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RESEARCH ARTICLE

Total global neighbourhood domination

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Abstract. A subset D of the vertex set of a connected graph G is called a total global neighbourhood dominating set*(*tgnd-*set)* of G if and only if D is a total dominating set of G as well as G^N, where G^N is the neighbourhood graph of G. The total global neighbourhood domination number*(*tgnd-*number)* is the minimum cardinality of a total global neighbourhood dominating set of G and is denoted by $\gamma_{\text{tgn}}(G)$. In this paper sharp bounds for γ_{tgn} are obtained. Exact values of this number for paths and cycles are presented as well. The characterization result for a subset of the vertex set of G to be a total global neighbourhood dominating set for G is given and also characterized the graphs of order $n(\geqslant 3)$ having tgnd-*numbers* $2, n-1, n$.

Introduction and preliminaries

Domination is an active topic in graph theory and has numerous applications to distributed computing, the web graph and adhoc networks. Haynes *et al.* gave a comprehensive introduction to the theoretical and applied facets of domination in graphs.

A subset D of the vertex set V is called a *dominating set* [8] of the graph G if and only if each vertex not in D is adjacent to some vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of the dominating set of G.

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Many variants of the domination number have been studied. For instance a dominating set S of graph G is called a *total dominating set* [3] if and only if every vertex in V is adjacent to a distinct vertex in D. The *total domination number* of G, denoted by $\gamma_t(G)$ is the smallest cardinality of the total dominating set of G. A set D is called a *connected dominating set* of G if and only if D is a dominating set of G and $\langle D \rangle$ is connected. The *connected domination number* [4] of G, denoted by $\gamma_c(G)$ is the smallest cardinality of the connected dominating set of G. A dominating set D of connected graph G is called a *connected dominating set*of G if the induced subgraph $\langle D \rangle$ is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$ is the least cardinality of the connected dominating set of G [7].

If G is a connected graph, then the *Neighbourhood Graph* [7] of G, denoted by $N(G)$ (or) G^N, is the graph having the same vertex set as that of G and edge set being $\{uv/u, v \in V(G)$, there is $w \in V(G)$ such that $uw, wv \in E(G)\}$ [2].

In [5], a new type of graphs, called *semi complete graphs*, are introduced as follows. A connected graph G is said to be *semi complete* if any two vertices in G have a common neighbour.

A subset D of the vertex set V is called a *global neighbourhood dominating set* [6] of the graph G if and only if D is a dominating set of G, as well as G^N. The *global neighbourhood domination number*, $\gamma_{gn}(G)$ is the minimum cardinality of the global neighbourhood dominating set of G.

In the present paper, we introduce a new graph parameter, the *total global neighbourhood domination number*, for a connected graph G. We call $D \subseteq V$ a total global neighbourhood dominating set(tgnd-*set*) of G if and only if D is a total dominating set for both G, G^N. The total global neighbourhood domination number is the minimum cardinality of a total global neighbourhood dominating set of G and is denoted by $\gamma_{\text{tgn}}(G)$. By a γ_{tgn}-set of G, we mean a total global neighbourhood dominating set for G of minimum cardinality.

All graphs considered in this paper are simple, finite, undirected and connected. For all graph theoretic terminology not defined here, the reader is referred to [1] and [8].

In this paper sharp bounds for γ_{tgn} are given. A characterization result for a proper subset of the vertex set of G to be a tgnd-*set* of G is obtained and also characterized the graphs whose tgnd-*numbers* are $2, n, n-1$.

Note. If G is a simple graph such that G has isolates, then clearly γ_{tgn}-set of G does not exist. So, unless otherwise stated, throughout this paper G stands for a connected graph such that G^N has no isolates.

1. Main results

We give the tgnd-*numbers* of some standard graphs.

Proposition 1. 1) $\gamma_{\text{tgn}}(K_n) = 2; n = 3, 4, \ldots,$
2) $\gamma_{\text{tgn}}(C_3) = 2$
3) $\gamma_{\text{tgn}}(C_4) = 4$
4) $\gamma_{\text{tgn}}(P_n) = 4; n = 4, 5$
5) $\gamma_{\text{tgn}}(P_n(or)C_n) = 4[\frac{n}{6}] + j; n = 6m + j; j = 0, 1, 2, 3.$
$= 4[\frac{n}{6}] + 4; n = 6m + j; j = 4, 5.$
6) $\gamma_{\text{tgn}}(K_{m,n}) = 4; m, n \geqslant 2.$
7) $\gamma_{\text{tgn}}(S_{m,n}) = 4.$
8) $\gamma_{\text{tgn}}(C_n o K_2) = n$
9) $\gamma_{\text{tgn}}(K_{1,n})$ *does not exist.*

Now, we give a characterization result for a total dominating set of G to be a total global neighbourhood dominating set of G. Also, we give a relation between connected dominating set and total global neighbourhood dominating set.

Theorem 1. *For a graph G the following holds.*
(i) *A total dominating set D of G is a total global neighbourhood dominating set of G if and only if from each vertex in D there is a path of length two to a vertex in D. (characterization result)*
(ii) *Any connected dominating set for G of cardinality atleast four is a total global neighbourhood dominating set for G.*

Proof. The proof of (i) is trivial.

The proof of (ii) is as follows. Let $D \subseteq V$ (vertex set of G) be a connected dominating set of G with $|D| \geqslant 4$. It is enough to prove that D is a total dominating set of G^N. If $D = V$, we are through. Otherwise, let v be any vertex in $V - D$. Suppose v is adjacent to all the vertices of D (in G). Since $\langle D \rangle$ is connected there are u, w in D such that $\langle uvw \rangle$ is a triangle in G. This implies uv, vw are in G^N (u, w are in D). If v is not adjacent to atleast one vertex in D, since D is connected there is w in D such that vw is in G^N. Hence in either case there is a w in D such that vw is in G^N.

Let v be an arbitrary vertex in D. Since D is a connected dominating set of G of cardinality atleast four, there is a v_1 in D such that vv_1 lies on C_3 (in G) or $d_G(v, v_1) = 2$. In either case vv_1 is in G^N.

Hence D is a total dominating set of G^N. □

Remark. For any connected graph G of order $n \geqslant 4$, we have $\gamma_t(G) \leqslant \gamma_{\text{tgn}}(G) \leqslant \gamma_c(G)$.

Lemma 1. *If H is a spanning subgraph of a connected graph G, then $\gamma_{\text{tgn}}(G) \leqslant \gamma_{\text{tgn}}(H)$.*

Lemma 2. *For a graph G with $n \geqslant 1$ vertices, we have $2 \leqslant \gamma_{\text{tgn}}(G) \leqslant n$.*

Proof. The proof follows by the characterization result. □

Now, we characterize the graphs attaining lower bound.

Theorem 2. *$\gamma_{\text{tgn}}(G) = 2$ if and only if there is an edge uv in G that lies on C_3 such that any vertex in $V - \{u, v\}$ is adjacent to atleast one of u, v.*

Proof. Assume that $\gamma_{\text{tgn}}(G) = 2$. So there is a pair of vertices u, v in V such that $\{u, v\}$ is a total dominating set for G, G^N. This implies u, v are adjacent in G, G^N. Hence uv lies on a cycle $C_3 = \langle uvwu \rangle$ in G. Since $\{u, v\}$ is a total dominating set for G, for $x \in V - \{u, v\}$, xv or xu is an edge in G.

The inverse implication is clear. □

Now, we characterize the graphs attaining upper bound.

Theorem 3. *$\gamma_{\text{tgn}}(G) = n$ if and only if $G = C_4$ or P_4.*

Proof. Assume that $\gamma_{\text{tgn}}(G) = n$. Suppose that $\text{diam}(G) \geqslant 4$. Then $d_G(u, v) \geqslant 4$ for some u, v in G. Clearly u or v is not a cut vertex in G. Hence $V - \{u\}$ or $V - \{v\}$ is connected dominating set of cardinality atleast four. By Theorem.1(ii), $V - \{u\}$ or $V - \{v\}$ is a tgnd-*set* of G of cardinality $n - 1$, a contrary to our assumption.

Suppose that $\text{diam}(G) = 3$. Without loss of generality assume that $d_G(u, v) = 3$ for some u, v in G. Let $P = \langle uv_1v_2v \rangle$ be a diammetral path in G. Form a spanning tree of G say G' by preserving the diammetral path. Clearly $\text{diam}(G') \geqslant 3$. If $G' \neq P$, then $V - \{w\}$ (w is a pendant vertex in G') is a tgnd-*set* of G'. By Lemma 1, $V - \{w\}$ is a tgnd-*set* of G of cardinality $n - 1$, a contrary to our assumption. If $G' = P$, then G is not cyclic. This implies that G is tree with diameter three. Clearly G cannot have more than two pendant vertices. Hence $G = P_4$.

Suppose that $\text{diam}(G) = 2$. By hypothesis, G cannot be acyclic. Also G cannot have pendant vertices. Therefore G is cyclic and each vertex lies on a cycle. Suppose that $g(G) = 3$. If $G = C_3$, $\gamma_{\text{tgn}}(G) = 2 < 3$ a contradiction. If $G \neq C_3$, then $V(C_3) \subset V$. If all the vertices in $V - V(C_3)$

are adjacent to C_3, then $\gamma_{\text{tgn}}(G) = 3 < n$, a contrary to our assumption. If there is atleast one vertex v_4 in $V - V(C_3)$ not adjacent to C_3, then $V - \{v_4\}$ is a tgnd-*set* of G which is again a contradiction. Hence $g(G) \neq 3$.

Suppose that $g(G) = 4$. Let $C_4 = \langle v_1 v_2 v_3 v_4 \rangle$ be a cycle in G. If $V = V(C_4)$, then we have two possibilities $G = C_4, G \neq C_4$. If $G \neq C_4$, we have $g(G) = 3$ which is not possible. If $G = C_4$, then $\gamma_{\text{tgn}}(G) = 4(= n)$. If $V \neq V(C_4)$ (i.e. $V(C_4) \subset V$). Notice that G has no pendant vertices. Since $g(G) = 4$, any vertex in $V - V(C_4)$ can be adjacent to exactly two non adjacent vertices of C_4. If all the vertices in $V - V(C_4)$ are adjacent to vertices in C_4, then $\gamma_{\text{tgn}}(G) < n$, a contrary to our assumption. If there is a vertex v_5 in $V - V(C_4)$ not adjacent to C_4, then $V - \{v_5\}$ is a tgnd-*set* of G, a contrary to our assumption. Hence by our assumption $g(G) = 4$ implies $G = C_4$.

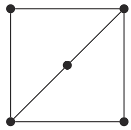
Suppose that $g(G) = 5$. Then we have two possibilities, $G = C_5$, $G \neq C_5$. If $G = C_5$, then $\gamma_{\text{tgn}}(G) = 4$, a contrary to our assumption. If $G \neq C_5$, then $V = V(C_5)$ or $V \neq V(C_5)$. If $V = V(C_5)$, then $g(G) < 5$ a contradiction to our supposition. If $V \neq V(C_5)(i.e. V(C_5) \subset V)$. Since $g(G) = 5$, each vertex in $V - V(C_5)$ is adjacent to at most one vertex in C_5. If all the vertices in $V - V(C_5)$ are adjacent to C_5, then $V(C_5)$ is a tgnd-*set* of G, a contrary to our assumption. Suppose that there is a vertex v_7 in $V - V(C_5)$ adjacent to $C_5(= \langle v_1 v_2 v_3 v_4 v_5 v_1 \rangle)$. Since $\text{diam}(G) = 2$, v_7 is at a distance two from each vertex of C_5. Then $V - \{v_6 v_7\}$ ($\langle v_1 v_6 v_7 \rangle$ is a path) is a tgnd-*set* of G, a contrary to our assumption. So $g(G) \neq 5$. Clearly $\text{diam}(G) \neq 1$. Hence we have $G = C_4$ or $G = P_4$. The inverse implication is clear. $\square$

Theorem 4. *If* $\text{diam}(G) \neq 2, 3$. *Then,* $\gamma_{\text{tgn}}(G) = n - 1$ *if and only if* $G = P_5$ *or* C_3.

Proof. Assume that $\gamma_{\text{tgn}}(G) = n - 1$. Suppose that $\text{diam}(G) \geqslant 5$. Forming a spanning tree G' of G, we get $\gamma_{\text{tgn}}(G) \leqslant \gamma_{\text{tgn}}(G') \leqslant n - m < n - 1$, a contrary to our assumption(here m is the number of pendant vertices in G'). Therefore $\text{diam}(G) \leqslant 4$.

Suppose that $\text{diam}(G) = 4$. If $G = P_5$, then $\gamma_{\text{tgn}}(G) = 4 = 5 - 1$. If $G \neq P_5$, forming a spanning tree G' of G, we have $\gamma_{\text{tgn}}(G) < n - 1$, a contrary to our assumption. Hence $G = P_5$. Suppose that $\text{diam}(G) = 1$. This implies $G = K_n (n \geqslant 3)$. By Theorem 2, $\gamma_{\text{tgn}}(G) = 2 < n - 1$ whenever $n \geqslant 4$, a contrary to our assumption. So $G = C_3$. The inverse implication is clear. $\square$

Theorem 5. *Suppose* $n \geqslant 5$ *and* $\operatorname{diam}(G) = 2$. *Then,* $\gamma_{\text{tgn}}(G) = n - 1$ *if and only if* $G = C_5$ *or* G *is isomorphic to* H *given by*

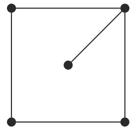


Proof. Assume that $\gamma_{\text{tgn}}(G) = n - 1$. By hypothesis $g(G) \leqslant 5$. Suppose that $g(G) = 5$. If $G = C_5$, then $\gamma_{\text{tgn}}(G) = n - 1 (n = 5)$. If $G \neq C_5$, then $V = V(C_5)$ or $V \neq V(C_5)$. If $V = V(C_5)$, then $g(G) < 5$, a contradiction. If $V \neq V(C_5)$ i.e., $V(C_5) \subset V$. Clearly G has no pendant vertices. By hypothesis any vertex in $V - V(C_5)$ is adjacent to atleast two non adjacent vertices of C_5 or at a distance two from each vertex of C_5. From the former case or later case we get $g(G) = 4$, a contradiction. So $V \neq V(C_5)$. Hence $g(G) = 5$ implies $G = C_5$.

Suppose that $g(G) = 4$. Then $G = C_4$ or $G \neq C_4$. If $G = C_4$, $\gamma_{\text{tgn}}(G) = 4 = n > n - 1$ a contrary to our assumption. If $G \neq C_4$, then we have $V = V(C_4)$ or $V \neq V(C_4)$. If $V = V(C_4)$, then $g(G) = 3$ a contradiction to our supposition. If $V \neq V(C_4)$ i.e., $V(C_4) \subset V$. By hypothesis and our supposition G has no pendant vertices and any vertex v in $V - V(C_4)$ is adjacent to exactly two non adjacent vertices of C_4 or v is at a distance two from each vertex of C_4 or v is at a distance two from a vertex of C_4 and adjacent to a vertex of C_4, non adjacent to the former. Except in the first case we can form a spanning tree G' of G with $\operatorname{diam}(G') \geqslant 5$. So $\gamma_{\text{tgn}}(G) \leqslant \gamma_{\text{tgn}}(G') \leqslant n - m < n - 1$, a contrary to our assumption(here m is the number of pendant vertices in G'). If G has more than one vertex of first kind, then $\gamma_{\text{tgn}}(G) < n - 1$ a contrary to our assumption. If G has exactly one vertex of first kind, then $\gamma_{\text{tgn}}(G) = n - 1$ and G is isomorphic to H.

Suppose that $g(G) = 3$. Clearly G has a cycle C_3 $(= \langle v_1 v_2 v_3 v_1 \rangle)$. If $G \neq C_3$, then $V(C_3) \subset V$. Clearly G cannot have more than one pendant vertex. Suppose G has exactly one pendant vertex, say v. Since $\operatorname{diam}(G) = 2$, there is a vertex w on C_3 such that vw is in G. Without loss of generality assume that $w = v_1$. Clearly $\{v_1, v_2\}$ or $\{v_1, v_3\}$ or $\{v_1, v\}$ is a tgnd-*set* for G a contrary to our assumption. This implies G has no pendant vertices. So any vertex in $V - V(C_3)$ is adjacent to C_3 or at a distance two from atleast one vertex of C_3. In either case $\gamma_{\text{tgn}}(G) < n - 1$. The inverse implication is clear. □

Theorem 6. *Suppose that* $n \geqslant 5$ *and* $\operatorname{diam}(G) = 3$*. Then* $\gamma_{\text{tgn}}(G) = n-1$ *if and only if* G *is isomorphic to* H *given by*



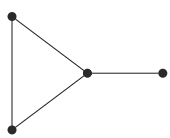
Proof. Assume that $\gamma_{\text{tgn}}(G) = n - 1$. Clearly $g(G)$ is not greater than 6.

Suppose that $g(G) = 5$. By hypothesis $G \neq C_5$. This implies $V(C_5) \subset V$. Clearly G cannot have more than two pendant vertices. Suppose G has exactly one pendant vertex, say v. By hypothesis v is adjacent to a vertex of $C_5(\langle v_1 v_2 v_3 v_4 v_5 v_1 \rangle)$, say v_1. Then clearly $V - \{v_2, v_3\}$ is a tgnd-*set* of G, a contrary to our assumption. Suppose G has exactly two pendant vertices. Since $\operatorname{diam}(G) = 3$ they are adjacent to a vertex on C_5 or adjacent to end vertices of an edge in C_5. In either case $\gamma_{\text{tgn}}(G) = n - 2 < n - 1$, a contrary to our assumption. So G cannot have pendant vertices. Since $\operatorname{diam}(G) = 3$ and $g(G) = 5$, G cannot have more than two cycles. Hence $g(G) \neq 5$.

Suppose $g(G) = 3$. Clearly $G \neq C_3$ (since $n \geqslant 5$). Also G cannot have pendant vertices. If $|V(G)| = 5$ or all the vertices in $V - V(C_3)$ are adjacent to C_3 or there is a vertex at a distance two from C_3, we get a contrary to our assumption. Hence $g(G) \neq 3$.

Suppose $g(G) = 4$. Since $\operatorname{diam}(G) = 3$, $G \neq C_4$.Clearly G cannot have more than two pendant vertices. If $|V(G)| = 5$, since $\operatorname{diam}(G) = 3$ the vertex in $V - V(C_4)$ is a pendant vertex. This implies $\gamma_{\text{tgn}}(G) = 4 = 5-1 = n-1$. Suppose $|V(G)| \geqslant 6$. If all the vertices in $V - V(C_4)$ are adjacent to C_4 (each vertex in $V - V(C_4)$ can be adjacent to exactly two non adjacent vertices in C_4 (since $g(G) = 4$)), then $\gamma_{\text{tgn}}(G) = 4 \leqslant n - 2 < n - 1$ a contrary to our assumption. If not, there is atleast one vertex in $V - V(C_4)$ at a distance two from C_4 (say v). Then $V - \{v, v_5\}$ is a tgnd-*set* of G ($C_4 = \langle v_1 v_2 v_3 v_4 \rangle, v_1 v_5$ is an edge in G) which is again a contradiction. So $|V(G)|$ is not greater than or equal to 6. Hence $G \cong H$. □

Theorem 7. *Suppose* $g(G) = 3$ *and* $\operatorname{diam}(G) = 2$*. Then* $\gamma_{\text{tgn}}(G) = n - 2$ *if and only if* $G = K_4$ *or* $G \cong K_4 - \{e\}$ *or* G *is isomorphic to* H *given by*



Proof. Assume that $\gamma_{\text{tgn}}(G) = n - 2$. Since $g(G) = 3$ there is a cycle $C_3 = \langle v_1 v_2 v_3 \rangle$ in G. Clearly $V(C_3) \subset V$. Suppose there is a vertex v in $V - V(C_3)$ which is not adjacent to C_3. Since $\operatorname{diam}(G) = 2$ there are paths of length 2 from v to each vertex of C_3, say $\langle v v_4 v_1 \rangle, \langle v v_5 v_2 \rangle, \langle v v_6 v_3 \rangle$.

Case 1: $v_4 \neq v_5 \neq v_6$. Then $V - \{v, v_4, v_5\}$ is a tgnd-*set* of G.

Case 2: two of them are equal. Without loss of generality assume that $v_4 = v_5$. Clearly G cannot have pendant vertices. Then $V - \{v, v_4, v_1\}$ is a tgnd-*set* of G.

Case 3: $v_4 = v_5 = v_6$. Clearly G cannot have pendant vertices. Then $V - \{v_1, v_2, v_3\}$ is a tgnd-*set* of G.

In each of the three cases, we get a contradiction with our assumption. So our supposition is false. Hence all the vertices in $V - V(C_3)$ are adjacent to C_3. Clearly C_3 has exactly one neighbour in $V - V(C_3)$, say v. If v is adjacent to exactly one vertex of C_3, then $\gamma_{\text{tgn}}(G) = 2 = 4 - 2$ and $G \cong H$. If v is adjacent to exactly two vertices of C_3, then $\gamma_{\text{tgn}}(G) = 2 = 4 - 2$ and $G \cong K_4 - \{e\}$. If v is adjacent to all vertices of C_3, then $\gamma_{\text{tgn}}(G) = 2 = 4 - 2$ and $G = K_4$.

The inverse implication is clear. □

Theorem 8. *If* $\delta(G) \geqslant 3$ *and* $g(G) > 4$*, then*

$$2e - n(n-3) \leqslant \gamma_{\text{tgn}}(G) \leqslant n - \Delta(G) + 1.$$

Proof. Suppose that D is a γ_{tgn}-set of G. Since $g(G) > 4$, for each vertex in V there is a vertex in D which is non adjacent to the former. This implies $e \leqslant n_{C_2} - [n - \gamma_{\text{tgn}}] - \frac{\gamma_{\text{tgn}}}{2}$. Hence $2e - n(n-3) \leqslant \gamma_{\text{tgn}}(G)$.

Suppose $d_G(v) = \Delta(G)$ for some v in V. Let $N_G(v) = \{v_1, v_2, \ldots, v_{\Delta(G)}\}$. Now consider the set $D = [V - N_G(v)] \bigcup \{v_i :$ i is exactly one of $1, 2, \ldots, \Delta(G)\}$. Without loss of generality assume that $D = [V - N_G(v)] \bigcup \{v_{\Delta(G)}\}$. Let $u_1 \in V$.

Case 1: $u_1 \in V - D$. This implies $u_1 \in \{v_1, v_2, \ldots, v_{\Delta(G)-1}\}$. Without loss of generality assume that $u_1 = v_1$. Clearly $u_1 v$ is in G.

Case 2: $u_1 \in D$. This implies $u_1 \notin \{v_1, v_2, \ldots, v_{\Delta(G)-1}\}$. If $u_1 = v$ or $u_1 = v_{\Delta(G)}$, then $u_1 v_{\Delta(G)}$ or $u_1 v$ is in G. If not since $\delta(G) \geqslant 3$ and $g(G) > 4$ there is $u_2 \notin \{v, v_1, v_2, \ldots, v_{\Delta(G)}\}$ such that $u_1 u_2$ is in G. Hence D is a total dominating set of G.

We now show that D is a total dominating set of G^N. Let $u_1 \in V$.

Case 1: $u_1 \in V - D$. This implies $u_1 \in \{v_1, v_2, \ldots, v_{\Delta(G)-1}\}$. Since $d_G(v_i, v_{\Delta(G)}) = 2$, $i = 1, 2, \ldots, \Delta(G) - 1$ we have $v_i v_{\Delta(G)}$ is in G^N. So $v_1, v_{\Delta(G)}$ is in G^N.

Case 2: $u_1 \in D$. This implies $u_1 \notin \{v_1, v_2, \ldots, v_{\Delta(G)-1}\}$. Suppose $u_1 = v$. Since $\delta(G) \geqslant 3$ and $g(G) > 4$ we have vu_2 is in G^N. for some $u_2 \in N(N(v))$ and $u_2 \in D$. If $u_1 = v_{\Delta(G)}$. Suppose $u_1 \notin \{v, v_1, v_2, \ldots, v_{\Delta(G)}\}$.If $u_1 \in N(v_i)$ for some $i = 1, 2, \ldots, \Delta(G)$, then $u_1 v$ is an edge in G^N. If $u_1 \notin N(v_i)$ for any i.

Subcase a: $u_1 \in N(N(v_i))$ for some $i = 1, 2, \ldots, \Delta(G)$. Without loss of generality assume that $u_1 \in N(N(v_i))$. Since $\delta(G) \geqslant 3$, there are u_2 and u_3 in G, adjacent to u_1. Since $g(G) > 4$, u_2 and u_3 cannot be adjacent to $\{v_1, v_2, \ldots, v_{\Delta(G)}\}$. This implies there is u_4 in D such that u_2u_4 or u_3u_4 is in G. Hence u_1u_4 is in G^N.

Subcase b: $u_1 \in V - [\{N(N(v_i)) : i{=}1, 2, \ldots, \Delta(G)\} \bigcup \{v_1, v_2, \ldots, v_{\Delta(G)}\}]$. By hypothesis there is a u_2 in $D - \{v, v_{\Delta(G)}\}$ such that u_1u_2 is in G^N. D is a total dominating set of G^N.

Hence D is a tgnd-*set* of G whose cardinality is $n - \Delta(G) + 1$. So $\gamma_{\rm tgn}(G) \leqslant n - \Delta(G) + 1$. This completes the proof. □

Notation. For $n \geqslant 4$ and $k = 2, 3$ define a family of graphs $\mathcal{G}_k$ as follows. $G \in \mathcal{G}_k$ if and only if there is $D \subset V$ such that $|D| = k$ satisfying:

(i) $\langle D \rangle$ is connected;
(ii) at least two vertices of D lie on the same C_3;
(iii) each vertex in $V - D$ is adjacent to a vertex in D.

Theorem 9. *For $n \geqslant 4$, $\gamma_{\rm tgn}(G) = 3$ if and only if $G \in \mathcal{G}_3 - \mathcal{G}_2$.*

Proof. Assume that $\gamma_{\rm tgn}(G) = 3$. Then there is a $\gamma_{\rm tgn}$-set of G such that $|D| = 3$ and $\langle D \rangle$ is connected. By the characterization result for tgnd-set there is a path of length 2 between a pair of adjacent vertices in D. This implies at least two vertices of D lie on the same C_3. So $G \in \mathcal{G}_3$. Since D is a $\gamma_{\rm tgn}$-set, $G \in \mathcal{G}_2$. Hence $G \in \mathcal{G}_3 - \mathcal{G}_2$. The inverse implication is clear. □

Before considering the next result, for convenience we introduce the following. For $n \geqslant 6$, define a family of trees $\mathcal{T}_k$ as $T \in \mathcal{T}_k$ if and only if there is a $D \subset V$ with $|D| = k$ satisfying:

(i) $\langle D \rangle$ is connected in G;
(ii) each vertex in $V - D$ is adjacent to a vertex in D (in G).

Theorem 10. $\gamma_{\rm tgn}(T) = 4$ *if and only if* $T \in \mathcal{T}_4 - \mathcal{T}_3$.

Proof. Suppose $\gamma_{\rm tgn}(T) = 4$. Then there is a $\gamma_{\rm tgn} - set$ of T (say D) such that D satisfies (i) and (ii) of the above mentioned family. This implies

$T \in \mathcal{T}_4$. Clearly by characterization theorem $T \notin \mathcal{T}_3$. Hence $T \in \mathcal{T}_4 - \mathcal{T}_3$. The inverse implication is clear. □

Theorem 11. *$\gamma_{\mathrm{tgn}}(T) = 5$ if and only if $T \in \mathcal{T}_5 - \mathcal{T}_4$.*

Theorem 12. *If G is a graph satisfying the following two conditions:*

(i) *each edge of G lies on C_3 or C_5;*

(ii) *there is no path of length four between any pair non adjacent vertices in G,then*

$$\frac{\gamma_t(G) + \gamma_t(G^N)}{2} \leqslant \gamma_{\mathrm{tgn}}(G) \leqslant \gamma_t(G) + \gamma_t(G^N)$$

Proof. By the hypothesis, we have $G = G^{NN}$. Clearly $\gamma_t(G) \leqslant \gamma_{\mathrm{tgn}}(G)$, $\gamma_t(G^N) \leqslant \gamma_{\mathrm{tgn}}(G^N) = \gamma_{\mathrm{tgn}}(G)$. Hence $\frac{\gamma_t(G)+\gamma_t(G^N)}{2} \leqslant \gamma_{\mathrm{tgn}}(G)$. Clearly $\gamma_{\mathrm{tgn}}(G) \leqslant \gamma_t(G) + \gamma_t(G^N)$. Thus the result follows. □

Theorem 13. *Assume that D is a γ_t-set of G. If there is a v in $V - D$ adjacent to all the vertices in D, then $\gamma_{\mathrm{tgn}}(G) \leqslant 1 + \gamma_t(G)$.*

Proof. Clearly $D \bigcup \{v\}$ is a tgnd-*set* of G. Hence, the theorem follows. □

Theorem 14. *If G is a semi complete graph, then $D \subseteq V$ is a total dominating set of G if and only if D is a* tgnd-*set of G.*

Proof. The proof follows from the fact that each edge in a semi complete graph lies on C_3. □

Theorem 15. *If G is a semi complete graph, then a set $D \subseteq V$ with $\delta_G(\langle D \rangle) \geqslant 1$ is a global dominating set of G if and only if D is a* tgnd-*set of G.*

Proof. The proof follows from the fact that, for a semi complete graph G, we have $G^c = G^N$. □

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