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RESEARCH ARTICLE

On one-sided Lie nilpotent ideals of associative rings

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Dedicated to Professor V. V. Kirichenko
on the occasion of his 65th birthday

Abstract. We prove that a Lie nilpotent one-sided ideal of an associative ring R is contained in a Lie solvable two-sided ideal of R. An estimation of derived length of such Lie solvable ideal is obtained depending on the class of Lie nilpotency of the Lie nilpotent one-sided ideal of R. One-sided Lie nilpotent ideals contained in ideals generated by commutators of the form $[\ldots[[r_1, r_2], \ldots], r_{n-1}], r_n]$ are also studied.

Introduction

It is well-known that if I is an one-sided nilpotent ideal of an associative ring R then I is contained in a two-sided nilpotent ideal of R. Hence the following question is of interest: for which one-sided ideal I of the ring R there exists a two-sided ideal J such that $J \supseteq I$ and J has properties like properties of I. In [5] it was noted that for an one-sided commutative ideal I of a ring R there exists a nilpotent-by-commutative two-sided ideal J of the ring R such that $J \supseteq I$.

Note that Lie nilpotent and Lie solvable associative rings were investigated by many authors (see, for example [4], [6], [7], [1]) and the structure of such rings is studied well enough.

In this paper we prove that a Lie nilpotent one-sided ideal I of an associative ring R is contained in a Lie solvable two-sided ideal J of

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R. An estimation (rather rough) of Lie derived length of the ideal J depending on Lie nilpotency class of I is also obtained (Theorem 1).

In case when the Lie nilpotent one-sided ideal I is contained in the ideal R_n of the ring R generated by all commutators of the form $[\ldots[[r_1, r_2], \ldots], r_{n-1}], r_n]$ and the Lie derived length of I is less then n it is proved that I is contained in a nilpotent two-sided ideal of R (Theorem 2).

The notations in the paper are standard. If S is a subset of an associative ring R then by $Ann^l_R(S)$ ($Ann^r_R(S)$) we denote the left (respectively right) annihilator of S in R. We also denote by $R^{(-)}$ the adjoint Lie ring of the associative ring R. Further, by $R_n^{(-)}$ we denote the n-th member of the lower central series of the Lie ring $R^{(-)}$. Then $R_n = R_n^{(-)} + R_n^{(-)} \cdot R = R_n^{(-)} + R \cdot R_n^{(-)}$ is a two-sided ideal of the (associative) ring R. In particular, R_2 is a two-sided ideal of the ring R generated by all commutators of the form $[r_1, r_2] = r_1 r_2 - r_2 r_1$, $r_1, r_2 \in R$. If R is a Lie solvable ring (i.e. such that $R^{(-)}$ is a solvable Lie ring) then we denote by $s(R)$ its Lie derived length. Analogously, by $c(R)$ we denote Lie nilpotency class of a Lie nilpotent ring R.

1. Lie nilpotent one-sided ideals

Lemma 1. *Let I be an one-sided ideal of an associative ring R and $Z = Z(I)$ be the center of I. Then there exists an ideal J in R such that $J^2 = 0$ and $[Z, R] \subseteq J$.*

Proof. Let, for example, I be a right ideal from R. Take arbitrary elements $z \in Z$, $i \in I$, $r \in R$. Then it holds $z(ir) - (ir)z = 0$ (since $ir \in I$). This implies the equality $i(zr - rz) = 0$ since $z \in Z(I)$. As elements z, i, r are arbitrarily chosen then we have $I[Z, R] = 0$. Consider the right annihilator $T = Ann^r_R(I)$. It is clear that T is a two-sided ideal of the ring R (since I is a right ideal of R) what implies that $[Z, R] \subseteq T$.

Further, for any element of the form $zr - rz$ from $[Z, R]$ and for any $t \in T$ it holds $(zr - rz)t = z(rt) - r(zt)$. Since $rt \in T$ then $z(rt) = 0$. Besides, $z \in I$ and therefore $zt = 0$ what brings the equality $(zr - rz)t = 0$. It means that $[Z, R] \cdot T = 0$.

Consider the left annihilator $J = Ann^l_T(T)$. It is easy to see that J is a two-sided ideal of the ring R. From relations $[Z, R] \subseteq T$ and $[Z, R] \cdot T = 0$ we have the inclusion $[Z, R] \subseteq J$. It is also clear that $J^2 = 0$. Analogously one can consider the case when I is a left ideal. $\square$

Theorem 1. *Let R be an associative ring and I be an one-sided ideal of R. If the subring I is Lie nilpotent then I is contained in a Lie solvable* *two-sided ideal J of R such that $s(J) \subseteq m(m+1)/2+m$ where $m = c(I)$ is Lie nilpotency class of the subring I.*

Proof. Let for example I be a right ideal. We prove our proposition by the induction on the class of Lie nilpotency $n = c(I)$ of the subring I. If $n = 1$ then I is a commutative right ideal and by Lemma 1 the ring R contains such an ideal T with zero square that it holds $(I+T)/T \subseteq Z(R/T)$ in the quotient ring R/T where $Z(R/T)$ is the center of R/T. It means that $I+T$ is a two-sided ideal of the ring R and $s(I+T) \leqslant 2$. Clearly $2 = n+n(n+1)/2$ if $n = 1$ and the statement of Theorem is true in case $n = 1$. Assume that the statement is true in case $c(I) \leqslant n-1$ and prove it when $c(I) = n$. Denote by Z the center of the subring I. By Lemma 1 there exists an ideal T of R with $T^2 = 0$ such that $[Z, R] \subseteq T$. Consider the quotient ring $\overline{R} = R/T$. Then $\overline{Z} = (Z+T)/T$ lies in the center of $\overline{R}$ and therefore $\overline{Z} + \overline{Z} \cdot \overline{R} = \overline{Z} + \overline{R} \cdot \overline{Z}$ is a two-sided ideal of the ring $\overline{R}$. Since $\overline{Z} \subseteq \overline{I} = (I+T)/T$ the ideal $\overline{Z} + \overline{Z} \cdot \overline{R}$ is Lie nilpotent of and its class of Lie nilpotency $\leqslant m$. Further, the quotient ring $\overline{R}/(\overline{Z} + \overline{Z} \cdot \overline{R})$ contains the right Lie nilpotent ideal $\overline{I} + (\overline{Z} + \overline{Z} \cdot \overline{R})/(\overline{Z} + \overline{Z} \cdot \overline{R})$ which is Lie nilpotent of class of Lie nilpotency $\leqslant m-1$. By the induction assumption the last right ideal is contained in some Lie solvable ideal of the ring $\overline{R}/(\overline{Z} + \overline{Z} \cdot \overline{R})$ of derived length $\leqslant \frac{(m-1)m}{2} + (m-1)$. Since $\overline{Z} + \overline{Z} \cdot \overline{R}$ is Lie solvable and its derived length $\leqslant m$ (even $\leqslant [log_2 m]+1$ but we take a rough estimation) and we consider the quotient ring R/T where T is Lie solvable of derived length 1, one can easily see that I is contained in some Lie solvable (two-sided) ideal of derived length which does not exceed

$$\frac{(m-1)m}{2} + (m-1) + (m+1) = \frac{(m+1)m}{2} + m.$$

Analogously one can consider the case when I is right ideal. $\square$

It seems to be unknown whether a sum of two Lie nilpotent associative rings is Lie solvable. So the next statement can be of interest (see also results about sums of PI-rings in [3]).

Corollary 1. *Let R be an associative ring which can be decomposed into a sum $R = A+B$ of its Lie nilpotent subrings A and B. If at least one of these subrings is an one-sided ideal of R then the ring R is Lie solvable.*

Remark 1. The statements of Theorem 1 and its Corollary become false when we replace Lie nilpotency of one-sided ideals by Lie solvability. Really, consider full matrix ring $R = M_2(\mathbb{K})$ over an arbitrary field $\mathbb{K}$ of characteristic $\neq 2$. It is clear that

$$I = \left\{ \left(\begin{array}{cc} x & y \\ 0 & 0 \end{array} \right) \middle| \, x, y \in \mathbb{K} \right\}$$

is a right Lie solvable ideal of the ring R but I is not contained in any Lie solvable ideal of R since R is a non-solvable Lie ring. It is also clear that

$$R = I + J \text{ where } J = \left\{ \left(\begin{array}{cc} 0 & 0 \\ z & t \end{array} \right) \middle| \, z, t \in \mathbb{K} \right\},$$

i.e. the simple associative ring R is a sum of two right Lie solvable ideals.

2. On embedding of Lie nilpotent ideals in rings

Lemma 2. *Let R be an associative ring, A be a Lie nilpotent subring of R of Lie nilpotency class $< m$. If Z_0 is a subring of A such that $Z_0 \subseteq Z(R)$ and $Z_0R \subseteq A$ then $Z_0^m R_m = 0$.*

Proof. Consider the two-sided ideal $J = Z_0 + Z_0R = Z_0 + RZ_0$ of the ring R. As $J \subseteq A$ then $\underbrace{[J, ..., J]}_{m} = 0$ by the condition $c(A) < m$. Further, it is easily to show that

$$[J, J] = [Z_0 + Z_0R, Z_0 + Z_0R] = Z_0^2[R, R].$$

By induction on k one can also show that $\underbrace{[J, ..., J]}_{k} = Z_0^k \underbrace{[R, ..., R]}_{k}$. Then we have from the condition on J that $\underbrace{[J, ..., J]}_{m} = Z_0^m \underbrace{[R, ..., R]}_{m} = 0$. This implies the equality

$$Z_0^m R_m = Z_0^m(\underbrace{[R, ..., R]}_{m} + \underbrace{[R, ..., R]}_{m} \cdot R) = \underbrace{[J, ..., J]}_{m} + \underbrace{[J, ..., J]}_{m} \cdot R = 0. \quad \square$$

Lemma 3. *Let R be an associative ring, I be an ideal of R. Then*

1) if J is a nilpotent ideal of the subring I then J lies in a nilpotent ideal J_I of the ring R such that $J_I \subseteq I$;

2) if $S = Ann_I^l(I)$ (or $Ann_I^r(I)$) then S is contained in a nilpotent ideal of the ring R which is contained in I.

The proof of this Lemma immediately follows from Lemma 1.1.5 from [2].

Theorem 2. *Let R be an associative ring and I be a Lie nilpotent one-sided ideal of R. If $I \subseteq R_n$ and Lie nilpotency class of I is less than n then I is contained in an (associative) nilpotent ideal of R.*

Proof. Let for example I be a right ideal of the ring R and $I \subseteq R_n$. One can assume that that $n \geqslant 2$ because the statement of Theorem is obvious in case $n = 1$. We fix $n \geqslant 2$ and prove the statement of Theorem by induction on the class of Lie nilpotency $c = c(I)$ of the subring I. If $c = 0$ then I is the zero ideal and the proof is complete. Assume that the statement is true for rings R with $c(I) \leqslant c-1$ and prove it in case $c(I) = c$. Since I is Lie nilpotent then by Lemma 1 there exists a nilpotent ideal T of the ring R such that in the quotient ring $\overline{R} = R/T$ it holds $[\overline{Z_0}, \overline{R}] = 0$ where Z_0 is the center of the subring I and $\overline{Z_0} = (Z_0 + T)/T$. Then by Lemma 2 it holds the relation $\overline{Z_0^n} \cdot \overline{R_n} = 0$. If $\overline{Z_0^n} = 0$ then $\overline{Z_0} + \overline{Z_0 R}$ is a nilpotent ideal of the ring $\overline{R}$ and then the subring Z_0 is contained in the nilpotent ideal $J = Z_0 + T$ of the ring R. Since in the quotient ring R/J for the right ideal $(I + J)/J$ it holds the inequality $c((I + J)/J) \leqslant c - 1$ then by the inductive assumption $(I + J)/J$ is contained in a nilpotent ideal S/J of the ring R/J. But then $I \subseteq S$ where S is nilpotent ideal of the ring R.

Let now $\overline{Z_0^n} \neq 0$. Then $\overline{Z_0^n} \subseteq Ann^l_{\overline{R_n}}(\overline{R_n})$ and since $\overline{Z_0} \subseteq \overline{R_n}$ then $\overline{Z_0^n}$ is contained in a nilpotent ideal $\overline{M}$ of the ring $\overline{R}$ by Lemma 3. It is obvious that $\overline{Z_0} + \overline{Z_0 R}$ is a nilpotent ideal of the ring $\overline{R}$. Repeating the above considerations we see that $I \subseteq S$ where S is a nilpotent ideal of the ring R. □

Corollary 2. *Let R be an associative ring with condition $R = [R, R]$. If I is a Lie nilpotent one-sided ideal of R then there exists a nilpotent (two-sided) ideal J of the ring R such that $I \subseteq J$*

Corollary 3. *Let R be a semiprime ring. Then every Lie nilpotent one-sided ideal is contained in the center $Z(R)$ of the ring R and has trivial intersection with the ideal R_2.*

Proof. Really since all nilpotent ideals of the ring R are zero then by Lemma 1 every Lie nilpotent one-sided ideal I is contained in $Z(R)$. Since $IR \subseteq Z$ then $[IR, R] = I[R, R] = 0$. Then from this equality we have $IR_2 = I([R, R] + [R, R] \cdot R) = 0$. Denote $J = I \cap R_2$. It is easily to show that $J \subseteq Ann^l_{R_2}(R_2)$ and by Lemma 3 the intersection J lies in a nilpotent ideal of the ring R. Because the ring R is semiprime we have $J = 0$. □

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