generated by the HK-filter $\mathcal{E}$. Moreover, the set $\mathcal{B}_{\mathcal{E}}$ of all differential left ideals which belong to some fixed radical Bland filter E is a basis for some HK-filter $\mathcal{F}_{\mathcal{E}}$ of the Δ-ring R. In this case different HK-filters correspond to different radical Bland filters.*

The proof of this statement involves a routine check of the corresponding conditions, therefore it is omitted.

2. Differential preradicals and kernel functors

A *preradical* in the category $R-Dmod$ is a functor

$$\sigma : R-Dmod \rightarrow R-Dmod$$

such that:

1. $\sigma(M)$ is differential submodule in M for each $M \in R-Dmod$;
2. For each differential homomorphism $f : M \rightarrow N$ we have

$$f(\sigma(M)) \subseteq \sigma(N).$$

Denote by $R-Dpr$ the complete big lattice of all differential preradicals in $R-Dmod$.

The most natural example of a preradical is the functor (differential Jacobson radical)
$J_D : R-Dmod \rightarrow R-Dmod$, where $J_D : M \mapsto J_D(M)$.

There are following classical operations in $R-Dpr$, namely $\wedge$, $\vee$, $\cdot$ which are defined as follows, for $\sigma, \tau \in R-Dpr$ and $M \in R-Dmod$:
$(\sigma \wedge \tau)(M) = \sigma(M) \cap \tau(M)$,
$(\sigma \vee \tau)(M) = \sigma(M) + \tau(M)$,
$(\sigma \cdot \tau)(M) = \sigma(\tau(M))$.

The meet $\wedge$ and the join $\vee$ can be defined for arbitrary families $\mathcal{C}$ of differential preradicals as follows:
$\wedge\{r \in \mathcal{C}\}(M) = \cap\{r(M) | r \in \mathcal{C}\}$,
$\vee\{r \in \mathcal{C}\}(M) = \sum\{r(M) | r \in \mathcal{C}\}$.

Notice that for each $M \in R-Dmod$, $\{r(M) | r \in \mathcal{C}\}$ is a set.

The operation $\cdot$ is called a *product*. It is well known that $r_1 \cdot r_2 \leqslant r_1 \wedge r_2 \leqslant r_1 \vee r_2$.

All these operations are associative and order-preserving.

The functor $\sigma \in R - Dpr$ is a *left exact differential preradical* if for each short exact sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

the sequence

$$0 \to r\left(L\right) \xrightarrow{\sigma(f)} r\left(M\right) \xrightarrow{\sigma(g)} r\left(N\right)$$

is exact.

The functor $\sigma \in R - \mathrm{Dpr}$ is a *differential radical* if $\sigma\left(M/\sigma\left(M\right)\right) = 0$ for each $M \in R - Dmod$.

For any $\sigma \in R - Dpr$, we will use the following four classes of differential modules:

$\mathrm{T}_\sigma = \left\{M \in R - Dmod \,|\, \sigma\left(M\right) = M\right\};$

$\mathrm{F}_\sigma = \left\{M \in R - Dmod \,|\, \sigma\left(M\right) = 0\right\};$

$\bar{\mathrm{T}}_\sigma = \left\{\sigma\left(M\right) |M \in R - Dmod\right\};$

$\mathrm{F}_\sigma = \left\{M/\sigma\left(M\right) |M \in R - Dmod\right\}.$

Recall that σ is differentially idempotent if and only if $\mathrm{T}_\sigma = \bar{\mathrm{T}}_\sigma$, σ is differential radical if and only if $\mathrm{F}_\sigma = \bar{\mathrm{F}}_\sigma$. The functror σ is a *left exact differential preradical* if it is differentially idempotent and its differential pretorsion class T_σ is closed under taking differential submodules. The functor σ is a *differential radical* if and only if it is radical and its differential pretorsion-free class F_σ is closed under taking differential quotient modules.

Let $\sigma, \tau, \eta \in R - Dpr$, $\left\{\sigma_\alpha\right\}_\alpha \subseteq R - Dpr$, $M \in R - Dmod$. Then the following properties hold:

1. $\sigma \leqslant \tau \Rightarrow \sigma \vee (\tau \wedge \eta) = \tau \wedge (\sigma \vee \eta)$(Modular law);
2. If $\left\{\sigma_\alpha\right\}_\alpha$ is a directed family, then $\tau \wedge \left(\vee_\alpha \sigma_\alpha\right) = \vee_\alpha \left(\tau \wedge \sigma_\alpha\right)$;
3. $\left(\wedge_\alpha \sigma_\alpha\right)\tau = \wedge_\alpha\left(\sigma_\alpha \tau\right);$
4. $\left(\vee_\alpha \sigma_\alpha\right)\tau = \vee_\alpha\left(\sigma_\alpha \tau\right).$

The classes of idempotent differential preradicals is closed under taking arbitrary joins, and the classes of differential radicals and left exact differential preradicals are closed under taking arbitrary meets.

A differential preradical $\sigma : R - Dmod \to R - Dmod$ is called a *differential kernel functor* if for every $M \in R - Dmod$ and any differential submodule N of M, $\sigma(N) = N \bigcap \sigma(M)$.

For example, by putting in correspondence to each differential modules M its differential socle $Soc_D(M) = \sum\{P|P$ is a differentially simple module $\}$, we obtain the kernel functor of taking the differential socle.

Another example of a differential kernel functor is $\sigma_{\mathcal{F}}$, obtained by using the differential preradical Bland filter or HK-filter $\mathcal{F}$.

Let σ be a differential kernel functor. Then the module $M \in R-Dmod$ is said to be *σ-torsion* if $\sigma(M) = M$, and it is called *σ-torsion-free* if $\sigma(M) = 0$. A (pre)torsion theory is, in fact, defined by giving the class of torsion objects and the class of torsion-free objects.

Denote by $K_D(R)$ the set of all differential kernel functors, and by $I_D(R)$ the set of all idempotent differential kernel functors. Then for an arbitrary ring R the following inclusion is satisfied: $\{0, \infty\} \subseteq I_D(R) \subseteq K_D(R)$.

To an arbitrary differential kernel functor σ there is associated a preradical filter $\mathrm{T}_D(\sigma)$ of the ring R consisting of left ideals I such that R/I is a submodule of some differential σ-torsion module.

Now we state the lemma to be proved in the ordinary way.

Lemma 1. *The filter $\mathrm{T}_D(\sigma)$ is preradical Bland filter.*

Proof. It is enough to prove the condition, defined by means of derivations.

Consider an arbitrary left ideal $I \in T_D(\sigma)$. Then there exists such a σ-torsion differential module M and an element $m \in M$ that $Ann_l(m) = I$. Since M is σ-torsion, $Ann_l(m') = J \in T_D(\sigma)$. Take $K = I \cap J \in T_D(\sigma)$. Then $0 = d_i(Km) \supseteq \delta_i(K)m + Km' = \delta_i(K)m$. It means that $\delta_i(K) \subseteq Ann_l(m) = I$ for each $i = 1, ..., n$. □

The kernel functor σ on category $R-Mod$ will be called an *extension* of differential kernel functor $\overline{\sigma}$ if the functorial diagram

$$\begin{array}{ccc} R-Dmod & \xrightarrow{\overline{\sigma}} & R-Dmod \\ \Phi\downarrow & & \downarrow\Phi \\ R-Mod & \xrightarrow[\sigma]{} & R-Mod \end{array}$$

is commutative. The set of all kernel functors σ which are extensions of some differential kernel functors will be denoted by $K_e(R)$.

The map $\Gamma : \sigma \mapsto T_D(\sigma)$ is not one-to-one as the following simple example shows.

Example. Let k be any universal differential field of characteristic 0. Then there only defined three different differential kernel functors on the category of all k-linear differential spaces $k-Dmod$, and all these kernel functors are, in fact, idempotent. However, there are only two kernel functors on the category of linear spaces $k-Mod$, therefore there exist only two trivial Bland filters over the ring k.

In case when all the derivations $\delta_i, i = 1, ..., n$, are zero, the notions introduced above transform into the well known notions of the torsion theory over rings with no additional structures. In the corresponding notations the index D is omitted.

For additional information on kernel functors see [6], [14].

The differential left module M will be called *uniformly differential* if each of its cyclic submodules is differential. A usual socle of any differential module is uniformly differential. It is clear that the sum of any two uniformly differential submodules is a uniformly differential. Since the union of any chain of uniformly differential submodules is a uniformly differential module, then each differential module contain the largest uniformly differential submodule $U(M)$ (by Zorn's lemma). It is easy to check that the functor $U(M)$ is an idempotent differential kernel functor.

A differential kernel functor will be called a *kernel HK-functor* if for each differential module M the inclusion $\sigma(M) \subseteq U(M)$ holds.

It is clear that for every HK-filter $\mathcal{F}$ the differential kernel functor $\sigma_{\mathcal{F}}$ is HK-functor.

The following theorem shows that the converse is true.

Theorem 1. *There exists a one-to-one correspondence between (idempotent) kernel functors on the category $R - Mod$, which are the extensions of some differential kernel functor defined on $R - Dmod$, and (radical) preradical Bland filters of the ring R .*

Proof. Let $\sigma \in K(R)$ and there is $\overline{\sigma} \in K_D(R)$ for which $\sigma \mid_{R-Dmod} = \overline{\sigma}$. Then, by Lemma 1, the filter $T_D(\overline{\sigma})$ is a Bland filter. Thus, the map

$$T_D : K_e(R) \to Fil_B(R)$$

is well defined. Conversely, if $\mathcal{F} \in fil_B(R)$, then the kernel functor

$$\sigma_{\mathcal{F}} : R - Mod \to R - Mod$$

has the property: $\sigma_{\mathcal{F}}(M) \in R - Dmod$ for every $M \in R - Dmod$ (this can by proved as in [2], p. 3). It means that $\sigma_{\mathcal{F}} \mid_{R-Dmod} = \overline{\sigma}$. Therefore, the map $\vartheta : Fil_B(R) \to K_e(R)$, were $\vartheta : \mathcal{F} \mapsto \sigma_{\mathcal{F}} \in K_e(R)$ is well defined.

Now we compute the compositions $\vartheta \circ T_D$ and $T_D \circ \vartheta$. For the kernel functor $\sigma : R - mod \to R - mod$, which is the extension of some differential kernel functor $\overline{\sigma}$, we have $(\vartheta \circ T_D)(\sigma) = \vartheta(T_D(\vartheta)) = \tau \in K_D(R)$. If M is some left R-module, then

$$\tau(M) = \{x \in M | \exists I \in I_D(\sigma), Ix = 0\} = \sigma(M).$$

Thus $\vartheta \circ T_D = 1|_{K_e(R)}$.

Conversely, if $\mathcal{F} \in Fil_B(R)$ then

$$(T_D \circ \vartheta)(\mathcal{F}) = T_D(\theta(\mathcal{F})) =$$

$$= \{I | I \in \mathcal{L}(R), \exists M \in R - Dmod, \exists x \in M, Ann_l(x) = I, M \in \mathcal{T}_{\vartheta(\mathcal{F})}\} =$$
$$= \{I | \exists M \in R - Dmod, \exists x \in M, Ann_l(x) \in \mathcal{F}, Ann_l(x) = I\} = \mathcal{F}.$$

Thus $T_D \circ \vartheta = 1|_{Fil_B(R)}$.

In the case of the idempotent kernel functor, the result is the consequence of the Lemma 1.5 in [2].

□

Theorem 2. *There exists a one-to-one correspondence between (idempotent) kernel HK-functors on the category $R - Dmod$ and (radical) HK-filters of the ring R.*

Proof. Let σ be an (idempotent) kernel HK-functor. Denote by $\mathcal{E}_\sigma$ the set of all left differential ideals of the ring R, for which there exists a uniformly differential σ-torsion module M and an element $x \in M$ such that $Ix = 0$. Then all the conditions of the definition of (idempotent) HK-filters are satisfied (i.e., $\mathcal{E}_\sigma$ is a HK-filter). Thus, we have the map

$$\varphi : K_{HK}(R) \to HKFil(R),$$

where $\varphi : \sigma \mapsto \mathcal{E}_\sigma$. Now we show that φ is bijective. If $\sigma \neq \tau$, $\sigma, \tau \in K_{HK}(R)$ then there is a differential module $M \neq 0$ such that $\sigma(M) \neq \tau(M)$. Since $\sigma(M)$ is uniformly differential, we can select an $x \in M$ such that $\sigma(Rx) \neq \tau(Rx)$ and $Ann_l(x) \in \mathcal{L}_D(R)$. Then $Ann_l(x) \in \mathcal{E}_\tau \setminus \mathcal{E}_\sigma$ or $Ann_l(x) \in \mathcal{E}_\sigma \setminus \mathcal{E}_\tau$ and we proved the injectivity of φ.

The surjectivity of this map is evident, since for every HK-filter $\mathcal{F} \in HKFil(R)$ there is a kernel functor $\sigma_\mathcal{F}$, defined by the rule $\sigma_\mathcal{F}(M) = \{x \in M | Ann_l(x) \in \mathcal{F}\}$, and it is a HK-functor over the ring R. Furthermore, $\varphi(\sigma_\mathcal{F}) = \mathcal{F}$. □

The set K(R) may be partially ordered by putting $\sigma \leqslant \tau$ if and only if $\sigma(M) \subseteq \tau(M)$ for all $M \in R - Mod$. Moreover, the set of differential kernel functors $K_D(R)$ is a complete lattice. In addition a product of kernel functors can be defined as follows: given σ and τ in $K_D(R)$, let $\mathcal{A}$ and $\mathcal{B}$ be their corresponding differential pretorsion classes. Consider the extension of $\mathcal{B}$ by $\mathcal{A}$, i. e., the collection $\mathcal{C} = \{M \in R - Dmod|$ there exists an exact sequence $0 \longrightarrow A \longrightarrow M \longrightarrow B \longrightarrow 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}\}$. Then $\mathcal{C}$ is a differential pretorsion class and we let $\tau\sigma$ denote the corresponding differential kernel functor. In terms of preradical filters, this product amounts to defining $\mathcal{F}_1\mathcal{F}_2 = \{I \in R | \exists J \in \mathcal{F}_2$ such that $I \subseteq J$ and $(I : a) \in \mathcal{F}_1$ for all $a \in J\}$. It can be verifies that if $I \in \mathcal{F}_1$ and $J \in \mathcal{F}_2$, then $IJ \in \mathcal{F}_1\mathcal{F}_2$.

With the natural order given by the inclusion of filters, the set of all preradical filters of left ideals $Fil - R$ becomes structurally identical to the set of all kernel functors $K(R)$. The bottom of $Fil(R)$ is the filter $\{R\}$ which we will denote by 0; its corresponding kernel functor is the zero kernel functor, which we will denote by 0, i. e. the functor $0 : R - Mod \to R - Mod$ defined by $0(M) = 0$ for all $M \in R - Mod$. The largest element of $Fil(R)$ is the set of all the left ideals of R; its corresponding kernel functor is the identity functor ∞, i. e. the functor $\infty : R - Mod \to R - Mod$ defined by $\infty(M) = M$ for all $M \in R - Mod$.

Conversely, any collection $\mathcal{C}$, closed under submodules, arbitrary direct sums, homomorphic images and extensions is of the form $\mathcal{C} = \mathcal{C}(\sigma)$ for a unique $\sigma \in I(R)$.

Theorem 3. *1. The set of (idempotent) kernel functors $\sigma \in K(R)$ on the category $R - Mod$ which are the extensions of some (idempotent) differential kernel functors $\overline{\sigma} \in K_D(R)$ defined on $R - Dmod$ is an complete lattice.*

2. The set of all (idempotent) kernel HK-functors is a complete lattice.

Proof. 1. Let $\{\sigma_i\}_{i \in I}$ be a family of (idempotent) kernel functors on category $R - Mod$, where every σ_i is the extension of the (idempotent) differential kernel functor $\overline{\sigma}_i$ on $R - Dmod$. Then for every left $R-$module M we have $(\bigwedge_{i \in I} \sigma_i)(M) = \bigcap_{i \in I} \sigma_i(M)$. It is clear that $(\bigwedge_{i \in I} \sigma_i)$ is an (idempotent) kernel functor. Denote by $\overline{\tau}$ an (idempotent) differential kernel functor $\bigwedge_{i \in I} \overline{\sigma}_i$. If M is differential module then we have $\overline{\tau}(M) = (\bigwedge_{i \in I} \overline{\sigma}_i)(M) = \bigcap_{i \in I} \overline{\sigma}_i(M) = \bigcap_{i \in I} \sigma_i(M) = (\bigwedge_{i \in I} \sigma_i)(M)$. It follow that $\bigwedge_{i \in I} \sigma_i$ is the extension of $\overline{\tau}$.

By analogy, we prove that $\bigvee_{i \in I} \sigma_i$ also exists and is the extension of some $\overline{\sigma} \in K_D(R)$.

2. For any family of kernel HK-functors $\{\sigma_i\}_{i \in I}$, the functors $\bigwedge_{i \in I} \sigma_i$ and $\bigvee_{i \in I} \sigma_i$ are defined as above. We prove that they are HK-functors. If $M \in R - Dmod$ then $\sigma_i(M)$ is uniformly differential for all $i \in I$. It follows that $\bigcap_{i \in I} \sigma_i(M)$ and $\bigcup_{i \in I} \sigma_i(M)$ also are uniformly differentials. Thus, $\bigwedge_{i \in I} \sigma_i(M)$ and $\bigvee_{i \in I} \sigma_i(M)$ are HK-functors. □

3. Lattices of differential pretorsion theories and Bland pretorsion theories

A *differential torsion theory* τ for the category $R - Dmod$ is a pair (T, F) of classes of differential R-modules such that

1. $\mathrm{T} \cap \mathrm{F} = 0$;

2. If $M \to N \to 0$ is an differentially exact sequence in $R - \mathrm{Dmod}$ and $M \in \mathrm{T}$, then $N \in \mathrm{T}$;

3. If $0 \to M \to N$ is an differentially exact sequence in $R - Dmod$ and $N \in \mathrm{F}$, then $M \in \mathrm{F}$;

4. For each differential R-module M there exists a short differentially exact sequence $0 \to T \to M \to F \to 0$ in $R - Dmod$ for which $T \in \mathrm{T}$ and $F \in \mathrm{F}$.

Remind that a class T is a torsion class for a differential torsion theory τ if and only if it is closed under differential quotient modules, direct sums and extensions. A class F is a torsion-free class for τ if and only if F is closed under differential submodules, direct products and extensions.

The differential modules in T are called *τ-torsion*, and the ones in F are *τ-torsion-free*.

A differential torsion theory is called *hereditary* if T is closed under differential submodules, and *cohereditary* if F is closed under differential factor modules. A differential torsion theory is hereditary if and only if a torsion-free class F is closed under differential injective envelopes.

Every differential R-module has a unique and the largest τ-torsion differential submodule given by $t_\tau(M) = \sum_{N \in \mathrm{S}} N$, where S is the set of all τ-torsion differential submodules of the differential module M.

In case when all the derivation are trivial we obtain the definition of a torsion theory for the category $R - Mod$.

A torsion theory τ for $R - Mod$ is called a *Bland torsion theory* if it is hereditary and the filter $\mathcal{E}_\tau = \{I | I$ is a left ideal of the ring R and $R/I \in \mathrm{T}\}$ determined by τ is a Bland filter.

In [2], Lemma 1.5) Bland has proved the following Proposition:

Proposition 3. *For a hereditary torsion theory τ on $R - Mod$ the following properties are equivalent:*

(1) $\mathcal{E}_\tau$ is a Bland filter;

(2) For each left R-module M and for each $x \in \tau(M)$ there is such $I \in \mathcal{E}_\tau$ that $\delta(I) \subseteq (0 : x)$;

(3) For each R-module M and for every derivation d defined on M, $d(t_\tau(M)) \subseteq t_\tau(M)$.

This proposition shows that a torsion theory in the category $R - Mod$ is differential if and only if for each differential module M, $d(t_\tau(M))$ is its differential submodule (i.e., $\tau \in K_e(R)$).

Proposition 4. *The intersection and union of an arbitrary family of preradical (radical) Bland filters of left ideals of the differential ring (R, Δ) is a preradical (radical) Bland filter of left ideals of (R, Δ).*

Proof. The assertions are direct consequences of the Theorems 1 and 2. □

Since the intersection of an arbitrary family of Bland filters is a Bland filter, we may see that the set of all Bland filters has the structure of a complete lattice, where the meet and the join of Bland filters are defined in the usual way.

We will also need the following fact.

Proposition 5. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are preradical (radical) Bland filters of left ideals of the differential ring (R, Δ), then their product $\mathcal{E}_1 \cdot \mathcal{E}_2$ is a preradical (radical) Bland filter of left ideals of (R, Δ).*

Proof. It is well known from [5] that the product of preradical filters is a preradical filter. We only need to show that the filter $\mathcal{E}_1 \cdot \mathcal{E}_2$ satisfies the second condition. Let $I \in \mathcal{E}_1 \cdot \mathcal{E}_2$. Then there exists such $J \in \mathcal{E}_2$ that $I \subseteq J$ and $(I : a) \in \mathcal{E}_1$ for all $a \in J$. The there exist such left ideals $K \in \mathcal{E}_2$ and $K_a \in \mathcal{E}_1$ with $a \in J$ that $\delta_i(K) \subseteq J$ and $\delta_i(K_a) \subseteq (I : a)$ for each $i = 1, 2, \ldots, n$. Consider the left ideal $T = I \bigcap K$ of R. It is clear that $\delta_i(T) \subseteq I$. Since $T \subseteq J \bigcap K$ and $J \bigcap K \in \mathcal{E}_2$, the inclusion $(T : a) = (I \bigcap K : a) \supseteq (I : a)$, for every $a \in J \bigcap K$, follows that $(T : a) \in \mathcal{E}_2$. It means that $T \in \mathcal{E}_1 \cdot \mathcal{E}_2$. □

4. Quantales of Bland and HK-filters

A *quantale* Q is a complete lattice with an associative binary multiplication $*$ satisfying

$$x * \left(\bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (x * x_i)$$

and

$$\left(\bigvee_{i \in I} x_i \right) * x = \bigvee_{i \in I} (x_i * x)$$

for all $x, x_i \in Q$, $i \in I$, I is a set. 1 denotes the greatest element of the quantale Q, 0 is the smallest element of Q. A quantale Q is said to be *unital* if there is an element $u \in Q$ such that $u * a = a * u = a$ for all $a \in Q$.

A meet of the preradical filters $\mathcal{F}_1$ and $\mathcal{F}_2$ is the preradical filter $\mathcal{F}_1 \bigwedge \mathcal{F}_2$ which is the intersection of $\mathcal{F}_1$ and $\mathcal{F}_2$.

A join of preradical filters $\mathcal{F}_1$ and $\mathcal{F}_2$ is the least preradical filter $\mathcal{F}_1 \bigvee \mathcal{F}_2$ which contain both $\mathcal{F}_1$ and $\mathcal{F}_2$.

A product of preradical filters $\mathcal{F}_1$ and $\mathcal{F}_2$ is a set $\mathcal{F}_1 \cdot \mathcal{F}_2$ of those left differential ideals of the differential ring R for which there exists an ideal $H \in \mathcal{F}_2$ such that $I \subseteq H$ and $(I : a^{(\infty)}) \in \mathcal{F}_1$ for all $a \in J$.

By a *subquantale* of a quantale Q is meant a subset K closed under joins and multiplication.

Proposition 6. *The set of all HK-filters of the differential ring R forms a quantale with respect to meets.*

Proof. If $I \in \mathcal{F} \cdot (\bigwedge_{i\in\Omega} \mathcal{F}_i)$ then there exists such $J \in \bigwedge_{i\in\Omega} \mathcal{F}_i$ that $I \subseteq J$ and $(I : a^{(\infty)}) \in \mathcal{F}$ for all $a \in J$. Then for all $i \in \Omega$ $J \in \mathcal{F}_i$ and $I \subseteq J$ and $(I : a^{(\infty)}) \in \mathcal{F}$, so for all $i \in \Omega$ $I \in \mathcal{F} \cdot \mathcal{F}_i$, that is $I \in \bigwedge_{i\in\Omega}(\mathcal{F} \cdot \mathcal{F}_i)$.

If $I \in (\bigwedge_{i\in\Omega} \mathcal{F}_i) \cdot \mathcal{F}$ then there exists such $J\mathcal{F}$ that $I \subseteq J$ and $(I : a^{(\infty)}) \in \bigwedge_{i\in\Omega} \mathcal{F}_i$ for all $a \in J$. Then for all $i \in \Omega$ $J \in \mathcal{F}$ and $I \subseteq J$ and $(I : a^{(\infty)}) \in \mathcal{F}_i$, so for all $i \in \Omega$ $I \in \mathcal{F}_i \cdot \mathcal{F}$, that is $I \in \bigwedge_{i\in\Omega}(\mathcal{F}_i \cdot \mathcal{F})$.

Check that $\mathcal{F} * (\wedge_{i\in I}\mathcal{F}_i) = \wedge_{i\in I} (\mathcal{F} * \mathcal{F}_i)$ and $(\wedge_{i\in I}\mathcal{F}_i) * \mathcal{F} = = \wedge_{i\in I} (\mathcal{F}_i * \mathcal{F})$. We have that $\mathcal{F} * (\wedge_{i\in I}\mathcal{F}_i) = \{I|\exists H \in \wedge_{i\in I}\mathcal{F}_i I \subseteq H \forall a \in H (I : a) \in \mathcal{F}\} = \{I|\exists H \in \bigcap_{i\in I} \mathcal{F}_i I \subseteq H \forall a \in H (I : a) \in \mathcal{F}\} = \{I|\exists \forall i \in I H \in \mathcal{F}_i I \subseteq H \forall a \in H (I : a) \in \mathcal{F}\} = \wedge_{i\in I} (\mathcal{F} * \mathcal{F}_i)$. □

Theorem 4. *The set of all preradical Bland filters of left ideals of the differential ring (R, Δ) is a quantale with respect to meets, which is a subquantale of the quantale of all preradical filters of the differential ring (R, Δ).*

Proof. As proved by J.Golan, for all preradical filters $\mathcal{E}_i \in Fill(R), i \in I, \mathcal{E} \in Fill(R)$ the following equality is valid:

$$\mathcal{E} \cdot (\bigwedge_{i\in I} \mathcal{E}_i) = \bigwedge_{i\in I} (\mathcal{E} \cdot \mathcal{E}_i)$$

(see [5], Proposition 3.13, p.37). It means that the complete lattice $Fill(R)$ is a quantale with respect to meets. (In fact, J.Golan stated his results in terms of semirings). It follows that, in order to finish the proof, it is enough to apply the Proposition 5 and the Theorems 1, 3. □

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References

[1] O. D. Artemovych, Differentially simple rings: survey, *Matematychni Studii*, vol. **23**, No. 2, 2005, 115-128.

[2] P. E. Bland, *Differential torsion theory, Journal of Pure and pplied Algebra*, **204**, 2006, pp. 1-8.

[3] F.Borceux, R.Cruciani, Generic Representation Theorem for Non-commutative Rings, *J. Algebra* , **167**, 1994, pp. 291-308.

[4] J. S. Golan, *Torsion Theories*, in: Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 29, Longman Scientific & Technical, Harlow, 1986, John Wiley & Sons Inc., New York.

[5] J. S. Golan, *Linear topologies on a ring: an overview*, in: Pitman Research Notes in Mathematics Series 159, Longman Scientific & Technical, Harlow, 1987, John Wiley & Sons Inc., New York.

[6] O. Goldman, Rings and modules of quotients, J. Algebra, **13**, 1969, pp. 10-47.

[7] H. E. Gorman, Differential rings and modules, *Scripta mathematica*, Vol. **29**, No. 1-2, 1979.

[8] M. Hazewinkel, N. Gubareni, V. V. Kirichenko *Algebras, Rings and Modules* Volume 1, Kluwer academic publisher, New York, Boston, Dortrecht, London, Moscow. 2004.

[9] O. L. Horbachuk, M. Ya. Komarnytskyi, On differential torsions, In: *Teoretical and applied problems of algebra and differential equations*, Kyev, Naukova Dumka, 1977 (in Russian).

[10] I. Kaplansky, *Introduction to differential algebra*, in: Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999.

[11] T. Y. Lam, *Lectures on Modules and Rings*, in: Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999.

[12] A. V. Mikhaliev, E. V. Pankratiev, Differential and difference algebra, *Itogi nauki i tehniki*, Seriya Algebra. Topology. Geometry, Vol. 25, 1987, pp. 67- .

[13] A. P. Mishina, L.A. Skorniakov, *Abelian groups and modules*, Nauka, Moskov, 1969.

[14] F. van Oystaeyen, *Primer spectra of non-commutative algebra*, Lect. Notes in Math. 444, Springer-Verlag, Berlin, 1975.

[15] J. Paseka, J. Rosicky, *Quantales*, in: B. Coecke, D. Moore, A. Wilce, (Eds.), Current Research in Operational Quantum Logic: Algebras, Categories and Languages, Fund. Theories Phys., vol. 111, Kluwer Academic Publishers, 2000, pp. 245-262.

[16] L. Vas, Differerntiability of torsion theories, *Journal of Pure and Applied Algebra*, **210**, 2007, pp. 847-853.

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