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Cyclic left and torsion-theoretic spectra of modules and their relations

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Abstract. In this paper, strongly-prime submodules of a cyclic module are considered and their main properties are given. On this basis, a concept of a cyclic spectrum of a module is introduced. This spectrum is a generalization of the Rosenberg spectrum of a noncommutative ring. In addition, some natural properties of this spectrum are investigated, in particular, its functoriality is proved.

Introduction

In this paper, we consider strongly-prime ideals and modules. The concept of strongly-prime ideal was introduced by Beachy in [1]. Also in that paper the author introduced and investigated the concept of a strongly-prime module. Independently, the concept of strongly-prime module and submodule were introduced and investigated by Dauns in his paper [3]. Also, the strongly-prime modules were investigated by Algirdas Kaučikas in [2], where the author studied strongly-prime submodules of cyclic modules, but he did not study the concept of the Rosenberg spectrum for modules. The concept of pre-order on ideals was introduced by Rosenberg, and this concept is a basic one in the definition of cyclic spectrum, whose functoriality is investigated in this paper. Also we consider the notion and some properties of torsion-theoretic spectra of rings and modules. The notion and main properties of torsion-theoretic spectra were introduced by Golan in [5]. The main result of this paper is the

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proof of the fact that there exists mapping from the cyclic spectrum to the torsion-theoretic spectrum of module is continuous and surjective.

1. Strongly-prime ideals and modules

Let R be an associative ring with $1 \neq 0$. To have a reference, recall some necessary concepts from the ring theory that are related to the concept of spectrum of a noncommutative ring.

A left ideal $\mathfrak{p}$ of a ring R is called *prime*, if for every $x, y \in R$, $xRy \subseteq \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Clearly, any left prime ideal is two-sided if and only if it is prime in the classical way. Set of all two-sided prime ideals is denoted by $\mathrm{Spec}(R)$ and is called a (prime) *spectrum* of a ring R.

Recall the definition of a *pre-order* $\leqslant$ on the set of left ideals of ring R in the following way: $\mathfrak{a} \leqslant \mathfrak{b}$ for left R-ideals $\mathfrak{a}$ and $\mathfrak{b}$ if and only if there exists a finite subset V of ring R such that $(\mathfrak{a} : V) \subseteq \mathfrak{b}$. A left prime ideal $\mathfrak{p}$ of a ring R is called a *left Rosenberg point* if $(\mathfrak{p} : x) \leqslant \mathfrak{p}$ for any $x \in R \backslash \mathfrak{p}$, [8]. The set of all left Rosenberg points of a ring R is called a left *Rosenberg spectrum* of R and is denoted by $\mathrm{spec}(R)$.

The space $\mathrm{spec}(R)$ may by defined in another way: this is the set of all strongly prime left ideals. Recall that left ideal $\mathfrak{p}$ of the ring R is called *strongly-prime*, if for every $x \in R \backslash \mathfrak{p}$ there exist a finite set V of ring R such that $(\mathfrak{p} : Vx) = \{r \in R\colon rVx \subseteq \mathfrak{p}\} \subseteq \mathfrak{p}$. Clearly, every strongly-prime left ideal of a ring R is a prime left ideal and every maximal left ideal is strongly-prime. It is known that if R is noetherian, then $\mathrm{Spec}(R) \subseteq \mathrm{spec}(R)$.

Now let us recover the information about corresponding analogues of the above concepts for left modules over a ring R.

The concept of strongly-prime module can be given in two ways.

A nonzero left module M over a ring R is called *strongly-prime*, if for any nonzero $x, y \in M$ there exists a finite subset $\{a_1, a_2, \ldots, a_n\} \subseteq R$ such that $\mathrm{Ann}_R\{a_1x, a_2x, \ldots, a_nx\} \subseteq \mathrm{Ann}_R\{y\}$, $(ra_1x = ra_2x = \cdots = ra_nx = 0)$, $r \in R$ implies $ry = 0$.

In [1], the authors introduced such a concept of strongly-prime submodule. A nonzero left module M over a ring R is called *strongly-prime*, if for any nonzero $x \in M$ there exists a finite subset $\{a_1, a_2, \ldots, a_n\} \subseteq R$ such that $\mathrm{Ann}_R\{a_1x, a_2x, \ldots, a_nx\} = 0$. If in this concept we put $M = R$, we obtain the concept of a strongly-prime ring. Such strongly-prime rings were studied in [4].

A submodule P of some module M is called *strongly-prime*, if the quotient module M/P is a strongly-prime R-module. The set of all strongly-prime submodules of module M is called the left prime spectrum of M and is denoted by spec(M). In particular, a left ideal $\mathfrak{p} \subset R$ is called *strongly-prime* if the quotient module $R/\mathfrak{p}$ is a strongly-prime R-module. In terms of elements, left ideal $\mathfrak{p} \subset R$ is strongly-prime if for every $u \notin \mathfrak{p}$ there exists such elements $\{a_1, ..., a_n\} \subseteq R$ and a natural number $n = n(u)$ such that $ra_1u, \ldots, ra_nu \in \mathfrak{p},\ r \in R$ implies $r \in \mathfrak{p}$.

2. Preorder on the set of modules and cyclic left spectrum of module

It is easy to see that if R is a left noetherian ring and $\mathfrak{p} \in \mathrm{Spec}(R)$, then $R/\mathfrak{p}$ is a left noetherian prime ring. This implies that it is sufficient to prove that in a left noetherian prime ring R zero ideal belongs to spec(R). But taking into account the assumption that R is a prime Goldie ring, for any $0 \neq x \in R$ any two-sided ideal RxR is essential, thus there exists a regular element $a = \sum_{i=1}^{n} r_i x s_i \in RxR$ (Using Goldie theorem). Let $V = \{r_1, \ldots, r_n\}$ and $y \in (0 : Vx)$, then $ya = \sum y r_i x s_i = 0$. Since a is regular, it follows that $y = 0$, hence $0 \in$ spec(R) indeed.

Clearly, it is necessary to demonstrate how to calculate prime left ideals in an easy example. For this purpose we use the following example.

Example 1. Consider the matrix ring $R = M_2(k)$ over a (commutative) field k.

It is well known that $\mathrm{Spec}(R) = \{0\}$. Let L be a nonzero left R-ideal and $0 \neq r \in L$. Since all nonzero left ideals of the ring R are maximal, $L = Rr$. Multiplying r by the matrix units e_{11} and e_{12} resp., it easily follows that we may assume r to be of the form $r = \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$, for some nonzero string $\left(\begin{smallmatrix} a & b \end{smallmatrix}\right) \in k^2$. One thus finds $L = R\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) = [a, b]_k$.

Moreover, $[a, b]_k = [a', b']_k$ if and only if there exists such $c \in k$ that $a = ca'$ and $b = cb'$. Then spec$(R) = \{[a, b]_k \mid a, b \in k\}$ may be identified with the projective line P_k^1 (with "generic point" $(0) = [0, 0]_k$). (See [6])

As in [8] we introduce a preorder $\leqslant$ on the set of all left ideals by putting $K \leqslant L$ for a pair of left R-ideals L and K if and only if there exists a finite subset V of the ring R such that $(K : V) \subseteq L$.

Let us try to establish a preorder on the modules. Let R be a regular module over itself with generator 1. Then $M = R \cdot 1$ is a cyclic module.

Theorem 1. *Every cyclic module is isomorphic to the quotient module of a regular module by the annihilator of a generator* $R \cdot m = R/\operatorname{Ann}(m)$, *where* $\operatorname{Ann}(m)$ *is the left annihilator of a generator* m.

Consider some submodules of a cyclic module M which is presented as $Rm = R/\operatorname{Ann}(m)$ for the generator m. Let L, K be some submodules. We can represent $L = \mathfrak{A}/\operatorname{Ann}(m)$ and $K = \mathfrak{B}/\operatorname{Ann}(m)$ for some left ideals $\mathfrak{A}$ and $\mathfrak{B}$ of a ring R. Then we define $L = \mathfrak{A}/\operatorname{Ann}(m) \leqslant K = \mathfrak{B}/\operatorname{Ann}(m)$ if and only if $\mathfrak{A} \leqslant \mathfrak{B}$ as the Rosenberg ideals. All properties are carried out. Thus the spectrum of a cyclic module is the set of all ideals that are in the spectrum of ring R.

It is well known that any module is the sum of its cyclic submodules. Then the *cyclic spectrum* of a arbitrary module M is defined as the union of all spectra of its cyclic submodules. The cyclic spectrum of module M is denoted by $\operatorname{Cspec}(M)$. Then we can define $L \leqslant K \Longleftrightarrow \operatorname{Cspec}(L) \subseteq \operatorname{Cspec}(K)$ for all submodules of the module M and obtain a preorder on the family of such submodules.

Example 2. Let $M = \{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) | a, b \in k\}$ be module of columns with height 2 over ring $R = M_2(k)$, where k is commutative field.

This module is cyclic with generator $e = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, that is, $M = R \times \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Then $\operatorname{Ann}(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) = \{\left(\begin{smallmatrix} 0 & b \\ 0 & d \end{smallmatrix}\right) | b, d \in k\}$, thus $M/\operatorname{Ann}(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) \cong \{\left(\begin{smallmatrix} a & 0 \\ c & 0 \end{smallmatrix}\right) | a, c \in k\}$. The maximal submodule is $\{(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))\}$, hence cyclic spectrum consists of one point.

Lemma 1. *Let L and K be left cyclic R-modules. Then $L \leqslant K$ if and only if there exists a cyclic left R-module X, a monomorphism $X \leftarrowtail L^n$ and an epimorphism $X \twoheadrightarrow K$. In other words, there exists a diagram $(L)^n \leftarrowtail X \twoheadrightarrow K$.*

Proof. Recall the definition of preorder for submodules of a cyclic module. Let L, K be some submodules. We can represent $L = \mathfrak{A}/\operatorname{Ann}(m)$ and $K = \mathfrak{B}/\operatorname{Ann}(m)$ for some left ideals $\mathfrak{A}$ and $\mathfrak{B}$ of the ring R. Then we define $L = \mathfrak{A}/\operatorname{Ann}(m) \leqslant K = \mathfrak{B}/\operatorname{Ann}(m)$ iff $\mathfrak{A} \leqslant \mathfrak{B}$ as Rosenberg ideals. Thus consider two cyclic modules L and K. They are fully represented by their ideals $\mathfrak{A}$ and $\mathfrak{B}$. Than if $\mathfrak{A} \leqslant \mathfrak{B}$ by the definition, than there exists a finite subset $V \subseteq R$, such that $(\mathfrak{A} : V) \leqslant \mathfrak{B}$. Put $V = \{v_1, \ldots, v_n\}$ and let $X = R\vec{v}$ be a cyclic module, where $\vec{v} = \{v_1, \ldots, v_n\} \in (L)^n$. Than we have

$$(0 : \vec{v}) = \cap_{i=1}^{n} (\mathfrak{A} : v_i) = (\mathfrak{A} : V) \subseteq \mathfrak{B},$$

which implies that there exists a surjection $X \twoheadrightarrow K$.

On the other hand, assume that there exists a diagram $(L)^n \leftarrowtail^{\alpha} X \twoheadrightarrow^{\beta} K$. Thus we can find such element $x \in X$, that $\beta(x) = \vec{1}$. Put

$\alpha(x) = (\vec{v_1}, \ldots, \vec{v_n}) \in (L)^n$, where $(\vec{v_1}, \ldots, \vec{v_n}) \in \mathfrak{A}$ for some $v_i \in R$. Put $V = \{v_1, \ldots, v_n\}$ and than we have

$$(\mathfrak{A} : V) = \cap_{i=1}^{n} (\mathfrak{A} : v_i) = (0 : \vec{v}) = (0 : x) \subseteq \mathfrak{B},$$

so $\mathfrak{A} \leqslant \mathfrak{B}$ and $L \leqslant K$. □

Usually from the preorder $\leqslant$ we obtain an *equivalence relation* $\sim$ as follows: $K \sim L$ iff $K \leqslant L$ and $L \leqslant K$. The equivalence class of the submodule L will be denoted by $[L]$.

Lemma 2 (11)**.** *If $\mathfrak{P}$ is a strongly-prime module, then for any element $x \in M$ the following properties are equivalent:*

(1) $x \notin \mathfrak{P}$;

(2) $(\mathfrak{P} : x) \leqslant \mathfrak{P}$;

(3) $(\mathfrak{P} : x) \in [\mathfrak{P}]$.

Lemma 3. *Let M be cyclic module. If $\mathfrak{P} \in \mathrm{Cspec}(M)$ and if L and K are submodules such that $L \cap K \leqslant \mathfrak{P}$, then either $L \leqslant \mathfrak{P}$ or $K \leqslant \mathfrak{P}$.*

Proof. Let $L \nleqslant \mathfrak{P}$ and $K \nleqslant \mathfrak{P}$ and let $L \cap K \leqslant \mathfrak{P}$. Thus, by the definition, there exist ideals $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{p}$ of the ring R, such that $L = \mathfrak{A}/\operatorname{Ann}(m)$, $K = \mathfrak{B}/\operatorname{Ann}(m)$ and $P = \mathfrak{p}/\operatorname{Ann}(m)$. Then there exists a finite subset V of the ring R, such that $(\mathfrak{A} \cap \mathfrak{B} : V) \subseteq \mathfrak{p}$. Since $\mathfrak{A} \nleqslant \mathfrak{p}$, this implies that $(\mathfrak{A} : F) \nsubseteq \mathfrak{p}$ for some finite subset F of the ring R. Thus, if we take $F = V$, we obtain the fact, that $(\mathfrak{A} : V) \nsubseteq \mathfrak{p}$. Now, if $x \in (\mathfrak{A} : V) - \mathfrak{p}$, then there exists a finite set $W \subseteq R$ with the property that $(\mathfrak{p} : Wx) \subseteq \mathfrak{p}$. Since $K \nleqslant \mathfrak{p}$, we have $\mathfrak{b} \nleqslant \mathfrak{p}$, get fact that $(\mathfrak{B} : F) \nleqslant \mathfrak{p}$ for any finite set $F \subseteq R$. In particular, this holds for $F = WxV$, thus we can find an element $y \in (\mathfrak{B} : WxV) - \mathfrak{p}$. Finally, $x \in (\mathfrak{A} : V)$ implies that $yWxV \subseteq \mathfrak{B}$, and y belongs to the set $(\mathfrak{B} : WxV)$. Certainly, $yWxV \subseteq \mathfrak{B}$, then $yWxV \subseteq \mathfrak{A} \cap \mathfrak{B}$ and $yWx \subseteq (\mathfrak{A} \cap \mathfrak{B} : V) \subseteq \mathfrak{p}$. Thus, $y \in (\mathfrak{p} : Wx) \subseteq \mathfrak{p}$, that contradicts to the fact, that $y \notin \mathfrak{p}$. □

Similarly

Lemma 4. *If $\mathfrak{P} \in \mathrm{Cspec}(R)$ and if L and K are submodules such that $LK \leqslant \mathfrak{P}$, then either $L \leqslant \mathfrak{P}$ or $K \leqslant \mathfrak{P}$.*

Recall the operation of multiplication of the submodules of cyclic module $R/\mathfrak{c}$. Any submodule of cyclic module can be viewed as the quotient-module of some left ideal by some other left ideal. Let we have two such submodules $L \cong \mathfrak{a}/\mathfrak{c}$ and $K \cong \mathfrak{b}/\mathfrak{c}$. Then $L \cdot K = \mathfrak{a}/\mathfrak{c} \cdot \mathfrak{b}/\mathfrak{c} = \mathfrak{a}\mathfrak{b}/\mathfrak{c}$.

Lemma 5. *Let $\mathfrak{P}$ and $\mathfrak{Q}$ be strongly-prime submodules of the cyclic module M. Then the following holds:*

(1) *If $\mathfrak{P} \sim \mathfrak{Q}$, then $\mathfrak{P} \cap \mathfrak{Q}$ is a strongly-prime module and $\mathfrak{P} \sim \mathfrak{P} \cap \mathfrak{Q}$;*

(2) *If $\mathfrak{P} \cap \mathfrak{Q}$ is a strongly-prime module, then either $\mathfrak{P} \subseteq \mathfrak{Q}$ or $\mathfrak{P} \supseteq \mathfrak{Q}$ or $\mathfrak{P} \sim \mathfrak{Q}$.*

Proof. Let $\mathfrak{P}$ and $\mathfrak{Q}$ be strongly-prime submodules of a cyclic module M. Thus, for every submodule of a cyclic module there exist ideals $\mathfrak{P} = \mathfrak{p}/\operatorname{Ann}(m)$ and $\mathfrak{Q} = \mathfrak{q}/\operatorname{Ann}(m)$, where $\mathfrak{P} \leqslant \mathfrak{Q}$ if and only if $\mathfrak{p} \leqslant \mathfrak{q}$ as Rosenberg ideals. Similarly, we can formulate the definition of equivalence relation. Thus let $\mathfrak{p} \sim \mathfrak{q}$ and $x \notin \mathfrak{p} \cap \mathfrak{q}$. Let $x \notin \mathfrak{p}$, thus there exists a finite subset $V \subseteq R$, such that $(\mathfrak{p} : Vx) \subseteq \mathfrak{p}$. If $x \notin \mathfrak{q}$, then $(\mathfrak{q} : Wx) \subseteq \mathfrak{q}$ for some finite subset W of the ring R. Let $U = V \cup W$, then $(\mathfrak{p} \cap \mathfrak{q} : Ux) \subseteq \mathfrak{p} \cap \mathfrak{q}$. If $x \in \mathfrak{q}$, then $(\mathfrak{q} : Vx) = R$, hence $(\mathfrak{p} \cap \mathfrak{q} : Vx) \subseteq \mathfrak{p}$. Since $\mathfrak{p} \sim \mathfrak{q}$ by the assumption, $\mathfrak{p} \leqslant \mathfrak{q}$, and thus $(\mathfrak{p} : U) \subseteq \mathfrak{q}$ for some finite subset $U \subseteq R$, and since we may assume that $1 \in U$, we obtain

$$(\mathfrak{p} \cap \mathfrak{q} : UVx) = ((\mathfrak{p} \cap \mathfrak{q} : Vx) : U) \subseteq (\mathfrak{p} : U) \subseteq \mathfrak{q}.$$

Moreover, since $V \subseteq UV$, we also have

$$(\mathfrak{p} \cap \mathfrak{q} : UVx) \subseteq (\mathfrak{p} \cap \mathfrak{q} : Vx) \subseteq \mathfrak{p},$$

hence $(\mathfrak{p} \cap \mathfrak{q} : UVx) \subseteq \mathfrak{p} \cap \mathfrak{q}$, thus $\mathfrak{p} \cap \mathfrak{q}$ is a strongly prime ideal. Clearly $\mathfrak{p} \cap \mathfrak{q} \leqslant \mathfrak{p}$. On the other hand, since $\mathfrak{p} \leqslant \mathfrak{q}$, there exists a finite subset $V \subseteq R$, with $(\mathfrak{p} : V) \subseteq \mathfrak{q}$. We may obviously assume that $1 \in V$, thus we have $(\mathfrak{p} : V) \subseteq \mathfrak{p}$. Hence $(\mathfrak{p} : V) \subseteq \mathfrak{p} \cap \mathfrak{q}$, so $\mathfrak{p} \leqslant \mathfrak{p} \cap \mathfrak{q}$ and $\mathfrak{p} \sim \mathfrak{p} \cap \mathfrak{q}$.

Let us now assume that $\mathfrak{p} \cap \mathfrak{q}$ is a strongly-prime ideal while $\mathfrak{p} \not\subseteq \mathfrak{q}$ and $\mathfrak{p} \not\supseteq \mathfrak{q}$. Sinc such a $\mathfrak{p} \not\subseteq \mathfrak{q}$ there exists an element $x \in \mathfrak{p} - \mathfrak{q}$. Thus $x \notin \mathfrak{p} \cap \mathfrak{q}$ and we may find a finite subset $V \subseteq R$ such that $(\mathfrak{p} \cap \mathfrak{q} : Vx) \subseteq \mathfrak{p} \cap \mathfrak{q}$. Since $(\mathfrak{p} : Vx) = R$, this yields $(\mathfrak{q} : Vx) \subseteq \mathfrak{p} \cap \mathfrak{q} \subseteq \mathfrak{p}$, hence $\mathfrak{p} \leqslant \mathfrak{q}$. By symmetry $\mathfrak{p} \geqslant \mathfrak{q}$, and thus $\mathfrak{p} \sim \mathfrak{q}$. □

We easy obtain the following corollary:

Corollary 1. *Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ be a finite family of strongly-prime modules, such that $\mathfrak{P}_1 \sim \cdots \sim \mathfrak{P}_n$, then $\cap_{i=1}^{n} \mathfrak{P}_i$ is a strongly-prime module and $\mathfrak{P}_1 \sim \cap_{i=1}^{n} \mathfrak{P}_i$.*

For any left module M, it's submodule N is called *strongly two-sided*, if left annihilator of every element of N is two-sided ideal. Clearly, new submodule is two-sided. Thus the set of such submodules is not empty, because the zero submodule is strongly two-sided submodule. The sum of all strongly two-sided submodules is called the *bound* of the submodule N. In other words, the *bound of the module* is the largest submodule among those that have two-sided left annihilators for all their elements. In the case when $M = N$ we are talking about the concept of a *bound of the module*. As follows, the *bound* of the module M is the largest strongly two-sided submodule of the module M. Denote the bound of a submodule N by $b(N)$, the bound of the module M by $b(M)$.

Lemma 6. *For every strongly-prime left submodule $\mathfrak{P}$ of the module M we have $b(\mathfrak{p}) \in \mathrm{Cspec}(M)$.*

Proof. Let $x, y \in M$ by elements, such that $xRy \subseteq b(\mathfrak{P})$. Assume that $y \notin b(\mathfrak{P})$. Then there exists such an element $s \in R$ with $ys \notin \mathfrak{P}$. For every $r \in R$, $(xr)R(ys) \subseteq (xRy)s \subseteq b(\mathfrak{P})s \subseteq b(\mathfrak{P}) \subseteq \mathfrak{P}$. Hence $rx \in \mathfrak{P}$. Thus $xR \subseteq b(\mathfrak{P})$, which proves the assertion. □

Lemma 7. *If $L \leqslant K$ are left R-modules, then $b(L) \subseteq b(K)$. Conversely, if R is a left noetherian fully-bounded ring, and if $b(L) \subseteq b(K)$, then $L \leqslant K$.*

Proof. Since $L \leqslant K$, there exists a representation $L = \mathfrak{A}/\operatorname{Ann}(m)$ and $K = \mathfrak{B}/\operatorname{Ann}(m)$ for some left ideals $\mathfrak{A}$ and $\mathfrak{B}$ of the ring R. Then $\mathfrak{A} \leqslant \mathfrak{B}$. Thus there exist a finite subset $V \subseteq R$, that $(\mathfrak{A} : V) \subseteq \mathfrak{B}$. Then for every elements $r \in b(L)$ and $s \in R$, we have $rs \in \mathfrak{A}$, therefor $r \in (\mathfrak{A} : s)$. Thus $r \in (\mathfrak{A} : V) = \cap_{s \in V} (\mathfrak{A} : s)$. Since the former is contained in $\mathfrak{B}$, we have $b(L) \subseteq K$, hence $b(L) \subseteq b(K)$.

On the other hand, if R is a left noetherian fully-bounded ring, then there exists a finite subset $V = \{v_1, \ldots, v_n\} \subseteq R$ such that $b(L) = \cap_{i=1}^{n} (\mathfrak{A} : v_i) = (\mathfrak{A} : V)$. Hence $(\mathfrak{A} : V) = b(\mathfrak{A}) \subseteq b(\mathfrak{B}) \subseteq \mathfrak{B}$, and $\mathfrak{A} \leqslant \mathfrak{B}$, therefore $L \leqslant K$. □

Corollary 2. *Let L and K be left modules such that $L \sim K$, then $b(L) = b(K)$. Moreover if R is a left noetherian fully-bounded ring, then the converse is also true.*

3. Functoriality of cyclic spectrum of module

The cyclic spectrum construction can be regarded as a contravariant functor from the category of modules to the category of sets,

$$\mathrm{CSpec}\colon \mathrm{Mod} \to \mathrm{Set}\,.$$

A contravariant functor CSpec is a rule assigning to each module M over an associative ring R the set $\mathrm{CSpec}(M)$, the cyclic spectrum, i.e. the set of submodules that are related in that spectrum, and to each module homomorphism $f\colon M_1 \to M_2$ the map of sets

$$\mathrm{Cspec}(M_1) \to \mathrm{Cspec}(M_2),$$
$$P \mapsto f^{-1}(P).$$

Consider the endomorphism ring $E = \mathrm{End}(M)$, and also consider the center of that ring, denoted by $C = \{c \in E \mid cr = rc, \forall r \in E\}$. Consider the construction of partial algebra over the ring C. It is the set Q with a reflexive, symmetric binary relation $\perp \subseteq Q \times Q$ (called commeasurability), partial addition and multiplication operations "+" and "$\cdot$", that are functions $I \to Q$, a scalar multiplication operation $E \times Q \to Q$, and elements $0, 1 \in C$, such that the following axioms are satisfied:

(1) for all $q \in Q$, $a \perp 0$ and $a \perp 1$;

(2) the relation $\perp$ is preserved by the partial binary operations: for all $q_1, q_2, q_3 \in Q$, with $q_i \perp q_j$ $(1 \leqslant i, j \leqslant 3)$ and for all $\lambda \in C$, one has $(q_1 + q_2) \perp q_3$, $(q_1 \cdot q_2) \perp q_3$ and $(\lambda q_1) \perp q_2$;

(3) if $q_i \perp q_j$ for $1 \leqslant i, j \leqslant 3$, then the values of all polynomials in q_1, q_2 and q_3 form a commutative algebra.

Commeasurability subalgebra of a partial C-algebra Q is a subset $Z \subseteq Q$ consisting of pairwise commeasurable elements that is closed under C-scalar multiplication and the partial binary operations of Q.

Given functors $K\colon \mathcal{A} \to \mathcal{B}$ and $S\colon \mathcal{A} \to \mathcal{C}$, we recall that the (right) Kan extension of S along K is a functor $L\colon \mathcal{B} \to \mathcal{C}$ with a natural transformation $\varepsilon\colon LK \to S$ such that for any other functor $F : \mathcal{B} \to \mathcal{C}$ with a natural transformation $\eta\colon FK \to S$ there is a unique natural transformation $\delta\colon F \to L$, such that $\eta = \varepsilon \circ (\delta K)$.

Theorem 2. *The functor* $\mathrm{Cspec}\colon \mathrm{Mod}^{\mathrm{op}} \to \mathrm{Set}$, *with the identity natural transformation* $\mathrm{Cspec}|_{\mathrm{Comm\,Mod}^{\mathrm{op}}} \to \mathrm{CSpec}$ *is the Kan extension of the functor* $\mathrm{Cspec}\colon \mathrm{Comm\,Mod}^{\mathrm{op}} \to \mathrm{Set}$ *along the embedding* $\mathrm{Comm\,Mod}^{\mathrm{op}} \subseteq \mathrm{Mod}^{\mathrm{op}}$.

Proof. Let $F\colon \mathrm{Mod} \to \mathrm{Set}$ be a contravariant functor with a fixed natural transformation $\eta\colon F|_{\mathrm{Comm\,Mod}} \to \mathrm{Spec}$. Consider functor C-Spec: $\mathrm{Comm\,Mod} \to \mathrm{CSpec}$. We need to show that there is a unique natural transformation $\delta\colon F \to \mathrm{CSpec}$, that induces $\eta\colon F|_{\mathrm{Comm\,Mod}} \to \mathrm{CSpec}$ upon a restriction to $\mathrm{Comm\,Mod} \subseteq \mathrm{Mod}$. To construct it, fix ring R and module M over it. For every submodule $N \subseteq M$ over ring R the inclusion $N \subseteq M$ given a morphisms of sets $F(M) \to F(N)$, and η provides a morphisms $\eta_N\colon F(N) \to \mathrm{CSpec}(N)$; these compose to give morphisms $F(M) \to \mathrm{CSpec}(N)$. By naturality of the morphisms involved, these maps of $F(M)$ collectively form a cone over the diagram obtained for submodules of module. By the universal property of limit, there exists a unique arrow making corresponding diagram commutative for all $N \subseteq M$.

Defined morphisms δ_M form the components of a natural transformation $\delta\colon F \to CSec$. By construction, δ induces η when restricted to Comm Mod. Uniquness of δ is guaranteed by the uniqueness of the indicated arrow used to define δ_M above. □

4. Localisations

Recall some definitions. By a *torsion-theoretic spectrum* we mean the space of all prime torsion theories (or prime Gabriel filters of a main ring) in the category of left R-modules with Zarisky topology. Recall that *prime torsion theory* $\pi \in R - tors$ is a torsion theory, for which $\pi = \chi(R/I)$ for some critical ideal I of the ring R, where $R - tors$ is class of all torsion theories of the category R-mod and $\chi(R/I)$ is the torsion theory, cogenerated by module $E(R/I)$. If τ is torsion theory of the category R-mod, then left R-module M is called *torsion free module* if and only if there exist R from M into some member of τ. Class of all torsion free modules for some τ is denote by $\mathfrak{F}_\tau$. Further information about the prime torsion theories can be fund in [5].

Remark 1. The class of all torsion theories R-tors can be partially ordered by setting $\tau \leqslant \tau'$ if and only if $\mathfrak{T}_\tau \subseteq \mathfrak{T}_{\tau'}$, namely, the class of all torsion modules of one torsion theory is contained in the class of all torsion modules of other torsion theory.

Introduce the notion of torsion-theoretic spectrum of a module M. Use the concepts of torsion-theoretic spectrum of a ring R introduced above. Introduce the concept of support of module M: $\mathrm{supp}(M) = \{\sigma | \sigma(M) \neq 0\}$. *Torsion-theoretic spectrum of module* M, $\text{R-Sp}(M)$ is defined as $\text{R-sp}(R) \cap \mathrm{supp}(M)$.

If M is a left R-module, denote by $\xi(M)$ the smallest torsion theory such that M will be a torsion module, by $\chi(M)$ the largest torsion theory, that M will be a torsion-free module. Clearly, $\mathcal{T}_{\chi(M)}$ consists of R-modules N such that $\mathrm{Hom}_R(N, E(M)) = 0$, where $E(M)$ is the injective hull of a module M.

Lemma 8. *If σ is a torsion theory and $\mathfrak{P}$ is a left Rosenberg point of a cyclic module M, then $M/\mathfrak{P}$ is either a σ-torsion module or a σ-torsion free module.*

Proof. Assume that $M/\mathfrak{P} \notin \mathcal{F}_\sigma$. If $\mathfrak{P}$ is a left Rosenberg point, then there exists ideal $\mathfrak{p}$ of a ring R such that $\mathfrak{P} = \mathfrak{p}/\operatorname{Ann}(m)$. Pick an element $0 = \bar{x} \in \sigma(R/\mathfrak{p})$. Thus, there exists a finite subset V of the ring R with $(\mathfrak{p} : Vx) \subseteq \mathfrak{p}$. Obviously, $V\bar{x} \subseteq \sigma(R/\mathfrak{p})$, hence, for every element $v \in V$ there exists left ideal $L_v \in \mathcal{L}(\sigma)$ such that $L_v vx \subseteq \mathfrak{p}$. Let $L = \cap_{v\in L} L_v$, then $L \in \mathcal{L}(\sigma)$ and $LVx \subseteq \mathfrak{p}$. Hence $L \subseteq (\mathfrak{p} : Vx) \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \mathcal{L}(\sigma)$, and therefore $M/\mathfrak{P}$ is σ-torsion module. □

Proposition 1. *If M is a fully bounded left noetherian module and $\mathfrak{P} \in \mathrm{Cspec}(M)$, then the torsion theory $\tau_{\mathfrak{P}} = \chi(M/\mathfrak{P})$ cogenerated by module $M/\mathfrak{P}$ is prime.*

Proof. Obviously, $\mathfrak{P} \notin \mathcal{L}(\tau_{\mathfrak{P}})$, therefore $M/\mathfrak{P}$ is a $\tau_{\mathfrak{P}}$-torsion free module. Thus, since $\chi(M/\mathfrak{P})$ is the largest torsion theory for which $M/\mathfrak{P}$ is torsion free module. We have $\chi(M/\mathfrak{P}) \leqslant \tau_{\mathfrak{P}}$. Conversely, assume that $\mathcal{L}(\chi(M/\mathfrak{P})) \nsubseteq \mathcal{L}(\tau_{\mathfrak{P}})$. Take $L \in \mathcal{L}(\chi(M/\mathfrak{P})) - \mathcal{L}(\tau_{\mathfrak{P}})$, then $L \leqslant \mathfrak{P}$. Thus, by the definition, $\mathfrak{A} \leqslant \mathfrak{p}$ for some ideals $\mathfrak{A}$ and $\mathfrak{p}$ of the ring R. Thus there exists a finite subset $U \subseteq R$ such that $\cap_{u\in U}(\mathfrak{A} : u) = (\mathfrak{A} : U) \subseteq \mathfrak{p}$. Hence $\mathfrak{p} \in \mathcal{L}(\chi(M/\mathfrak{P}))$, contradicting the definition of $\chi(M/\mathfrak{P})$. □

The previous statements imply the following result.

Theorem 3. *The mapping $\Phi\colon \mathrm{Cspec}(M) \to \text{M-sp}$, where $\Phi(\mathfrak{P}) = \chi(M/\mathfrak{P})$ is continuous and surjective.*

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