

## Groups whose non-normal subgroups have small commutator subgroup

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**ABSTRACT.** The structure of groups whose non-normal subgroups have a finite commutator subgroup is investigated. In particular, it is proved that if  $k$  is a positive integer and  $G$  is a locally graded group in which every non-normal subgroup has finite commutator subgroup of order at most  $k$ , then the commutator subgroup of  $G$  is finite. Moreover, groups with finitely many normalizers of subgroups with large commutator subgroup are studied.

### Introduction

It is well known that a group has only normal subgroups (i.e. is a *Dedekind group*) if and only if it is either abelian or the direct product of a quaternion group of order 8 and a periodic abelian group with no elements of order 4. The structure of groups for which the set of non-normal subgroups is small in some sense has been studied by many authors in several different situations. A group  $G$  is called *metahamiltonian* if every non-normal subgroup of  $G$  is abelian. In a series of relevant papers, G.M. Romalis and N.F. Sesekin investigated the behaviour of (generalized) soluble metahamiltonian groups and proved in particular that such groups have finite commutator subgroup (see [14],[15],[16]). However, since every group with commutator subgroup of prime order is obviously metahamiltonian, there is no bound for the order of the commutator subgroup of soluble metahamiltonian groups.

The aim of this paper is to study groups in which every subgroup either is normal or has finite commutator subgroup. The first step in this

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direction is of course the case of groups whose proper subgroups have a finite commutator subgroup; such groups were considered by V.V. Belyaev and N.F. Sesekin (see [1],[2]) and their results were later improved by B. Bruno and R.E. Phillips [3]. In order to avoid Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) and other similar pathologies, we will restrict our attention to locally graded groups. Recall here that a group  $G$  is said to be *locally graded* if every finitely generated non-trivial subgroup of  $G$  contains a proper subgroup of finite index; of course, all locally (soluble-by-finite) groups are locally graded. We will prove that if  $k$  is a positive integer and  $G$  is a locally graded group in which every non-normal subgroup has finite commutator of order at most  $k$ , then the commutator subgroup of  $G$  is finite. A similar result holds if the restriction on commutator subgroups is imposed only to infinite subgroups, provided that  $G$  is not a Černikov group. Note that the case  $k = 1$  of our theorem is precisely the above quoted result by Romalis and Sesekin. If  $G$  is any group whose commutator subgroup is of type  $p^\infty$  for some prime number  $p$ , all non-normal subgroups of  $G$  have a finite commutator subgroup; thus the bound assumption in the previous statement cannot be removed. Finally, in the last section we extend our results to groups with finitely many normalizers of subgroups with large commutator subgroup.

Most of our notation is standard and can be found in [13].

## 1. Groups with small non-normal subgroups

In the investigation concerning the structure of groups whose non-normal subgroups are small in some sense, one has to consider the case of groups in which every subgroup of finite index is normal. Of course, in this situation information can be obtained only on the factor group with respect to the finite residual. Recall that the *finite residual* of a group  $G$  is the intersection of all (normal) subgroups of finite index of  $G$ , and  $G$  is *residually finite* if its finite residual is trivial; residually finite groups are obviously locally graded.

It is easy to show that every periodic residually finite group whose finite homomorphic images are Dedekind groups is likewise a Dedekind group. The situation is different in the case of non-periodic groups; in fact, the direct product  $Q_8 \times \mathbb{Q}_2$  (where  $Q_8$  is the quaternion group of order 8 and  $\mathbb{Q}_2$  is the additive group of rational numbers whose denominators are powers of 2) is a residually finite non-abelian group and all its subgroups of finite index are normal. Our first lemma describes residually finite groups whose finite homomorphic images are Dedekind groups.

**Lemma 1.** *Let  $G$  be a non-abelian residually finite group. Then all subgroups of finite index of  $G$  are normal if and only if  $Z(G) = G' \times A$ , where  $|G'| = 2$ ,  $|G/Z(G)| = 4$ ,  $A^2 = A^4$  and  $G/A^2$  is a Dedekind group.*

*Proof.* Suppose first that every subgroup of finite index of  $G$  is normal, and let  $\mathcal{L}$  be the set of all subgroups of finite index of  $G$ . If  $H$  is any element of  $\mathcal{L}$ , the factor group  $G/H$  is a Dedekind group and so  $G'H/H$  has order at most 2. As

$$\bigcap_{H \in \mathcal{L}} H = \{1\},$$

it follows that  $G'$  has exponent 2. Let  $X$  be any finite subgroup of  $G'$ , and let  $H$  be a subgroup of finite index of  $G$  such that  $X \cap H = \{1\}$ ; then

$$X \simeq XH/H \leq (G/H)'$$

and hence  $|X| \leq 2$ . Therefore  $G'$  has order 2. By hypothesis, the group  $G$  contains a normal subgroup  $A$  such that  $G/A$  is a quaternion group of order 8. Clearly,  $A \cap G' = \{1\}$ , so that  $A \leq Z(G)$  and  $Z(G) = G' \times A$ ; in particular  $|G/Z(G)| = 4$ . The factor group  $G/A^4$  is periodic and residually finite, so that it is a Dedekind group; as  $A \cap G' = \{1\}$ , it follows that  $A/A^4$  has exponent 2. Therefore  $A^2 = A^4$  and  $G/A^2$  is a Dedekind group.

Conversely, suppose that  $G$  has the structure described in the statement, and let  $X$  be any subgroup of finite index of  $G$ . Since  $A^2 = A^4$ , the subgroup  $X \cap A^2$  has odd index in  $A^2$ ; moreover,  $G/A^2$  is a Dedekind 2-group, so that  $G/X \cap A^2$  is likewise a Dedekind group, and hence the subgroup  $X$  is normal in  $G$ .  $\square$

The consideration of the direct product  $Q_8 \times Q_8$  proves that the above lemma cannot be extended to the case of groups having a residual system whose factors are finite Dedekind groups.

Our next result deals with locally graded groups in which every non-normal subgroup has locally finite commutator subgroup.

**Lemma 2.** *Let  $G$  be a locally graded group whose non-normal subgroups have a locally finite commutator subgroup. Then the commutator subgroup  $G'$  of  $G$  is locally finite.*

*Proof.* Let  $T$  be the largest normal subgroup of  $G$  which is locally finite, and consider any subgroup  $X$  of  $G$  such that  $X'$  is not locally finite. Then  $X$  is normal in  $G$  and  $G/X$  is a Dedekind group, so that in particular  $G'' \leq X$ . It follows that every proper subgroup of  $G''$  has locally finite commutator subgroup. If  $G''$  is a finitely generated non-trivial group, it

contains a proper normal subgroup  $K$  of finite index; then  $K'$  is a locally finite subnormal subgroup of  $G$ , so that  $K'$  is contained in  $T$ . On the other hand, if  $G''$  is not finitely generated, its commutator subgroup  $G^{(3)}$  is locally finite and hence  $G^{(3)} \leq T$ . Therefore  $G/T$  is a soluble-by-finite group in any case, and replacing  $G$  by  $G/T$  it can be assumed without loss of generality that  $G$  is a soluble-by-finite group.

As the hypotheses are inherited by subgroups, in order to prove the statement we may also suppose that  $G$  is finitely generated and by a further reduction that it contains no locally finite non-trivial normal subgroups. Assume for a contradiction that  $G$  is not torsion-free. Then  $G$  is not nilpotent, so that it has a finite non-nilpotent homomorphic image (see [13] Part 2, Theorem 10.51) and in particular  $G$  contains a non-normal subgroup  $H$  such that the index  $|G : H|$  is finite. As the subgroup  $H'$  is locally finite, also the core  $A = H_G$  of  $H$  in  $G$  has locally finite commutator subgroup. Thus  $A' = \{1\}$ , so that  $A$  is an abelian normal subgroup of  $G$  such that  $G/A$  is finite, and in particular  $A$  is torsion-free and finitely generated. For each positive integer  $n$ , the subgroup  $A^n$  has finite index in  $G$  and so its centralizer  $C_G(A^n)$  is central-by-finite; it follows from Schur's theorem that the commutator subgroup  $C_G(A^n)'$  is finite, and hence  $C_G(A^n)$  is torsion-free abelian. Let  $x \neq 1$  be an element of finite order of  $G$ ; then the subgroup  $\langle x, A^n \rangle$  is not abelian and  $\langle x, A^n \rangle'$  is torsion-free, so that  $\langle x, A^n \rangle$  must be normal in  $G$ . Therefore

$$\langle x \rangle = \bigcap_{n>0} \langle x, A^n \rangle$$

is likewise a normal subgroup of  $G$  and hence  $[A, x] = \{1\}$ . This contradiction shows that  $G$  is torsion-free, so that all its non-normal subgroups are abelian and hence  $G$  itself is abelian. The lemma is proved.  $\square$

The following lemma shows that we actually work within the universe of finite-by-soluble groups.

**Lemma 3.** *Let  $G$  be a locally graded group whose non-normal subgroups have a finite commutator subgroup. Then either  $G$  is soluble or the subgroup  $G^{(3)}$  is finite.*

*Proof.* Assume that  $G$  is not soluble, and let  $\mathcal{L}$  be the set of all subgroups of  $G$  with infinite commutator subgroup. If  $X$  is any element of  $\mathcal{L}$ , then  $X$  is normal in  $G$  and  $G/X$  is a Dedekind group; it follows in particular that

$$M = \bigcap_{X \in \mathcal{L}} X$$

is a normal subgroup of  $G$  and  $G''$  is contained in  $M$ . Moreover, every proper subgroup of  $M$  has finite commutator subgroup and hence either  $M'$  is finite or  $M$  is a Černikov group (see [3], Theorem 2). In order to prove that  $G^{(3)}$  is finite, we may obviously suppose that  $M$  is a Černikov group. Let  $J$  be the largest divisible subgroup of  $M$ , and let  $x$  be any element of  $M$ . As the subgroup  $\langle x, J \rangle$  is soluble, it is properly contained in  $M$  and hence  $\langle x, J \rangle'$  is finite. Thus  $x$  has finitely many conjugates in  $\langle x, J \rangle$  and so also in  $M$ ; it follows from Dietzmann's lemma that  $M = EJ$ , where  $E$  is a finite normal subgroup of  $M$ . Therefore  $M'$  is finite and the lemma is proved.  $\square$

For our purposes, we also need the following lemma, which is a special case of a result concerning groups whose non-normal subgroups have finite conjugacy classes (see [9], Proposition 4.1).

**Lemma 4.** *Let  $G$  be an abelian-by-finite group whose non-normal subgroups have a finite commutator subgroup. Then the commutator subgroup  $G'$  of  $G$  is a Černikov group.*

**Corollary 1.** *Let  $G$  be a residually finite group whose non-normal subgroups have a finite commutator subgroup. Then the commutator subgroup  $G'$  of  $G$  is finite.*

*Proof.* If every subgroup of finite index of  $G$  is normal, we have that all finite homomorphic images of  $G$  are Dedekind groups and it follows from Lemma 1 that  $G'$  has order at most 2. Suppose now that  $G$  contains a non-normal subgroup  $X$  of finite index; then  $X'$  is finite and hence also the core  $X_G$  of  $X$  has finite commutator subgroup. As the factor group  $G/X_G$  is abelian-by-finite, its commutator subgroup  $G'/X_G$  is a Černikov group by Lemma 4. Therefore  $G'$  is likewise a Černikov group, and hence it is finite.  $\square$

**Lemma 5.** *Let  $G$  be a locally graded group whose non-normal subgroups have a finite commutator subgroup. Then either  $G$  is locally nilpotent or every finitely generated subgroup of  $G$  has finite commutator subgroup.*

*Proof.* Assume that  $G$  contains a finitely generated non-nilpotent subgroup  $H$ , and let  $E$  be any finitely generated subgroup of  $G$ . As  $G$  is soluble-by-finite by Lemma 3, the subgroup  $K = \langle E, H \rangle$  has a finite non-nilpotent homomorphic image (see [13] Part 2, Theorem 10.51) and in particular  $K$  contains a non-normal subgroup  $X$  such that the index  $|K : X|$  is finite. By hypothesis the commutator subgroup  $X'$  of  $X$  is finite, and hence  $K$  is finite-by-abelian-by-finite. As finitely generated

abelian-by-finite groups are residually finite, it follows from Lemma 1 that  $K'$  is finite, so that  $E'$  is likewise finite and the lemma is proved.  $\square$

**Corollary 2.** *Let  $G$  be a locally graded group whose non-normal subgroups have a finite commutator subgroup. Then  $G$  locally satisfies the maximal condition on subgroups.*

We can now prove the main result of this section.

**Theorem 1.** *Let  $k$  be a positive integer, and let  $G$  be a locally graded group whose infinite non-normal subgroups have a finite commutator subgroup of order at most  $k$ . Then either  $G$  is a Černikov group or its commutator subgroup  $G'$  is finite.*

*Proof.* Clearly, every non-normal subgroup of  $G$  has finite commutator subgroup, so that in particular  $G'$  is locally finite by Lemma 2; moreover,  $G$  locally satisfies the maximal condition on subgroups by Corollary 2 and hence every finitely generated subgroup of  $G$  has finite commutator subgroup. It follows from Lemma 3 that  $G$  is finite-by-soluble, so that without loss of generality we may suppose that  $G$  is a soluble group. Assume for a contradiction that  $G$  is neither finite-by-abelian nor a Černikov group, so that in particular  $G$  is not finitely generated. Let  $E$  be any infinite finitely generated subgroup of  $G$ , and suppose that  $|E'| > k$ ; then  $E$  is normal in  $G$  and  $G/E$  is a Dedekind group, so that  $|G'E/E| \leq 2$ . Moreover,  $E$  satisfies the maximal condition on subgroups, so that  $E \cap G'$  is finite, and hence  $G'$  is likewise finite. This contradiction shows that  $|E'| \leq k$  for each infinite finitely generated subgroup  $E$  of  $G$ . On the other hand, as  $G'$  is infinite, there exists a finite subgroup  $X$  of  $G$  such that  $|X'| > k$ , and every finitely generated subgroup of  $G$  containing  $X$  must be finite. Therefore the group  $G$  is locally finite.

Assume that the normal closure  $N = X^G$  of  $X$  is infinite, and let

$$H_1 \geq H_2 \geq \dots \geq H_n \geq H_{n+1} \geq \dots$$

be any sequence of infinite normal subgroups of  $N$ . Since every infinite subgroup containing  $X$  is normal in  $G$ , we have that all proper subgroups of  $N$  containing  $X$  are finite, and so  $N = H_n X$  for each  $n$ ; if  $m$  is a positive integer such that  $H_n \cap X = H_m \cap X$  for all  $n \geq m$ , it follows that  $H_n = H_m$  for all  $n \geq m$ . Therefore the group  $N$  satisfies the minimal condition on normal subgroups. Let  $i$  be the largest positive integer such that the  $i$ -th term  $N^{(i)}$  of the derived series of  $N$  is infinite. Then  $N^{(i+1)}$  is finite and  $N = XN^{(i)}$ , so that the index  $|N : N^{(i)}|$  is finite and hence  $N$  is finite-by-abelian-by-finite. As the minimal condition on

normal subgroups is inherited by subgroups of finite index (see [13] Part 1, Theorem 5.21), it follows that  $N$  is a Černikov group. Then  $G/C_G(N)$  is likewise a Černikov group (see [13] Part 1, Theorem 3.29), so that  $C_G(N)$  is not a Černikov group and hence there exists an abelian subgroup  $A$  of  $C_G(N)$  which does not satisfy the minimal condition on subgroups (see [13] Part 1, Theorem 3.32). In particular,  $A$  contains a subgroup of the form  $B = B_1 \times B_2$ , where both  $B_1$  and  $B_2$  are infinite and  $B \cap X = \{1\}$ . Clearly, the subgroups  $B_1X$  and  $B_2X$  are normal in  $G$ , so that also  $X = B_1X \cap B_2X$  is normal in  $G$  and this contradiction shows that  $X^G$  is finite. On the other hand, every infinite subgroup of  $G/X^G$  is normal and hence  $G/X^G$  is either a Černikov group or a Dedekind group (see [4]). Therefore either  $G$  is a Černikov group or its commutator subgroup  $G'$  is finite, and this last contradiction completes the proof of the theorem.  $\square$

The description of locally graded group whose non-normal subgroups have boundedly finite commutator subgroups is an easy consequence of the above result.

**Theorem 2.** *Let  $k$  be a positive integer, and let  $G$  be a locally graded group whose non-normal subgroups have a finite commutator subgroup of order at most  $k$ . Then the commutator subgroup  $G'$  of  $G$  is finite.*

*Proof.* Assume for a contradiction that  $G'$  is infinite. It follows from Theorem 1 that  $G$  is a Černikov group; in particular,  $G$  is locally finite and hence there exists a finite subgroup  $E$  of  $G$  such that  $|E'| > k$ . On the other hand, by hypothesis  $E$  is normal in  $G$  and  $G/E$  is a Dedekind group, so that  $G'$  is finite, a contradiction.  $\square$

## 2. Groups with few normalizers of large subgroups

In a famous paper of 1955, B.H. Neumann [11] proved that each subgroup of a group  $G$  has finitely many conjugates if and only if the centre  $Z(G)$  has finite index, and hence central-by-finite groups are precisely those groups in which the normalizers of subgroups have finite index. This result suggests that the behaviour of normalizers has a strong influence on the structure of the group. In fact, it follows easily from a result of Y.D. Polovickiĭ [12] that a group has finitely many normalizers of abelian subgroups if and only if it is central-by-finite. More recently, groups with finitely many normalizers of  $\chi$ -subgroups have been studied for various choices of the property  $\chi$  (see for instance [5],[6],[7],[8]). The aim of this section is to study groups with finitely many normalizers of subgroups with infinite commutator subgroup; the first step of this investigation is

of course the case of groups whose non-normal subgroups have a finite commutator subgroup.

It follows in particular from Lemma 1 that every residually finite group whose subgroups of finite index are normal is central-by-finite; this latter property actually characterizes residually finite groups with finitely many normalizers of subgroups of finite index. In fact, we have:

**Lemma 6.** *Let  $G$  be a residually finite group with finitely many normalizers of subgroups of finite index. Then the factor group  $G/Z(G)$  is finite.*

*Proof.* Let  $N_G(X_1), \dots, N_G(X_m)$  be the proper normalizers of subgroups of finite index of  $G$ . The intersection

$$N = \bigcap_{i=1}^m N_G(X_i)$$

normalizes each subgroup of finite index of  $G$  and the index  $|G : N| = r$  is obviously finite. If  $\bar{G}$  is any finite homomorphic image of  $G$ , it follows that each subgroup of  $\bar{G}$  has at most  $r$  conjugates and so the order of  $\bar{G}'$  is bounded by a positive integer  $s = s(r)$  depending only on  $r$  (see [10] and [13] Part 1, p.102). Let  $\mathcal{L}$  be the set of all normal subgroups of finite index of  $G$ . Then

$$(G')^{s!} \leq \bigcap_{H \in \mathcal{L}} H = \{1\}$$

and in particular  $G'$  is periodic. Moreover, all subgroups of finite index of the residually finite group  $N$  are normal, and hence  $N/Z(N)$  is finite by Lemma 1; thus  $G$  is abelian-by-finite and  $G'$  is locally finite. Assume for a contradiction that  $G'$  is infinite, so that it contains a finite subgroup  $E$  with  $|E| > r$ . As  $G$  is residually finite, there exists a normal subgroup  $H$  of  $G$  such that  $G/H$  is finite and  $H \cap E = \{1\}$ ; then the commutator subgroup of  $G/H$  contains more than  $r$  elements, and this contradiction shows that  $G'$  must be finite. Finally, it is clear that any residually finite group with finite commutator subgroup is central-by-finite.  $\square$

Next lemma is an extension of Corollary 1.

**Lemma 7.** *Let  $G$  be a residually finite group with finitely many normalizers of subgroups with infinite commutator subgroup. Then the commutator subgroup  $G'$  of  $G$  is finite.*



*Proof.* Assume that the statement is false, and choose a counterexample  $G$  such that the set

$$\{N_G(X_1), \dots, N_G(X_m)\},$$

consisting of all proper normalizers of subgroups with infinite commutator subgroup, has smallest order  $m$ . Then  $m > 0$  by Corollary 1, and for each  $i = 1, \dots, m$  the group  $N_G(X_i)$  has less than  $m$  proper normalizers of subgroups with infinite commutator subgroup. It follows that  $N_G(X_i)$  has finite commutator subgroup, and this contradiction proves the statement.  $\square$

It is quite easy to prove the following result, which improves Theorem 2.

**Theorem 3.** *Let  $k$  be a positive integer, and let  $G$  be a locally graded group with finitely many normalizers of subgroups whose commutator subgroup has more than  $k$  elements. Then the commutator subgroup  $G'$  of  $G$  is finite.*

*Proof.* Assume that the statement is false, and choose a counterexample  $G$  such that the set

$$\{N_G(X_1), \dots, N_G(X_m)\},$$

consisting of all proper normalizers of subgroups of  $G$  whose commutator subgroup contains more than  $k$  elements, has smallest order  $m$ . Of course,  $m > 0$  by Theorem 2, and for every  $i = 1, \dots, m$  the group  $N_G(X_i)$  has fewer than  $m$  proper normalizers of subgroups whose commutator subgroup contains more than  $k$  elements, so that  $N_G(X_i)'$  is finite. Since the subgroup  $N_G(X_i)$  has obviously finitely many conjugates in  $G$ , also the conjugacy class of  $N_G(X_i)'$  is finite, and hence the normal subgroup

$$K = \langle N_G(X_1)', \dots, N_G(X_m)' \rangle^G$$

is finite by Dietzmann's lemma; in particular, the factor group  $G/K$  is likewise locally graded. Moreover, every subgroup of  $G/K$  whose commutator subgroup has more than  $k$  elements must be normal and so  $G/K$  has finite commutator subgroup by Theorem 2. Thus also  $G'$  is finite, and this contradiction proves the theorem.  $\square$

In order to generalize Theorem 1 to the case of groups with finitely many normalizers of infinite subgroups with unbounded commutator subgroups, we need some more lemmas.

**Lemma 8.** *Let  $G$  be a group having finitely many normalizers of subgroups with infinite commutator subgroup. Then  $G$  contains a characteristic subgroup  $M$  of finite index such that  $N_M(X)$  is normal in  $M$  for each subgroup  $X$  with infinite commutator subgroup.*

*Proof.* If  $X$  is any subgroup of  $G$  with infinite commutator subgroup, the normalizer  $N_G(X)$  has obviously finitely many images under automorphisms of  $G$ ; in particular, the index  $|G : N_G(N_G(X))|$  is finite. It follows that also the characteristic subgroup

$$M(X) = \bigcap_{\alpha \in \text{Aut } G} N_G(N_G(X))^\alpha$$

has finite index in  $G$ . Let  $\mathcal{H}$  be the set of all subgroups of  $G$  with infinite commutator subgroup. If  $X$  and  $Y$  are elements of  $\mathcal{H}$  such that  $N_G(X) = N_G(Y)$ , then  $M(X) = M(Y)$ , and hence also

$$M = \bigcap_{X \in \mathcal{H}} M(X)$$

is a characteristic subgroup of finite index of  $G$ . Let  $X$  be any subgroup of  $M$  such that  $X'$  is infinite. Then

$$M \leq M(X) \leq N_G(N_G(X)),$$

and so the normalizer  $N_M(X) = N_G(X) \cap M$  is a normal subgroup of  $M$ .  $\square$

**Lemma 9.** *Let  $G$  be a soluble-by-finite group having finitely many normalizers of subgroups with infinite commutator subgroup. Then  $G$  locally satisfies the maximal condition on subgroups.*

*Proof.* As the hypotheses are inherited by subgroups, it can be assumed without loss of generality that the group  $G$  is finitely generated. It follows from Lemma 8 that  $G$  contains a normal subgroup  $M$  such that  $G/M$  is finite and  $N_M(X)$  is normal in  $M$  for each subgroup  $X$  of  $M$  with infinite commutator subgroup. Suppose first that every subgroup of finite index of  $M$  is subnormal. Then all finite homomorphic images of  $M$  are nilpotent, so that  $G$  is nilpotent-by-finite (see [13] Part 2, Theorem 10.51) and hence it satisfies the maximal condition on subgroups. On the other hand, if  $M$  contains a non-subnormal subgroup  $X$  of finite index, the commutator subgroup  $X'$  must be finite; then  $G$  is finite-by-abelian-by-finite and so it satisfies the maximal condition on subgroups also in this case.  $\square$

**Lemma 10.** *Let  $G$  be a group having finitely many normalizers of subgroups with infinite commutator subgroup. If  $G$  contains a soluble-by-finite normal subgroup  $N$  such that  $N'$  is infinite, then  $G$  is likewise soluble-by-finite.*

*Proof.* As  $N'$  is infinite, the factor group  $G/N$  has only finitely many normalizers of subgroups, and hence it is central-by-finite by Polovickii's theorem. Therefore  $G$  is soluble-by-finite.  $\square$

**Theorem 4.** *Let  $k$  be a positive integer, and let  $G$  be a locally graded group with finitely many normalizers of infinite subgroups whose commutator subgroup has more than  $k$  elements. Then either  $G$  is a Černikov group or its commutator subgroup  $G'$  is finite.*

*Proof.* Assume that the statement is false, and choose a counterexample  $G$  such that the set

$$\{N_G(X_1), \dots, N_G(X_m)\},$$

consisting of all proper normalizers of infinite subgroups of  $G$  whose commutator subgroup contains more than  $k$  elements, has smallest order  $m$ . Of course,  $m > 0$  by Theorem 1, and for every  $i = 1, \dots, m$  the group  $N_G(X_i)$  has less than  $m$  proper normalizers of infinite subgroups whose commutator subgroup has more than  $k$  elements, so that either  $N_G(X_i)'$  is finite or  $N_G(X_i)$  is a Černikov group. Suppose that  $N_G(X_i)'$  is finite for some  $i$ . The factor group  $G/(N_G(X_i)')^G$  has fewer than  $m$  proper normalizers of infinite subgroups whose commutator subgroup contains more than  $k$  elements, and hence  $G/(N_G(X_i)')^G$  is either finite-by-abelian or a Černikov group. On the other hand, the subgroup  $N_G(X_i)'$  has finitely many conjugates in  $G$  and so its normal closure  $(N_G(X_i)')^G$  is finite by Dietzmann's lemma; this is a contradiction, since  $G$  is a counterexample to the statement. Therefore  $N_G(X_i)$  is a Černikov group for each  $i = 1, \dots, m$  and in particular all non-normal subgroups of  $G$  have locally finite commutator subgroup, so that  $G'$  is locally finite by Lemma 2. Moreover, the index  $|G : N_G(X_i)|$  must be infinite for every  $i$ ; it follows that

$$N_G(N_G(X_i)) \neq N_G(X_j)$$

for all  $i$  and  $j$ , and so every  $N_G(X_i)$  is a normal subgroup of  $G$ . As  $G/N_G(X_i)$  has less than  $m$  proper normalizers of infinite subgroups whose commutator subgroup contains more than  $k$  elements, the group  $G$  is soluble-by-finite by Lemma 10, and hence it locally satisfies the maximal condition on subgroups by Lemma 9.

The factor group  $G/C_G(N_G(X_i))$  is isomorphic to a group of automorphisms of the Černikov group  $N_G(X_i)$  and obviously  $C_G(N_G(X_i)) \leq N_G(X_i)$ , so that  $G/C_G(N_G(X_i))$  cannot be periodic (see [13] Part 1, Theorem 3.29) and in particular

$$N_G(X_1) \cup \dots \cup N_G(X_m)$$

is a proper subset of  $G$ . Let  $x$  be an element of the set

$$G \setminus \bigcup_{i=1}^m N_G(X_i),$$

so that the commutator subgroup of every infinite non-normal subgroup of  $G$  containing  $x$  has order at most  $k$ . As  $G'$  is infinite, there exists a finitely generated subgroup  $E$  of  $G$  such that  $|E'| > k$ , and so  $\langle x, E \rangle$  is an infinite group whose commutator subgroup contains more than  $k$  elements. Therefore  $\langle x, E \rangle$  is a normal subgroup of  $G$  and  $G/\langle x, E \rangle$  is a Dedekind group, so that  $G'/G' \cap \langle x, E \rangle$  is finite. Finally,  $\langle x, E \rangle$  satisfies the maximal condition on subgroups, so that  $G' \cap \langle x, E \rangle$  is finite and hence  $G'$  is likewise finite. This last contradiction completes the proof of the theorem.  $\square$

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