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On differential preradicals

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Dedicated to Professor V. V. Kirichenko on the occasion of his 65th birthday

Abstract. Differential preradicals and differential preradical filters are considered. Differentially closed fields are investigated.

1. Preliminaries

Throughout the whole text, all rings are considered to be associative with $1 \neq 0$. All modules are unitary left modules.

Let A be a ring.

Definition. A derivation of A is a mapping $\delta : A \to A$ such that

$$\forall a, b \in A : \delta(a + b) = \delta(a) + \delta(b),$$

$$\forall a, b \in A : \delta(ab) = \delta(a)b + a\delta(b).$$

Definition. A derivation δ of A is trivial if

$$\forall a \in A : \delta(a) = 0.$$

Definition. A ring R with a derivation δ of R is called a differential ring.

Definition. Let A be a ring with a derivation δ, and let $A[\delta]$ be the set of all polynomials over ring A. Then $A[\delta]$ is a ring under addition of polynomials and multiplication induced by the rule

$$\delta a = a\delta + \delta(a)$$

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and its consequences. This ring will be denoted by D_A and it is called a ring of linear differential operators of A [4, p.43].

Definition. Let M be a left A-module, where A is a differential ring with the derivation δ. A mapping $d : M \to M$ is said to be a derivation of M in case

$$\forall m_1, m_2 \in M : d(m_1 + m_2) = d(m_1) + d(m_2),$$

$$\forall a \in A \forall m \in M : d(am) = \delta(a)m + ad(m).$$

Definition. Let A be a differential ring. A left A-module M with a derivation d of M is called a differential (left A-)module.

Definition. Let A be a differential ring and let (M_i, d_i) be differential left A-module for every $i \in \{1, 2\}$. An A-homomorphism $f : M_1 \to M_2$ is said to be a differential $(A-)$homomorphism in case

$$\forall m \in M_1 : f(d_1(m)) = d_2(f(m)).$$

Definition. Let A be a differential ring and let M be a differential left A-module with a derivation d. A submodule N of the left A-module M is called a differential submodule of the differential left A-module M in case $\forall n \in N : d(n) \in N$.

Let A be a differential ring and let M be a differential left A-module with a derivation d.

If N is a differential submodule of M then we define a mapping $\tilde{d} : M/N \to M/N$ such that $\forall m \in M : \tilde{d}(m + N) = d(m) + N$.

It is clear because

$$\forall m \in M \forall n \in N : \tilde{d}((m + n) + N) = d(m + n) + N =$$

$$= (d(m) + d(n)) + N = (d(m) + N) + (d(n) + N) =$$

$$= (d(m) + N) + N = d(m) + N = \tilde{d}(m + N).$$

It is easy to see that $\tilde{d}$ is a derivation of M/N. Therefore we have the differential left A-module M/N.

Given two differential left A-modules M, N, the set of all differential A-homomorphisms from M to N will be denoted by

$$Hom_{A-Dmod}(M, N) = DHom_A(M, N).$$

Definition. Given a differential ring A the category of differential left A-modules is the system

$$A - DMod = (A - Dmod, DHom_A, \circ),$$

where $A - Dmod$ is the class of all differential left A-modules,

$$Hom_{A-Dmod}(M,N) = Hom_{A_D}(M,N),$$

and $\circ$ is the usual composition of functions.

Let A be a differential ring.

It is well known that the categories $A - Dmod$ and $D_A - Mod$ are isomorphic (L.A. Skorniakov). We shall define the isomorphism in the following way.

Let M be a differential left A-module with the derivation d. Setting

$$\left(\sum_{i=0}^{n} a_i \delta^i\right) m := \sum_{i=0}^{n} a_i d^i(m),$$

where

$$\sum_{i=0}^{n} a_i \delta^i \in D_A, m \in M,$$

we obtain a D_A-module structure on M. It is clear that if $f : M_1 \to M_2$ is a differential A-homomorphism then f is a D_A-homomorphism.

Conversely, let M be a left D_A-module. Then we define a mapping $d : M \to M$ by the rule: $d(m) := \delta m$ $(m \in M)$. It is easily seen that d is a derivation of M. Moreover, if $f : M_1 \to M_2$ is a D_A-homomorphism then taking into consideration the natural structures of differential modules on M_1, M_2 we have that f is a differential A-homomorphism from differential module M_1 to differential module M_2.

Definition. A differential preradical r of $A - Dmod$ assigns to each differential A-module C its differential submodule $r(C)$ in such a way that

(DT1) for every differential A-homomorphism $f : N \to M$

$$f(r(N)) \subseteq r(M).$$

A differential preradical r of $A - Dmod$ is said to be idempotent in case

(DT2) for every differential left A-module N

$$r(r(N)) = r(N).$$

A differential preradical r of $A - Dmod$ is said to be a differential radical in case

(DT3) for every differential left A-module N

$$r(N/r(N)) = 0.$$

A differential preradical r of $A - Dmod$ is said to be hereditary in case

(DT4) for every differential left A-module M and for every its differential submodule N

$$r(N) = N \cap r(M).$$

Definition. A differential preradical r of $A - Dmod$ is said to be trivial in case either for every differential left A-module M we have $r(M) = M$ or for every differential left A-module M we have $r(M) = 0$.

Let A be a differential ring with the derivation δ. Let $a \in A$. We shall denote the elements

$$\delta(a), \delta(\delta(a)), ..., \delta(\delta(...\delta(a)...)), ...$$

by

$$a', a'', ..., a^{(n)}, ...$$

respectivelly.

If $a \in A$ then the set $\{a', a'', ..., a^{(n)}, ...\}$ will be denoted by a^{∞}.

Definition. A left ideal I of A is called a differential left ideal if

$$\forall a \in I : a' \in I.$$

Remark. Let I be an idempotent ideal of A. Then I is a differential ideal. Let $a \in I$. Hence

$$a = b_1c_1 + b_2c_2 + ... + b_nc_n,$$

where

$$\{b_1, c_1, b_2, c_2, ..., b_n, c_n\} \subseteq I.$$

It follows from this

$$a' = b_1'c_1 + b_1c_1' + b_2'c_2 + b_2c_2' + ... + b_n'c_n + b_nc_n' \in I.$$

Put

$$(I : a^{\infty}) = \{u \,|u \in A \& \forall\, n \in \{0, 1, 2...\} : ua^{(n)} \in I\}.$$

Remark. Let I be a differential left ideal of a differential ring A and let $a \in A$. Then $(I : a^{\infty})$ is a differential left ideal.

It is clear that $(I : a^{\infty})$ is a left ideal of A. Let $b \in (I : a^{\infty})$. Then $\forall n \in \{0, 1, 2...\} : ba^{(n)} \in I$. Since I is a differential left ideal,

$$\forall n \in \{0, 1, 2...\} : b'a^{(n)} + ba^{(n+1)} = (ba^{(n)})' \in I.$$

But $\forall n \in \{0, 1, 2...\} : ba^{(n+1)} \in I$. Therefore $\forall n \in \{0, 1, 2...\} : b'a^{(n)} \in I$. Hence $b' \in (I : a^{\infty})$.

Remark. Let M be a differential left A-module, where A is a differential ring, and let $m \in M$. Then the left ideal

$$I = \{a \,| a \in A \& \forall n \in \{0, 1, 2, ...\} : a\, m^{(n)} = 0\}$$

is a differential ideal of A.

Let $a \in I$. Hence $\forall n \in \{0, 1, 2, ...\} : am^{(n)} = 0$. Therefore

$$\forall n \in \{0, 1, 2, ...\} : a'm^{(n)} = a'm^{(n)} + am^{(n+1)} = (am^{(n)})' = 0' = 0.$$

Hence $a' \in I$.

Remark. Let I, J be differential left ideals of a differential ring A. Then $I \cap J$ is a differential left ideal of A.

Remark. Let I, J be differential left ideals of a differential ring A. Then the left ideal of AIJ is a differential left ideal of A.

Let $a \in IJ$. Then $a = i_1 j_1 + i_2 j_2 + ... + i_n j_n$, where

$$\{i_1, i_2, ..., i_n\} \subseteq I, \{j_1, j_2, ..., j_n\} \subseteq J.$$

Hence

$$a = i'_1 j_1 + i'_2 j_2 + ... + i'_n j_n + i_1 j'_1 + i_2 j'_2 + ... + i_n j'_n \in IJ$$

because

$$\{i'_1, i'_2, ..., i'_n\} \subseteq I, \{j'_1, j'_2, ..., j'_n\} \subseteq J.$$

Definition. Let A be a differential ring with the derivation δ. A differential preradical filter of A is a collection F of differential left ideals of A possessing the following properties:

DF1. $I \in F \& I \subseteq J \& J$ is a differential left ideal of $A \to J \in F$;

DF2. $I \in F \& a \in A \to (I : a^{\infty}) \in F$;

DF3. $I \in F \& J \in F \to I \cap J \in F$.

Definition. Let A be a differential ring with the derivation δ. A differential radical filter of A is a differential preradical filter F of A possessing the following property:

DF4. I is a differential left ideal of $A \&$ $I \subseteq J \&$ $J \in F \&$

$$\& (\forall a \in J : (I : a^{\infty}) \in F) \to I \in F.$$

Lemma 1.1. Let A be a differential ring. If F is a collection of differential left ideals of A possessing DF2, DF4 then F possesses DF3.

Proof. Let F be a collection of differential left ideals of A possessing DF2, DF4. Let $I \in F \& J \in F$. By DF2, $\forall j \in J : (I : j^{\infty}) \in F$. It is clear that $\forall j \in J : (I \cap J : j^{\infty}) \subseteq (I : j^{\infty})$. Let $j \in J, c \in (I : j^{\infty})$. Hence

Definition. A base of a differential radical (preradical) filter F is a collection B of differential left ideals from F possessing the following property:

$$\forall I \in F \exists H \in B : H \subseteq I.$$

All necessary definitions and results about preradicals and differential rings can be found in many papers and books [2, 3, 6].

Remark. The above definitions can be formulated for differential rings with many derivations. We shall consider differential rings with one derivation.

2. Examples of differential radical filters

We shall give some lemmas in order to construct differential radical filters of differential rings. It is easy to see that the following lemmas with trivial derivations are well-known theorems about radical filters.

The following lemma is a differential variant of Maranda's Lemma (see [2]).

Lemma 2.1. Let A be a differential ring and B be a multiplicatively closed system of differential two-sided ideals, which are finitely generated as differential left ideals. Then the set

$F_B = \{T | T$ is a differential left ideal of A &$\exists$L $\in$ B:L $\subseteq$ T $\}$
is a differential radical filter of A.

Proof. Recall that a differential left ideal I is generated by elements $a_1, a_2, ..., a_n$ if it is generated by the sets

$$a_1^{\infty}, a_2^{\infty}, ..., a_n^{\infty}$$

as a left ideal.

We shall verify DF1, DF2, DF4.

DF1. This is clear.

DF2. Let $I \in F_B$. Then there exists $H \in B$ such that $H \subseteq I$. But H is a differential two-sided ideal. Hence $\forall a \in A : H \subseteq (H : a^{\infty})$. It follows from this that $\forall a \in A : (I : a^{\infty}) \in F_B$.

DF4. Let $I \subseteq J \in F_B$, where I is a differential left ideal of A such that

$$\forall a \in J : (I : a^{\infty}) \in F_B.$$

Then there exists $H \in F_B$ such that $H \subseteq J$ and $H = [a_1, a_2, ..., a_n]$ for some $a_1, a_2, ..., a_n \in A$.

Let

$$H_1 \subseteq (I : a_1^{\infty}),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$H_n \subseteq (I : a_n^\infty),$$

where $H_1, ..., H_n \in B$. Let $v \in H, v_1 \in H_1, ..., v_n \in H_n$. Then

$$v = (r_{01}a_1 + r_{11}a_1' + ...r_{p_1 1}a_1^{(p_1)}) + ... + (r_{0n}a_n + r_{1n}a_n' + ...r_{p_n n}a_n^{(p_n)}),$$

where

$$\{r_{01}, r_{11}, ..., r_{p_1 1}, r_{0n}, r_{1n}, ..., r_{p_n n}\} \subseteq A$$

and

$$v_1...v_n r_{mi} \in H_i, \forall i \in \{1, ..., n\}.$$

Since $H_i \subseteq (I : a_i^\infty)$, $v_1...v_n r_{mi} a_i^{(m)} \in I$, therefore $v_1...v_n v \in I$. Hence $H_1...H_n H \subseteq I$. But $H_1...H_n H \in B$. Hence $I \in F_B$.

Lemma 2.2. Let A be a differential ring. If I is an idempotent ideal of A then the set $F = \{T | T$ is a differential left ideal of $A \& I \subseteq T\}$ is a differential radical filter of A.

Proof. DF1. This is clear.

DF2. Let $T \in F, a \in A$. Therefore $I \subseteq T$. Hence $I \subseteq (T : a^\infty)$. It follows from this that $(T : a^\infty) \in F$.

DF4. Let $K \subseteq J, J \in F, \forall a \in J : (K : a^\infty) \in F$. Hence $\forall a \in J : I \subseteq (K : a^\infty)$. Therefore $IJ \subseteq K$. But $I \subseteq J$. Hence $I = II \subseteq IJ \subseteq K$. It follows from this that $K \in F$.

Lemma 2.3. Let A be a differential ring and S be a differential two-sided ideal of A. If every differential left ideal of A is two-sided then the set

$F_S = \{T | T$ is a differential left ideal of $A \& S + T = A\}$

is a differential radical filter of A.

Proof. DF1. This is clear.

DF2. Let $I \in F_S, a \in A$. Since I is two-sided, $I \subseteq (I : a^\infty)$. By DF1, $(I : a^\infty) \in F_S$.

DF4. Let $I \subseteq J, J \in F_S, \forall a \in J : (I : a^\infty) \in F_S$. Hence $S + J = A$. Hence $s + j = 1$ for some $s \in S, j \in J$. Since $(I : j^\infty) \in F_S$, $\exists j_0 \in (I : j^\infty) \exists s_0 \in S : s_0 + j_0 = 1$. Hence $1 = (s_0 + j_0)(s + j) = (s_0 s + s_0 j + j_0 s) + j_0 j$. But $s_0 s + s_0 j + j_0 s \in S$. Since $j_0 \in (I : j^\infty)$, $j_0 j \in I$. Therefore $S + I = A$. Hence $I \in F_S$.

3. Differential hereditary radicals

Theorem 3.1. Let A be a differential ring. If F is a differential preradical filter of A then the function $\sigma, \sigma(M) = \{m \,| m \in M \& \exists I \in F : Im^\infty = 0\}$ (M is a differential left A-module) is a differential hereditary preradical.

Proof.

1). Let $m_1, m_2, m \in \sigma(M), a \in A$. It is easy to see that

$$\exists I_1, I_2, I \in F : I_1 m_1^\infty = 0 \& I_2 m_2^\infty = 0 \& I m^\infty = 0.$$

Hence $(I_1 \cap I_2)(m_1 + m_2)^\infty = 0$. But $I_1 \cap I_2 \in F$ (DF3). Therefore $m_1 + m_2 \in \sigma(M)$. By DF2, $(I : a^\infty) \in F$. It is clear that

$$(am)^\infty = \{am, a'm + am', ..., \sum_{s=0}^{n} C_n^s a^{(n-s)} m^{(s)}, ...\}.$$

It is obvious that $u \in (I : a^\infty)$

$$\forall n \in \{0, 1, 2...\} : u(am)^{(n)} = u(\sum_{s=0}^{n} C_n^s a^{(n-s)} m^{(s)}) =$$

$$= \sum_{s=0}^{n} C_n^s$$

Hence $(I : a^\infty)(am)^\infty = 0$. Therefore $am \in \sigma(M)$. It is clear that $I(m')^\infty = 0$.

2). Let $f : N \to M$ be a differential A-homomorphism and $n \in \sigma(N)$.

Hence $\exists I \in F : In^\infty = 0$. It is clear that $I(f(n))^\infty = 0$. Hence $f(n) \in \sigma(M)$.Therefore $f(\sigma(N)) \subseteq \sigma(M)$.

3). Let M be a differential left A-module and let N be its differential submodule.

It is clear that $\sigma(N) \subseteq N \cap \sigma(M)$. Let $n \in N \cap \sigma(M)$. Hence $\exists I \in F : In^\infty = 0$. Therefore $N \cap \sigma(M) \subseteq \sigma(N)$.

Theorem 3.2. Let A be a differential ring. If I is an idempotent ideal of A then the function

$$\sigma : M \mapsto \sigma(M)$$

($\sigma(M) = \{m \in M \,|\forall i \in I \forall n \in \{0, 1, 2, ...\} : im^{(n)} = 0\}$, M is a differential left A-module) is a differential hereditary radical.

Proof. 1). Let $m, m_1, m_2 \in \sigma(M), a \in A$. Then

$$\forall i \in I \forall n \in \{0, 1, 2, ...\} : i(m_1 + m_2)^{(n)} = im_1^{(n)} + im_2^{(n)} = 0.$$

Hence $m_1 + m_2 \in \sigma(M)$ and

$$\forall i \in I \forall n \in \{0, 1, 2, ...\} : i(am)^{(n)} = i(\sum_{k=0}^{n} C_n^k a^{(k)} m^{(n-k)}) =$$

$$= \sum_{k=0}^{n} C_n^k i a^{(k)} m^{(n-k)} = 0.$$

Hence $am \in \sigma(M)$.

$\forall i \in I \forall n \in \{0,1,2,...\} : i(m')^{(n)} = im^{(n+1)} = 0$.Hence $m' \in \sigma(M)$.

Therefore $\sigma(M)$ is a differential submodule of M.

2). Let $f : N \to M$ be a differential A-homomorphism and $m \in \sigma(N)$. Then

$$\forall i \in I \forall n \in \{0,1,2,...\} : i(f(m))^{(n)} = if(m^{(n)}) = f(im^{(n)}) = f(0) = 0.$$

Hence $f(m) \in \sigma(M)$. Therefore $f(\sigma(N)) \subseteq \sigma(M)$.

3). It is clear that

$$r(N) = N \cap r(M)$$

for a differential submodule N of a differential left A-module M.

4). Let M be a differential left A-module and

$$m + \sigma(M) \in \sigma(M/\sigma(M)).$$

Then

$\forall i \in I \forall n \in \{0,1,2,...\} : im^{(n)} \in \sigma(M)$. But

$$\forall i \in I \exists \{j_{i,1}, ..., j_{i,k_i}, u_{i,1}, ..., u_{i,k_i}\} \subseteq I : i = j_{i,1}u_{i,1} + ... + j_{i,k_i}u_{i,k_i}.$$

Hence

$$im^{(n)} = (j_{i,1}u_{i,1} + ... + j_{i,k_i}u_{i,k_i})m^{(n)} =$$

$$= j_{i,1}(u_{i,1}m^{(n)}) + ... + j_{i,k_i}(u_{i,k_i}m^{(n)}) = 0 + ... + 0 = 0.$$

Therefore $m \in \sigma(M)$. It follows from this that $\sigma(M/\sigma(M)) = 0$.

4. Differentially closed fields

Let A be a differential ring and σ be a differential preradical of $A-Dmod$. We shall say that σ splits if for every differential left A-module M there exists a differential submodule K of M such that $M = \sigma(M) \oplus K$.

Definition. A differential field F is said to be universal if for every differential equation $x^{(n)} + a_1x^{(n-1)} + ... + a_nx = 0$ ($n \in N, \{a_1, ..., a_n\} \subseteq F$) there exists its non zero solution belonging to F.

The differential hereditary preradical of $A - Dmod$ corresponding to the singular hereditary preradical of $D_A - Mod$ is called a differential singular hereditary preradical.

Let F be a differential field. It is not difficult to see that a singular hereditary preradical of D_F-Mod is a hereditary radical.

Theorem 4.1. Let F be a differential field with $char(F) = 0$. Then the following properties are equivalent:

Every differential hereditary radical of $F - Dmod$ splits;

The differential singular hereditary radical of $F - Dmod$ splits;

F is universal.

Proof.

(i) $\Rightarrow$(ii). This is clear.

(ii) $\Rightarrow$(iii). The singular hereditary radical of D_F-Mod splits. Taking into consideration that D_F is a simple principal ideal domain (see Theorem 3.6 [4]), apply Theorem 10 [5] . It follows from this that every simple left D_F-module is injective. Apply Theorem 5.21 [4].

(iii) $\Rightarrow$(i). By Theorem 5.21 [4], every cyclic left D_F-module is injective. Therefore every singular left D_F-module is injective (see [1, p.366]). Let r be a hereditary radical of D_F-Mod such that $T(r) \neq D_F - Mod$ and let M be a left D_F-module. Let E_r be the corresponding radical filter. Since every non-zero left ideal of D_F is essential, every left ideal belonging to E_r is essential. Hence $r(M)$ is singular and, therefore, $r(M)$ is injective.

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