Algebra and Discrete Mathematics
Number 4. **(2007).** pp. 84 – 101

RESEARCH ARTICLE

Discrete limit theorems for Estermann zeta-functions. I

Antanas Laurinčikas and Renata Macaitienė

Communicated by V. V. Kirichenko

In honour of the 65th birthday of Professor V. V. Kirichenko

Abstract. A discrete limit theorem in the sense of weak convergence of probability measures on the complex plane for the Estermann zeta-function is obtained. The explicit form of the limit measure in this theorem is given.

Introduction

As usual, denote by $\mathcal{P}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ the sets of all primes, positive integers, non-negative integers, integers, real and complex numbers, respectively. For arbitrary $\alpha \in \mathbb{C}$ and $m \in \mathbb{N}$, the generalized divisor function $\sigma_\alpha(m)$ is defined by

$$\sigma_\alpha(m) = \sum_{d/m} d^\alpha.$$

If $\alpha = 0$, then $\sigma_\alpha(m)$ becomes the divisor function

$$\sigma_0(m) = d(m) = \sum_{d/m} 1.$$

2000 Mathematics Subject Classification: *11M41.*
Key words and phrases: *compact topological group, Estermann zeta-function, Haar measure, probability measure, limit theorem, weak convergence.*

It is well known that, for every positive ϵ,

$$d(m) \ll_\epsilon m^\epsilon, \quad m \in \mathbb{N}.$$

Here and in the sequel $f(x) \ll_\eta g(x)$ with a positive function $g(x)$, $x \in I$, means that there exists a constant $c = c(\eta) > 0$ such that $|f(x)| \le cg(x)$, $x \in I$. Since

$$\sigma_\alpha(m) = m^\alpha \sigma_{-\alpha}(m), \tag{1}$$

hence we have that

$$\sigma_\alpha(m) \ll_\epsilon m^{\epsilon+\max(\Re\alpha,0)}. \tag{2}$$

Let $s = \sigma + it$ be a complex variable, and k and l be coprime integers. For $\sigma > \max(1, 1 + \Re\alpha)$, the Estermann zeta-function $E(s; \frac{k}{l}, \alpha)$ with parameters α and $\frac{k}{l}$ is defined by

$$E\left(s; \frac{k}{l}, \alpha\right) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)}{m^s} \exp\left\{2\pi i m \frac{k}{l}\right\}.$$

The function $E(s; \frac{k}{l}, \alpha)$ is analytically continuable to the whole complex plane, except for two simple poles at $s = 1$ and $s = 1 + \alpha$ if $\alpha \neq 0$, and a double pole at $s = 1$ if $\alpha = 0$.

The function $E(s; \frac{k}{l}, \alpha)$ with parameter $\alpha = 0$ was introduced by T. Estermann in [2] for needs of the representation of a number as the sum of two products. I. Kiuchi investigated [6] $E(s; \frac{k}{l}, \alpha)$ for $\alpha \in (-1, 0]$. The paper [12] is devoted to zero distribution of the Estermann zeta-function. The mean-square of $E(s; \frac{k}{l}, \alpha)$ was considered in [14], while the universality for $E(s; \frac{k}{l}, \alpha)$ was proved in [3]. The mentioned results also can be found in [13].

In view of [1], we have the functional equation

$$E\left(s; \frac{k}{l}, \alpha\right) = E\left(s - \alpha; \frac{k}{l}, -\alpha\right).$$

Therefore, without loss of generality, we can suppose that $\Re\alpha \le 0$.

The first attempt to characterize the asymptotic behaviour of the function $E(s; \frac{k}{l}, \alpha)$ by probabilistic terms was made in [9]. Here a limit theorem in the sense of weak convergence of probability measures on the complex plane was proved. To state this theorem, we need some notation.

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane, and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime p. By the Tikhonov theorem, with the product topology and pointwise multipilication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(S)$ denotes the class of Borel sets of the space S, the probability Haar measure m_H can be defined, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathcal{P}$. We extend the function $\omega(p)$ to the set $\mathbb{N}$ by the formula

$$\omega(m) = \prod_{p^r \| m} \omega^r(p), \quad m \in \mathbb{N},$$

where $p^r \| m$ means that $p^r \mid m$ but $p^{r+1} \nmid m$. Now on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ we define, for $\sigma > \frac{1}{2}$, the complex-valued random element $E(\sigma; \frac{k}{l}, \alpha; \omega)$ by the series

$$E\left(\sigma; \frac{k}{l}, \alpha; \omega\right) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)\omega(m)}{m^\sigma} \exp\left\{2\pi i m \frac{k}{l}\right\},$$

and denote by $P^{\mathbb{C}}_{E,\sigma}$ its distribution, i.e.,

$$P^{\mathbb{C}}_{E,\sigma}(A) = m_H\left(\omega \in \Omega : E\left(\sigma; \frac{k}{l}, \alpha; \omega\right) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}).$$

Denote by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then in [9] the following result has been obtained.

Theorem 1. *Suppose that $\sigma > \frac{1}{2}$ and $\Re\alpha \leq 0$. Then the probability measure*

$$\frac{1}{T}\text{meas}\left\{t \in [0, T] : E\left(\sigma + it; \frac{k}{l}, \alpha\right) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the measure $P^{\mathbb{C}}_{E,\sigma}$ as $T \to \infty$.

In [10] a generalization of Theorem 1 was given, a limit theorem in the space of meromorphic functions for the Estermann zeta-function was obtained. Let $D = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$, and let $M(D)$ denote the space of meromorphic on D functions equipped with the topology of uniform convergence on compacta. Moreover, by $H(D)$ denote the space of analytic on D functions equipped with the topology of $M(D)$. $H(D)$ is a subspace of $M(D)$. On $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element

$$E\left(s; \frac{k}{l}, \alpha; \omega\right) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)\omega(m)}{m^s} \exp\left\{2\pi i m \frac{k}{l}\right\}, \quad s \in D, \quad \omega \in \Omega,$$

and denote by P_E^H its distribution, i.e.,

$$P_E^H(A) = m_H\left(\omega \in \Omega : E\left(s; \frac{k}{l}, \alpha; \omega\right) \in A\right), \quad A \in \mathcal{B}(H(D)).$$

Then in [10] the following theorem has been proved.

Theorem 2. *Suppose that $\Re\alpha \leq 0$. Then the probability measure*

$$\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : E\left(s + i\tau; \frac{k}{l}, \alpha\right) \in A\right\}, \quad A \in \mathcal{B}(M(D)),$$

converges weakly to P_E^H as $T \to \infty$.

Theorems 1 and 2 are of continuous type, the measures in them are defined by shifts $E(\sigma + it; \frac{k}{l}, \alpha)$ and $E(s + i\tau; \frac{k}{l}, \alpha)$, when t and τ vary continuously in the interval $[0, T]$. The aim of this paper is to obtain a discrete limit theorem on the complex plane for the Estermann zeta-function, when t in $E(\sigma + it; \frac{k}{l}, \alpha)$ takes values from some discrete set.

Let, for brevity, for $N \in \mathbb{N}_0$,

$$\mu_N(...) = \frac{1}{N+1} \sum_{\substack{0 \leq m \leq N \\ ...}} 1,$$

where in place of dots a condition satisfied by m is to written.

Theorem 3. *Suppose that $\sigma > \frac{1}{2}$ and $\Re\alpha \leq 0$. Moreover, let $h > 0$ be a fixed number such that $\exp\left\{\frac{2\pi r}{h}\right\}$ is irrational for all $r \in \mathbb{Z} \setminus \{0\}$. Then the probability measure*

$$P_{N,\sigma} \stackrel{\text{def}}{=} \mu_N\left(E\left(\sigma + imh; \frac{k}{l}, \alpha\right) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P_{E,\sigma}^{\mathbb{C}}$ as $N \to \infty$.

1. Limit theorems for absolutely convergent series

Let, for fixed $\sigma_1 > \frac{1}{2}$,

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}.$$

For $n \in \mathbb{N}$ and $\sigma > \frac{1}{2}$, define

$$E_n\left(s; \frac{k}{l}, \alpha\right) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m) v_n(m)}{m^s} \exp\left\{2\pi i m \frac{k}{l}\right\},$$

and, for $\widehat{\omega} \in \Omega$,

$$E_n\left(s; \frac{k}{l}, \alpha; \widehat{\omega}\right) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m) v_n(m) \widehat{\omega}(m)}{m^s} \exp\left\{2\pi i m \frac{k}{l}\right\}.$$

Since, by (2), for $\Re\alpha \leq 0$, the estimate $\sigma_\alpha(m) \ll m^\epsilon$ is true, it is easily seen that the series for $E_n\left(s; \frac{k}{l}, \alpha\right)$ and $E_n\left(s; \frac{k}{l}, \alpha; \omega\right)$ converge absolutely in the half-plane $\sigma > \frac{1}{2}$. The details are similar to those given in Chapter 5 of [8].

On $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, define two probability measures

$$P_{N,n,\sigma} = \mu_N\left(E_n\left(\sigma + imh; \frac{k}{l}, \alpha\right) \in A\right)$$

and

$$\widehat{P}_{N,n,\sigma} = \mu_N\left(E_n\left(\sigma + imh; \frac{k}{l}, \alpha; \widehat{\omega}\right) \in A\right).$$

Theorem 4. *Suppose that $\sigma > \frac{1}{2}$ and $\Re\alpha \leq 0$. Let $h > 0$ be a fixed number such that $\exp\left\{\frac{2\pi r}{h}\right\}$ is irrational for all $r \in \mathbb{Z} \setminus \{0\}$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure $P_{n,\sigma}$ such that the measures $P_{N,n,\sigma}$ and $\widehat{P}_{N,n,\sigma}$ both converge weakly to $P_{n,\sigma}$ as $N \to \infty$.*

The proof of Theorem 4 is based on a discrete limit theorem on the torus Ω. Define

$$Q_N(A) = \mu_N\left((p^{-imh} : p \in \mathcal{P}) \in A\right), \quad A \in \mathcal{B}(\Omega).$$

Lemma 1. *Let $h > 0$ be a fixed number such that $\exp\left\{\frac{2\pi r}{h}\right\}$ is irrational for all $r \in \mathbb{Z} \setminus \{0\}$. Then the probability measure Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. The dual group of Ω is

$$\mathcal{D} \stackrel{\text{def}}{=} \bigoplus_p \mathbb{Z}_p,$$

where $\mathbb{Z}_p = \mathbb{Z}$ for each prime p. An element $\underline{k} = (k_2, k_3, k_5, ...) \in \mathcal{D}$, where only a finite number of integers k_p, $p \in \mathcal{P}$, are distinct from zero, acts on Ω by

$$\omega \to \omega^{\underline{k}} = \prod_p \omega^{k_p}(p).$$

Therefore, the Fourier transform $g_N(\underline{k})$ of the measure Q_N is of the form

$$\begin{aligned} g_N(\underline{k}) &= \int_\Omega \prod_p \omega^{k_p}(p) \mathrm{d}Q_N = \frac{1}{N+1} \sum_{m=0}^{N} \prod_p p^{-imhk_p} \\ &= \frac{1}{N+1} \sum_{m=0}^{N} \exp\left\{ -imh \sum_p k_p \log p \right\}, \end{aligned} \tag{3}$$

where only a finite number of integers k_p, $p \in \mathcal{P}$, are distinct from zero. It is well known that the system $\{\log p : p \in \mathcal{P}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Moreover,

$$\prod_p p^{k_p} = \exp\left\{ \sum_p k_p \log p \right\}$$

is a rational number, while, by the hypothesis of the lemma, the number

$$\exp\left\{ \frac{2\pi r}{h} \right\}$$

is irrational for all $r \in \mathbb{Z} \setminus \{0\}$. Hence, we obtain that

$$\exp\left\{ -ih \sum_p k_p \log p \right\} \neq 1$$

for $\underline{k} \neq \underline{0}$. Thus, we deduce from (3) that

$$g_N(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ \frac{1}{N+1} \frac{1 - \exp\left\{ -i(N+1)h \sum_p k_p \log p \right\}}{1 - \exp\left\{ -ih \sum_p k_p \log p \right\}} & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

This shows that

$$\lim_{N \to \infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}, \end{cases}$$

and in view of Theorem 1.4.2 of [4] the lemma is proved, since the limit Fourier transform corresponds the measure m_H. □

Proof of Theorem 4. Define the function $u_{n,\sigma} : \Omega \to \mathbb{C}$ by the formula

$$u_{n,\sigma}(\omega) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m) \omega(m) v_n(m)}{m^\sigma} \exp\left\{ 2\pi i m \frac{k}{l} \right\}.$$

Then the function $u_{n,\sigma}$ is continuous, and

$$u_{n,\sigma}\left((p^{-imh}: p \in \mathcal{P})\right) = E_n\left(\sigma + imh; \frac{k}{l}, \alpha\right).$$

Therefore, $P_{N,n,\sigma} = Q_N u_{n,\sigma}^{-1}$. Thus, by Lemma 1 and Theorem 5.1 of [1] we obtain that the measure $P_{N,n,\sigma}$ converges weakly to $m_H u_{n,\sigma}^{-1}$ as $N \to \infty$.

Now let the function $\widehat{u}_{n,\sigma}: \Omega \to \mathbb{C}$ be given by the formula

$$\widehat{u}_{n,\sigma}(\omega) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)\widehat{\omega}(m)\omega(m)v_n(m)}{m^\sigma} \exp\left\{2\pi i m \frac{k}{l}\right\}.$$

Then, similarly as above, we find that the measure $\widehat{P}_{N,n,\sigma}$ converges weakly to $m_H \widehat{u}_{n,\sigma}^{-1}$ as $N \to \infty$. However,

$$\widehat{u}_{n,\sigma}(\omega) = u_{n,\sigma}(\omega\widehat{\omega}) = u_{n,\sigma}(u(\omega)),$$

where $u(\omega) = \omega\widehat{\omega}$, $\omega \in \Omega$. Hence, $m_H \widehat{u}_{n,\sigma}^{-1} = m_H(u_{n,\sigma}u)^{-1} = (m_H u^{-1})u_{n,\sigma}^{-1} = m_H u_{n,\sigma}^{-1}$, since the Haar measure is invariant. □

2. Approximation in the mean

To prove Theorem 3, we have to pass from the function $E_n(s; \frac{k}{l}, \alpha)$ to $E(s; \frac{k}{l}, \alpha)$. For this, we need the estimate for the mean

$$\frac{1}{N+1} \sum_{m=0}^{N} \left| E\left(\sigma + imh; \frac{k}{l}, \alpha\right) - E_n\left(\sigma + imh; \frac{k}{l}, \alpha\right) \right|.$$

If $\sigma > \frac{1}{2}$ and $\Re\alpha \leq 0$, then it is known [14] that

$$\int_1^T \left| E\left(\sigma + it; \frac{k}{l}, \alpha\right) \right|^2 \mathrm{d}t \ll T, \quad T \to \infty. \tag{4}$$

In our case, a discrete version of estimate (4) is necessary. To prove an estimate of such a kind, we use the Gallagher lemma, see [11], Lemma 1.4.

Lemma 2. *Let T_0 and $T \geq \delta > 0$ be real numbers, $\mathcal{T}$ be a finite set in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$, and*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x| < \delta}} 1.$$

Moreover, let $S(x)$ *be a complex-valued continuous function on* $[T_0, T_0+T]$ *having a continuous derivative on* (T_0, T_0+T)*. Then*

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}|S(t)|^2 \le \frac{1}{\delta}\int_{T_0}^{T_0+T}|S(x)|^2\mathrm{d}x + \left(\int_{T_0}^{T_0+T}|S(x)|^2\mathrm{d}x\right)^{\frac12}\left(\int_{T_0}^{T_0+T}|S'(x)|^2\mathrm{d}x\right)^{\frac12}.$$

Lemma 3. *Suppose that* $\sigma > \frac12$*,* $\sigma \neq 1$*,* $\sigma \neq 1+\Re\alpha$*, if* $\alpha \neq 0$*,* $\Re\alpha \le 0$ *and* $N \to \infty$*. Then*

$$\sum_{m=0}^{N}\left|E\left(\sigma+imh+i\tau;\frac{k}{l},\alpha\right)\right|^2 \ll N+|\tau|.$$

Proof. A simple application of the integral Cauchy formula and (4) show that

$$\int_1^T\left|E'\left(\sigma+it;\frac{k}{l},\alpha\right)\right|^2\mathrm{d}t \ll T.$$

Hence, and from (4), using Lemma 2, we have that

$$\begin{aligned}\sum_{m=0}^{N}\left|E\left(\sigma+imh+i\tau;\frac{k}{l},\alpha\right)\right|^2 &\le \frac1h\int_0^{hN}\left|E\left(\sigma+it+i\tau;\frac{k}{l},\alpha\right)\right|^2\mathrm{d}t\\
&+ \left(\int_0^{hN}\left|E\left(\sigma+it+i\tau;\frac{k}{l},\alpha\right)\right|^2\mathrm{d}t\right)^{\frac12}\left(\int_0^{hN}\left|E'\left(\sigma+it+i\tau;\frac{k}{l},\alpha\right)\right|^2\mathrm{d}t\right)^{\frac12}\\
&\ll \int_{-|\tau|}^{hN+|\tau|}\left|E\left(\sigma+it;\frac{k}{l},\alpha\right)\right|^2\mathrm{d}t\\
&+ \left(\int_{-|\tau|}^{hN+|\tau|}\left|E\left(\sigma+it;\frac{k}{l},\alpha\right)\right|^2\mathrm{d}t\right)^{\frac12}\left(\int_{-|\tau|}^{hN+|\tau|}\left|E'\left(\sigma+it;\frac{k}{l},\alpha\right)\right|^2\mathrm{d}t\right)^{\frac12}\\
&\ll N+|\tau|.\end{aligned}$$

□

Theorem 5. *Suppose that* $\sigma > \frac{1}{2}$ *and* $\Re\alpha \leq 0$. *Then*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \left| E\left(\sigma + imh; \frac{k}{l}, \alpha\right) - E_n\left(\sigma + imh; \frac{k}{l}, \alpha\right) \right| = 0.$$

Proof. Let σ_1 the same as in Section 1. For $n \in \mathbb{N}$, define

$$l_n(s) = \frac{s}{\sigma_1} \Gamma\left(\frac{s}{\sigma_1}\right) n^s.$$

Then, see [9], for $\sigma > \frac{1}{2}$,

$$E_n\left(s; \frac{k}{l}, \alpha\right) = \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} E\left(s + z; \frac{k}{l}, \alpha\right) l_n(z) \frac{\mathrm{d}z}{z}.$$

Define σ_2 by

$$\sigma > \sigma_2 > \begin{cases} \frac{1}{2} & \text{if } \alpha = 0 \text{ or } 1 + \Re\alpha - \sigma > 0, \\ 1 + \Re\alpha & \text{otherwise.} \end{cases}$$

Thus, we obtain by the residue theorem that

$$\begin{aligned} E_n\left(s; \frac{k}{l}, \alpha\right) &= \frac{1}{2\pi i} \int_{\sigma_2 - \sigma - i\infty}^{\sigma_2 - \sigma + i\infty} E\left(s + z; \frac{k}{l}, \alpha\right) l_n(z) \frac{\mathrm{d}z}{z} \\ &\quad + E\left(s; \frac{k}{l}, \alpha\right) + R\left(s; \frac{k}{l}, \alpha\right), \end{aligned}$$

where

$$R\left(s; \frac{k}{l}, \alpha\right) = \begin{cases} \operatorname*{Res}_{z=1-s} E(s+z; \frac{k}{l}, \alpha) \frac{l_n(z)}{z} & \text{if } \alpha = 0, \\ \operatorname*{Res}_{z=1-s} E(s+z; \frac{k}{l}, \alpha) \frac{l_n(z)}{z} & + \operatorname*{Res}_{z=1+\alpha-s} E(s+z; \frac{k}{l}, \alpha) \frac{l_n(z)}{z} \\ & \text{if } 1 + \Re\alpha - \sigma > 0. \end{cases}$$

Hence, we have

$$\begin{aligned} &\frac{1}{N+1} \sum_{m=0}^{N} \left| E\left(\sigma + imh; \frac{k}{l}, \alpha\right) - E_n\left(\sigma + imh; \frac{k}{l}, \alpha\right) \right| \\ &\ll \int_{-\infty}^{\infty} \left(\frac{|l_n(\sigma_2 - \sigma + i\tau)|}{|\sigma_2 - \sigma + i\tau|} \frac{1}{N+1} \sum_{m=0}^{N} \left| E\left(\sigma_2 + imh + i\tau; \frac{k}{l}, \alpha\right) \right| \right) \mathrm{d}\tau \end{aligned}$$

$$+\frac{1}{N+1}\sum_{m=0}^{N}\left|R\left(\sigma_2-\sigma+imh;\frac{k}{l},\alpha\right)\right|. \tag{5}$$

We can choose $\sigma_2 \neq 1$ and $\sigma_2 \neq 1+\Re\alpha$. Thus, by Lemma 3

$$\begin{aligned}
&\frac{1}{N+1}\sum_{m=0}^{N}\left|E\left(\sigma_2+imh+i\tau;\frac{k}{l},\alpha\right)\right| \\
&\ll \frac{1}{N}\left(\sum_{m=0}^{N}1\right)^{\frac{1}{2}}\left(\sum_{m=0}^{N}\left|E\left(\sigma_2+imh+i\tau;\frac{k}{l},\alpha\right)\right|^2\right)^{\frac{1}{2}} \\
&\ll 1+|\tau|.
\end{aligned} \tag{6}$$

Applying Lemma 2 again, we find that

$$\begin{aligned}
&\sum_{m=0}^{N}\left|R\left(\sigma_2-\sigma+imh;\frac{k}{l},\alpha\right)\right| \\
&\qquad\ll \sqrt{N}\left(\sum_{m=0}^{N}\left|R\left(\sigma_2-\sigma+imh;\frac{k}{l},\alpha\right)\right|^2\right)^{\frac{1}{2}} \\
&\qquad\ll \sqrt{N}\Bigg(\int_0^{Nh}\left|R\left(\sigma_2-\sigma+it;\frac{k}{l},\alpha\right)\right|^2 \mathrm{d}t \\
&+\left(\int_0^{Nh}\left|R\left(\sigma_2-\sigma+it;\frac{k}{l},\alpha\right)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^{Nh}\left|R'\left(\sigma_2-\sigma+it;\frac{k}{l},\alpha\right)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\Bigg)^{\frac{1}{2}}.
\end{aligned} \tag{7}$$

Since the function $l_n(s)$ contains the Euler gamma-function, we obtain the estimate

$$\int_0^{Nh}\left|R\left(\sigma_2-\sigma+it;\frac{k}{l},\alpha\right)\right|^2 \mathrm{d}t \ll 1. \tag{8}$$

This and application of the Cauchy integral formula give the bound

$$\int_0^{Nh}\left|R'\left(\sigma_2-\sigma+it;\frac{k}{l},\alpha\right)\right|^2 \mathrm{d}t \ll 1.$$

This and (7), (8) lead to the estimate

$$\frac{1}{N+1}\sum_{m=0}^{N}\left|R\left(\sigma_2-\sigma+imh;\frac{k}{l},\alpha\right)\right|^2 \mathrm{d}t \ll \frac{1}{\sqrt{N}}.$$

Therefore, in view of (5) and (6)

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\left|E\left(\sigma+imh;\frac{k}{l},\alpha\right)-E_n\left(\sigma+imh;\frac{k}{l},\alpha\right)\right|$$
$$\ll \lim_{n\to\infty}\int_{-\infty}^{\infty}|l_n(\sigma_2-\sigma+i\tau)|\,(1+|\tau|)\mathrm{d}t. \tag{9}$$

However, since $\sigma_2-\sigma<0$,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|l_n(\sigma_2-\sigma+i\tau)|\,(1+|\tau|)\mathrm{d}t=0,$$

and the theorem is a consequence of estimate (9). □

We also need an analogue of Theorem 5 for the functions $E(s;\frac{k}{l},\alpha;\omega)$ and $E_n(s;\frac{k}{l},\alpha;\omega)$

Theorem 6. *Let $\sigma>\frac{1}{2}$ and $\Re\alpha\le 0$. Then, for almost all $\omega\in\Omega$,*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\left|E\left(\sigma+imh;\frac{k}{l},\alpha;\omega\right)-E_n\left(\sigma+imh;\frac{k}{l},\alpha;\omega\right)\right| = 0.$$

Proof. In [9], Lemma 5, it was obtained that, under the hypotheses of the theorem,

$$\int_0^T\left|E\left(\sigma+it;\frac{k}{l},\alpha;\omega\right)\right|^2\mathrm{d}t\ll T$$

for almost all $\omega\in\Omega$. Hence, similarly to the proof of Lemma 3, we obtain that

$$\sum_{m=0}^{N}\left|E\left(\sigma+imh+i\tau;\frac{k}{l},\alpha;\omega\right)\right|^2\ll N+|\tau| \tag{10}$$

for almost all $\omega\in\Omega$.

The random variables $\omega(m)$, $m\in\mathbb{N}$, are pointwise orthogonal, that is

$$\mathbb{E}\big(\omega(m)\overline{\omega(n)}\big)=\begin{cases}1 & \text{if } m=n,\\ 0 & \text{if } m\neq n,\end{cases}$$

where $\mathbb{E}(X)$ denotes the expectation of X. Hence, we have that

$$\mathbb{E}\left(\frac{\sigma_\alpha(m)\omega(m)}{m^\sigma}\,\frac{\overline{\sigma_\alpha(n)}\overline{\omega}(n)}{n^\sigma}\exp\left\{2\pi i\frac{k}{l}(m-n)\right\}\right)$$
$$=\begin{cases}\frac{|\sigma_\alpha(m)|^2}{m^{2\sigma}} & \text{if } m=n,\\ 0 & \text{if } m\neq n.\end{cases}$$

Thus, in view of (2), the series

$$\sum_{m=1}^{\infty}\mathbb{E}\left|\frac{\sigma_\alpha(m)\omega(m)}{m^\sigma}\exp\left\{2\pi im\frac{k}{l}\right\}\right|^2\log^2 m$$

converges for any fixed $\sigma>\frac{1}{2}$. Therefore, by the Rademacher theorem, see, for example [11], the series, for any fixed $\sigma>\frac{1}{2}$,

$$\sum_{m=1}^{\infty}\frac{\sigma_\alpha(m)\omega(m)}{m^\sigma}\exp\left\{2\pi im\frac{k}{l}\right\}$$

converges for almost all $\omega\in\Omega$. Hence, the series

$$\sum_{m=1}^{\infty}\frac{\sigma_\alpha(m)\omega(m)}{m^\sigma}\exp\left\{2\pi im\frac{k}{l}\right\},$$

for almost all $\omega\in\Omega$, converges uniformly on compact subsets of the half-plane $\{s\in\mathbb{C}:\sigma>\frac{1}{2}\}$. This shows that, for almost all $\omega\in\Omega$, the function $E(s;\frac{k}{l},\alpha;\omega)$ is analytic in the region $\{s\in\mathbb{C}:\sigma>\frac{1}{2}\}$. Therefore, using the representation

$$E_n\left(s;\frac{k}{l},\alpha;\omega\right)=\frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}E\left(s+z;\frac{k}{l},\alpha;\omega\right)l_n(z)\frac{\mathrm{d}z}{z},$$

we obtain that, for $\frac{1}{2}<\sigma_2<\sigma$,

$$E_n\left(s;\frac{k}{l},\alpha;\omega\right)=\frac{1}{2\pi i}\int_{\sigma_2-\sigma-i\infty}^{\sigma_2-\sigma+i\infty}E\left(s+z;\frac{k}{l},\alpha;\omega\right)l_n(z)\frac{\mathrm{d}z}{z}+E\left(s;\frac{k}{l},\alpha;\omega\right)$$

for almost all $\omega\in\Omega$. Using the latter formula and (9), we complete the proof in the same way as in the case of Theorem 5. □

3. Proof of Theorem 3

Define one more probability measure

$$\widehat{P}_{N,\sigma} = \mu_N\left(E\left(\sigma + imh; \frac{k}{l}, \alpha; \omega\right) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}).$$

We begin the proof of Theorem 3 with the following statement.

Theorem 7. *Suppose that $\sigma > \frac{1}{2}$ and $\Re\alpha \leq 0$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the measures $P_{N,\sigma}$ and $\widehat{P}_{N,\sigma}$ both converge weakly to P_σ as $N \to \infty$.*

Proof. By Theorem 4, for $\sigma > \frac{1}{2}$, the measures $P_{N,n,\sigma}$

$$\widehat{P}_{N,n,\sigma} = \mu_N\left(E_n\left(\sigma + imh; \frac{k}{l}, \alpha; \omega\right) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}),$$

for every $\omega \in \Omega$, both converge weakly to the same measure $P_{n,\sigma}$ as $N \to \infty$.

For any positive M, by the Chebyshev inequality

$$\begin{aligned} P_{N,n,\sigma}\big(\{z \in \mathbb{C} : |z| > M\}\big) &= \mu_N\left(\left|E_n\left(\sigma + imh; \frac{k}{l}, \alpha\right)\right| > M\right) \\ &\leq \frac{1}{M(N+1)} \sum_{m=0}^{N} \left|E_n\left(\sigma + imh; \frac{k}{l}, \alpha\right)\right|. \end{aligned} \tag{11}$$

As we have observed above, the series for $E_n(s; \frac{k}{l}, \alpha)$ converges absolutely for $\sigma > \frac{1}{2}$. Also, the latter property holds for $E_n'(s; \frac{k}{l}, \alpha)$. Therefore, for $\sigma > \frac{1}{2}$,

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \int_1^T \left|E_n\left(\sigma + it; \frac{k}{l}, \alpha\right)\right|^2 \mathrm{d}t &= \sum_{m=1}^{\infty} \frac{|\sigma_\alpha(m)|^2 v_n^2(m)}{m^{2\sigma}} \\ &\leq \sum_{m=1}^{\infty} \frac{|\sigma_\alpha(m)|^2}{m^{2\sigma}} < \infty, \end{aligned} \tag{12}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \int_1^T \left|E_n'\left(\sigma + it; \frac{k}{l}, \alpha\right)\right|^2 \mathrm{d}t = \sum_{m=1}^{\infty} \frac{|\sigma_\alpha(m)|^2 v_n^2(m) \log^2 m}{m^{2\sigma}}$$

$$\leq \sum_{m=1}^{\infty} \frac{|\sigma_\alpha(m)|^2 \log^2 m}{m^{2\sigma}} < \infty. \quad (13)$$

An application of Lemma 2 yields

$$\frac{1}{N+1}\sum_{m=0}^{N}\Big|E_n\Big(\sigma+imh;\frac{k}{l},\alpha\Big)\Big| \ll \frac{1}{\sqrt{N}}\Bigg(\sum_{m=0}^{N}\Big|E_n\Big(\sigma+imh;\frac{k}{l},\alpha\Big)\Big|^2\Bigg)^{\frac{1}{2}}$$

$$\ll \frac{1}{\sqrt{N}}\Bigg(\frac{1}{Nh}\int_0^{Nh}\Big|E_n\Big(\sigma+it;\frac{k}{l},\alpha\Big)\Big|^2 \mathrm{d}t$$

$$+\Bigg(\frac{1}{N}\int_0^{Nh}\Big|E_n\Big(\sigma+it;\frac{k}{l},\alpha\Big)\Big|^2\mathrm{d}t\Bigg)^{\frac{1}{2}}\Bigg(\frac{1}{N}\int_0^{hN}\Big|E_n'\Big(\sigma+it;\frac{k}{l},\alpha\Big)\Big|^2\mathrm{d}t\Bigg)^{\frac{1}{2}}\Bigg)^{\frac{1}{2}}.$$

This, (12) and (13) show that

$$\sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\Big|E_n\Big(\sigma+imh;\frac{k}{l},\alpha;\omega\Big)\Big| \leq C(h)R, \quad (14)$$

where

$$R=\Bigg(\sum_{m=1}^{\infty}\frac{|\sigma_\alpha(m)|^2}{m^{2\sigma}}+\Bigg(\sum_{m=1}^{\infty}\frac{|\sigma_\alpha(m)|^2}{m^{2\sigma}}\Bigg)^{\frac{1}{2}}\Bigg(\sum_{m=1}^{\infty}\frac{|\sigma_\alpha(m)|^2\log^2 m}{m^{2\sigma}}\Bigg)^{\frac{1}{2}}\Bigg)^{\frac{1}{2}}<\infty.$$

For arbitrary $\epsilon>0$, let $M_\epsilon = C(h)R\epsilon^{-1}$. Then, taking into account (11) and (14), we find that

$$\limsup_{N\to\infty} P_{N,n,\sigma}\big(\{z\in\mathbb{C}:|z|>M_\epsilon\}\big)\leq\epsilon. \quad (15)$$

The function $u:\mathbb{C}\to\mathbb{R}$, $z\to|z|$, is continuous. Therefore, by Theorem 4 and Theorem 5.1 of [1] we have that, for $\sigma>\frac{1}{2}$, the probability measure

$$\mu_N\Bigg(\Big|E_n\Big(\sigma+imh;\frac{k}{l},\alpha\Big)\Big|\in A\Bigg), \quad A\in\mathcal{B}(\mathbb{R}),$$

converges weakly to $P_{n,\sigma}u^{-1}$ as $N\to\infty$. This together with Theorem 2.1 of [1] and (15) implies

$$\begin{aligned} P_{n,\sigma}\big(\{z\in\mathbb{C}:|z|>M_\epsilon\}\big) &\leq \liminf_{N\to\infty} P_{N,n,\sigma}\big(\{z\in\mathbb{C}:|z|>M_\epsilon\}\big)\\ &\leq \limsup_{N\to\infty} P_{N,n,\sigma}\big(\{z\in\mathbb{C}:|z|>M_\epsilon\}\big)\leq\epsilon \end{aligned}$$

(16)

for all $n \in \mathbb{N}$. Define $K_\epsilon = \{z \in \mathbb{C} : |z| \leq M_\epsilon\}$. Then the set K_ϵ is compact, and by (16)

$$P_{n,\sigma}(K_\epsilon) \geq 1 - \epsilon$$

for all $n \in \mathbb{N}$. This means that the family of probability measures $\{P_{n,\sigma} : n \in \mathbb{N}\}$ is tight, and by the Prokhorov theorem, see Theorem 6.1 of [1], it is relatively compact. Therefore, there exists a subsequence $\{P_{n_k,\sigma}\} \subset \{P_{n,\sigma}\}$ such that $P_{n_k,\sigma}$ converges weakly to some measure P_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $k \to \infty$.

Let θ_N be a random variable defined on a certain probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mathbb{P})$ with the distribution

$$\mathbb{P}(\theta_N = mh) = \frac{1}{N+1}, \quad m = 0, 1, ..., N.$$

Define

$$X_{N,n} = X_{N,n}(\sigma) = E_n\left(\sigma + i\theta_N; \frac{k}{l}, \alpha\right)$$

and denote by $X_n = X_n(\sigma)$ the complex-valued random variable with the distribution $P_{n,\sigma}$. Then by Theorem 4

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} X_n, \tag{17}$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution. Moreover, from the above remark

$$X_{n_k}(\sigma) \xrightarrow[k\to\infty]{\mathcal{D}} P_\sigma. \tag{18}$$

Define

$$X_N(\sigma) = E\left(\sigma + i\theta_N; \frac{k}{l}, \alpha\right).$$

Then in view of Theorem 5, for $\sigma > \frac{1}{2}$ and any $\epsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(|X_N(\sigma) - X_{N,n}(\sigma)| \geq \epsilon\right) \\
= &\lim_{n\to\infty} \limsup_{N\to\infty} \mu_N \left(\left|E\left(\sigma + imh; \frac{k}{l}, \alpha\right) - E_n\left(\sigma + imh; \frac{k}{l}, \alpha\right)\right| \geq \epsilon\right) \\
\leq &\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\epsilon(N+1)} \sum_{m=1}^{\infty} \left|E\left(\sigma + imh; \frac{k}{l}, \alpha\right) - E_n\left(\sigma + imh; \frac{k}{l}, \alpha\right)\right| = 0.
\end{aligned}$$

This, (17), (18) and Theorem 4.2 of [1] show that

$$X_N(\sigma) \xrightarrow[N\to\infty]{\mathcal{D}} P_\sigma, \tag{19}$$

and this is equivalent to weak convergence of $P_{N,\sigma}$ to P_σ as $N \to \infty$.

Relation (19) shows that the measure P_σ is independent of the choice of the sequence $P_{n_k,\sigma}$. Hence, we obtain that

$$X_n(\sigma) \xrightarrow[n\to\infty]{\mathcal{D}} P_\sigma. \tag{20}$$

Now define

$$\widehat{X}_{N,n} = \widehat{X}_{N,n}(\sigma) = E_n\left(\sigma + i\theta_N; \frac{k}{l}, \alpha; \omega\right)$$

and

$$\widehat{X}_N = \widehat{X}_N(\sigma) = E\left(\sigma + i\theta_N; \frac{k}{l}, \alpha; \omega\right).$$

Then in the same way as above, using (20) and Theorem 6, we find that the measure $\widehat{P}_{N,\sigma}$ also converges weakly to P_σ as $N \to \infty$. □

Proof of Theorem 3. In view of Theorem 7, it remains to identify the limit measure P_σ.

Let $A \in \mathcal{B}(\mathbb{C})$ be a fixed continuity set of the limit measure P_σ in Theorem 7. Then we have that

$$\lim_{N\to\infty} \mu_N\left(E\left(\sigma + imh; \frac{k}{l}, \alpha\right) \in A\right) = P_\sigma(A). \tag{21}$$

Now on $(\Omega, \mathcal{B}(\Omega))$ define the random variable θ by the formula

$$\theta = \theta(\omega) = \begin{cases} 1 & \text{if } E\big(\sigma; \frac{k}{l}, \alpha; \omega\big) \in A, \\ 0 & \text{if } E\big(\sigma; \frac{k}{l}, \alpha; \omega\big) \notin A. \end{cases}$$

Then we have that

$$\mathbb{E}\theta = \int_\Omega \theta \mathrm{d}m_H = m_H\left(\omega \in \Omega : E\left(s; \frac{k}{l}, \alpha; \omega\right) \in A\right) = P^{\mathbb{C}}_{E,\sigma}. \tag{22}$$

Let $a_h = \{p^{-ih} : p \in \mathcal{P}\}$. Define the transformation f_h on Ω by $f_h(\omega) = a_h\omega$, $\omega \in \Omega$. Then f_h is a measurable measure preserving transformation on $(\Omega, \mathcal{B}(\Omega), m_H)$. In [5] it was obtained that the transformation f_h is ergodic. Then by the classical Birkhoff-Khinchine theorem, see, for example [7], we obtain that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \theta\big(f_h^m(\omega)\big) = \mathbb{E}\theta \tag{23}$$

for almost all $\omega \in \Omega$. However, by the definition of f_h, we have that

$$\frac{1}{N+1} \sum_{m=0}^{N} \theta\big(f_h^m(\omega)\big) = \mu_N \left(E\Big(\sigma + imh; \frac{k}{l}, \alpha; \omega\Big) \in A \right).$$

From this, (22) and (23) we obtain that

$$\lim_{N\to\infty} \mu_N \left(E\Big(\sigma + imh; \frac{k}{l}, \alpha; \omega\Big) \in A \right) = P_{E,\sigma}^{\mathbb{C}}(A).$$

Therefore, by (21), $P_\sigma(A) = P_{E,\sigma}^{\mathbb{C}}(A)$. Since A is arbitrary continuity set of P_σ, the latter equality is true for any continuity set A. However, all continuity sets constitute the determining class, and we have that $P_\sigma(A) = P_{E,\sigma}^{\mathbb{C}}(A)$ for all $A \in \mathcal{B}(\mathbb{C})$. The theorem is proved. □

References

[1] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

[2] T. Estermann, *On the representation of a number as the sum of two products*, Proc. London Math. Soc., N.**31**, 1930, pp.123–133.

[3] R. Garunkštis, A. Laurinčikas, R. Šleževičienė, J. Steuding, *On the universality of Estermann zeta-functions*, Analysis, N.**22**, 2002, pp.285–296.

[4] H. Heyer, *Probability Measures on Locally Compact Groups*. Springer-Verlag, Berlin, 1977.

[5] R. Kačinskaitė, *A discrete limit theorem for the Matsumoto zeta-function on the complex plane*, Lith. Math. J., N.**40**(4), 2000, pp. 364–378.

[6] I. Kiuchi, *On an exponentials sum involving the arithmetic function* $\sigma_a(n)$, Math. J. Okayama Univ., N.**29**, 1987, pp.193–205.

[7] U. Krengel, *Ergodic Theorems*, Walter de Gruyter, Berlin, 1985.

[8] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1996.

[9] A. Laurinčikas, *Limit theorems for the Estermann zeta-function. I*, Statist. Probab. Lett., N.**72**, 2005, pp.227–237.

[10] A. Laurinčikas, *Limit theorems for the Estermann zeta-function. II*, Cent. Eur. J. Math., N.**3**(4), 2005, pp.580–590.

[11] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Springer-Verlag, Berlin, 1971.

[12] R. Šleževičienė, J. Steuding, *On the zeros of the Estermann zeta-function*, Integral Transforms and Special Functions, N.**13**, 2002, pp.363–371.

[13] R. Šleževičienė, *On some aspects in the theory of the Estermann zeta-function*, Fiz. Mat. Fak. Moksl. Semin. Darb., N.**5**, 2002, pp.115–130.

[14] R. Šleževičienė, J. Steuding, *The mean–square of the Estermann zeta-functon*, Faculty of Mathematics and Informatics, Vilnius University, Preprint 2002-32, 2002.

Contact information

A. Laurinčikas
Department of Mathematics
and Informatics,
Vilnius University,
Naugarduko 24,
LT-03225 Vilnius,
Lithuania
E-Mail: antanas.laurincikas@mif.vu.lt

R. Macaitienė
Department of Mathematics
and Informatics,
Šiauliai University,
P. Višinskio 19,
LT-77156 Šiauliai,
Lithuania
E-Mail: renata.macaitiene@mi.su.lt

Received by the editors: 09.07.2007
and in final form 07.04.2008.