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RESEARCH ARTICLE

On sum of a nilpotent and an ideally finite algebras

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Abstract. We study associative algebras R over arbitrary fields which can be decomposed into a sum $R = A + B$ of their subalgebras A and B such that $A^2 = 0$ and B is ideally finite (is a sum of its finite dimensional ideals). We prove that R has a locally nilpotent ideal I such that R/I is an extension of ideally finite algebra by a nilpotent algebra. Some properties of ideally finite algebras are also established.

Introduction

Properties of associative rings (and algebras) R which are sums $R = A+B$ of their subrings A and B were studied by many authors (see, for example, a survey [6]). Rings which are sums of two nilpotent subrings were studied by O.Kegel in [2] and many further results about sums of rings were connected with his results. Rings with zero multiplication are the simplest nilpotent rings. But even for such rings there exist problems about sums $R = A + B$ which are quite complicated (for example, a question whether a sum $R = A + B$ of an algebra A with zero multiplication and a nil-algebra B is a nil-algebra is equivalent to the famous Koethe problem which was recently solved by A.Smoctunowicz).

In this paper we study associative algebras R over an arbitrary field, which can be decomposed into a sum $R = A + B$ of subalgebras A and B such that $A^2 = 0$ and B contains a nonzero finite dimensional ideal. It

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is proved that the algebra R under such restriction has a nonzero ideal which is an extension of a nilpotent ideal by a finite dimensional algebra. Note that some properties of algebras $R = A + B$, where $A^2 = 0$ and B is an arbitrary ring, were studied in the paper [5]. One can only state in this case that R either contains a nonzero nilpotent ideal, or R has an ideal which lies in B, or R contains an ideal similar to the matrix ring $M_2(I)$ for some ideal I of the subring B.

Notations in the paper are standard. All algebras are associative (not necessarily with 1) over an arbitrary field $\mathbb{K}$. If A and B are $\mathbb{K}$-subspaces of R, then AB denotes a $\mathbb{K}$-subspace which is spanned on elements ab, $a \in A, b \in B$. For an arbitrary subset $X \subseteq R$ we denote by $Ann^l_R(X)$ the left annihilator of the set X in R, the right annihilator is denoted by $Ann^r_R(X)$. The two-sided annihilator $Ann_R(X)$ is, obviously, the intersection of the left and the right annihilators. For shortness, we will call an associative algebra R over a field nilpotent-by-finite provided that R has a nilpotent ideal I such that $\dim R/I < \infty$. Further, $\beta(R)$ denotes the the prime radical of an associative ring R.

1. Ideally finite associative algebras

On the analogy of the Lie theory where ideally finite Lie algebras were studied (see, for example [7]) we introduce for convenience the following definition

Definition 1. *An associative algebra R over a field will be called ideally finite provided that R is a sum of its finite dimensional ideals.*

Lemma 1. *An associative algebra R is ideally finite if and only if the two-sided annihilator of every element of R has finite codimension in R.*

Proof. If R is an ideally finite associative algebra, then every its element x lies evidently in some finite dimensional ideal I_x. Since the annihilator $S_x = Ann_R(I_x)$ is of finite codimension in R and $S_x \subseteq Ann_R(x)$, then $Ann_R(x)$ is of finite codimension in R. Let now the annihilator $Ann_R(x)$ be of finite codimension for every element $x \in R$. Take the set of coset representatives $x_1, \dots, x_n$ of $Ann_R(x)$ in R. One can immediately check out that the vector subspace of R spanned on elements $x, x_i x, x x_j, x_i x x_j, i, j = 1, \dots, n$ is an ideal of the algebra R. Therefore the element x belongs to a finite dimensional ideal of the algebra R. $\square$

Remark 1. It was noted in the paper [4] (Lemma 12) that if an element $x \in R$ has left (right) annihilator in R of finite codimension then x belongs to a left (right) finite dimensional ideal of the algebra R.

Lemma 2. *Let R be an associative algebra over a field $\mathbb{K}$ and V be a $\mathbb{K}$-subspace of R of finite codimension in R. Then for every element $a \in R$ there exists $\mathbb{K}$-subspace $V_a \subseteq V$ of finite codimension in V such that $aV_a \subseteq V$ and $V_a a \subseteq V$.*

Proof. Consider the linear map $l_a : R \to R/V$, defined by the rule $x \mapsto ax + V$. Since $\dim R/V < \infty$, then $\operatorname{Ker} l_a$ has finite codimension in R. Denote $V_1 = \operatorname{Ker} l_a \cap V$. Then it holds obviously $V_1 \subseteq V$ and $aV_1 \subseteq V$. Analogously one can define the linear map $r_a : R \to R/V$ by the rule $x \mapsto xa+V$ and after putting $V_2 = \operatorname{Ker} r_a \cap V$ we get $V_2 \subseteq V$ and $V_2 a \subseteq V$. Because V_1 and V_2 are subspaces of finite codimension of R, the subspace $V_a = V_1 \cap V_2$ is of finite codimension in R and satisfies conditions of the Lemma. □

The next result specifies Lemma 11 from [4].

Lemma 3. *Let R be an associative algebra over an arbitrary field and J be a right (left) finite dimensional ideal of the algebra R. Then the algebra R has a two-sided idea I with zero square such that $I + J$ is contained in the smallest two-sided ideal T_J of the algebra R which contains J and $dimT_J/(I + J) < \infty$.*

Proof. Let J be for example a right ideal. Then $S = Ann^r_R(J)$ is a two-sided ideal of the algebra R and $\dim R/S < \infty$. Take a set of coset representatives $x_1, \ldots, x_n$ of S in R. Then the smallest ideal T_J of the algebra R containing J is of the form

$$T_J = J + RJ = J + SJ + x_1 J + \ldots x_n J$$

Denote $I = SJ$. It is clear that I is a two-sided ideal of the algebra R and $I^2 = 0$. The subalgebra $J + I$ is, obviosly, of finite codimension in T_J. □

Lemma 4. *Let R be an associative algebra over an arbitrary field $\mathbb{K}$ and A be its subalgebra such that the right annihilator $I = Ann^r_R(A)$ (left annihilator $J = Ann^l_R(A)$) is of finite codimension in R. Then R has a nilpotent ideal T of nilpotency index ≤ 2 such that $\dim A/(A \cap T) < \infty$.*

Proof. Consider the case when the right annihilator I is of finite dimension in R. Let $x_1, \ldots, x_n$ be a set of coset representatives of I in R. Then

$$A + AR = A + A(\sum_{k=1}^{n}(x_k + I) = A + \sum_{k=1}^{n} Ax_k$$

is a right ideal of the algebra R, containing A. For any element x_k, $k = 1, \ldots, n$ there exists (by Lemma 2) a $\mathbb{K}$-subspace I_k, $I_k \subseteq I$ such that

$x_k I_k \subseteq I$ and the subspace I_k is of finite codimension in I. Denote $T = \cap_{k=1}^{n} I_k$. Then T is a subspace of finite codimension in I (and therefore in R) such that $x_k T \subseteq I$ for all $k = 1, \ldots, n$. But then $Ax_k T \subseteq AI = 0$ and therefore $(A + AR)T = 0$. Thus, without loss of generality, one can assume that the subalgebra A from conditions of Lemma is a right ideal of the algebra R. In this case, the right annihilator of A is, obvious a two-sided ideal of R and the intersection $I_0 = A \cap I$ is a right ideal of the algebra R such that $I_0^2 = 0$ and $\dim A/I_0 < \infty$. It is easily to see that $S = I_0 + RI_0$ is a two-sided ideal of the algebra R and $S^2 = 0$. Besides, $A \cap S$ is of finite codimension in A. Analogously, one can consider the case of the left annihilator J. □

Corollary 1. *Under conditions of Lemma 4 the subalgebra A is contained in some nilpotent-by-finite ideal of the algebra R.*

Really, by Lemma 4 the subalgebra $(A + T)/T$ is finite dimensional and its left (right) annihilator in R/T is of finite codimension. Then by Remark 1 $(A + T)/T$ is contained in some nilpotent by-by-finite ideal of the algebra R/T. It follows from this that A is contained in some nilpotent-by-finite ideal of the algebra R.

In the next Lemma we collect some known results about nilpotent and finite dimensional ideals.

Lemma 5. *(see, for example, [4]). Let A be an associative algebra and I an ideal of A. If J is an ideal of the subalgebra I then it holds:*

(1) if subalgebra J is nilpotent then J lies in a nilpotent ideal J_I of the algebra A and $J_I \subseteq I$;

(2) if subalgebra J is finite dimensional then J lies in an ideal J_I of the algebra A such that $J_I \subseteq I$ and J_I possesses a nilpotent ideal T of the algebra A with $\dim J_I/T < \infty$.

3) if an associative algebra R has an ideal I which is a nilpotent-by-finite algebra and the quotient algebra Q/I is the same then the algebra R is also nilpotent-by-finite.

Proof. 1) See, for example, [1], Lemma 1.1.5; 2) See [4], Lemma 3; 3) The proof of this part can be easily obtained from the proof of part 3 of Proposition 1 in [4]. □

Proposition 1. *Let R be an associative algebra over an arbitrary field and $RF(R)$, $(LF(R))$ be the sum of all finite dimensional right (respectively, left) ideals of R. Then $RF(R)$ and $LF(R)$ are two-sided ideals of the algebra R and the sum $F(R)$ of all finite dimensional (two-sided) ideals of R is $F(R) = RF(R) \cap LF(R)$.*

Proof. We can write the sum $I = RF(R)$ of all right ideals of R in the form $RF(R) = \sum_{\lambda\in\Lambda} I_\lambda$ where $\{I_\lambda\}, \lambda \in \Lambda$ is the set of all finite dimensional right ideals of R. Then for any element $a \in R$ we have that $aI = \sum_{\lambda\in\Lambda} aI_\lambda$ is a sum of finite dimensional right ideals aI_λ. Therefore $aRF(R) \subseteq RF(R)$ i.e. $RF(R)$ is a two-sided ideal of the algebra R. Analogously one can show that $LF(R)$ is a two-sided ideal of R.

Let now $x \in RF(R) \cap LF(R)$ be an arbitrary element. Then its left and right annihilators in R are of finite codimension in R and hence its two-sided annihilator in R is of finite codimension. By Lemma 1 the element x is contained in some two-sided finite dimensional ideal of R, i.e. $x \in F(R)$. The inclusion $F(R) \subseteq LF(R) \cap RF(R)$ is clear. □

Corollary 2. *Let I be an ideal of the algebra R which is an ideally finite algebra. Then $I + \beta(R)/\beta(R)$ is contained in $F(R/\beta(R))$. In particular, the ideal $F(R)$ of a semi-prime algebra R contains all ideals which are ideally finite algebras.*

2. On sum of an algebra with zero square and an ideally finite algebra

Theorem 1. *Let R be an associative algebra over an arbitrary field, which is a sum $R = A + B$ of its subalgebras A and B with $A^2 = 0$. If the subalgebra B possesses a nonzero finite dimensional ideal J then the algebra R has a nilpotent ideal I such that $(J + I)/I$ is contained in some finite dimensional ideal of the quotient algebra R/I.*

Proof. 1. Without loss of generality, one can assume that $ABA \subseteq A$. Indeed, denote $A_1 = A + ABA$. Then, obviously, $R = A_1 + B$ and $A_1^2 = 0$. Besides,

$$A_1BA_1 = (A + ABA)B(A + ABA) = ABA + ABABA + ABABABA.$$

Note that $ABABA \subseteq A(A+B)A = ABA$ (because $ABA \subseteq R = A+B$), analogously $ABABABA \subseteq ABA$. Since $ABA \subseteq A_1$, it follows from above relations that $A_1BA_1 \subseteq A_1$.

2. Show that the subalgebra $T = A + AJ + JA + J$ of R possesses a nilpotent ideal S (of the subalgebra T) such that $\dim A/(A \cap S) < \infty$. Note first that the subspace T is a subalgebra because of conclusions

$$AJA \subseteq A,\ JAJ \subseteq RJ = (A + B)J \subseteq AJ + J.$$

Besides, the subalgebra T has a subalgebra $T_1 = A + AJ + JA$ of finite codimension in T (because $\dim J < \infty$). Then as well known (see for example, [3]) the subalgebra T_1 contains an ideal I_1 of the subalgebra T of finite codimension in T. As the subalgebra A is a nilpotent ideal of the subalgebra T_1 then $A \cap I_1$ is a nilpotent ideal of the subalgebra I_1. By Lemma 5 $A \cap I_1$ is contained in some nilpotent ideal S of the subalgebra T which lies in I_1 (since $I_1 \subseteq T_1$, then $S \subseteq T_1$). As I_1 is of finite codimension in T and $A \subseteq T$, then $A \cap I_1$ is of finite codimension in A. It follows from this (taking into account the inclusion $(A \cap I_1) \subseteq S$) that $\dim A/(A \cap S) < \infty$.

3. Prove on this step that R has a nilpotent ideal N such that $\dim JAJ/(JAJ \cap N) < \infty$. Take the least ideal $M = J + JR + RJ + RJR$ of the algebra R containing J. After inserting $A + B$ instead R in the expression for M we obtain

$$M = J + (A+B)J + J(A+B) + (A+B)J(A+B) = J + AJ + JA + AJA$$

(because J is an ideal of the algebra B). By the first step of the proof $AJA \subseteq A$ and therefore $M \subseteq T$ where T is the subalgebra of R defined on the step 2. Because the subalgebra S from the step 2 is a nilpotent ideal of the subalgebra T then $M \cap S$ is a nilpotent ideal of the subalgebra M and therefore by the Lemma 5 $M \cap S$ is contained in some nilpotent ideal N of the algebra R which is contained in M. By the previous step of the proof the subalgebra $A \cap S$ is of finite codimension in A and taking in account the inclusion $AJA \subseteq A$ we see that $S \cap (AJA)$ is of finite codimension in AJA.

Further, by construction $AJA \subseteq M$ and $(M \cap S) \subseteq N$ and therefore $(S \cap AJA) \subseteq N$. It follows from the previous consideration that $\dim AJA/(AJA \cap N) < \infty$ i.e. the subalgebra $(AJA + N)/N$ of the quotient algebra R/N is finite dimensional.

4. Show that the algebra R has such a nilpotent ideal I that $(J+I)/I$ is contained in some finite dimensional ideal of the quotient algebra R/I. Denote for convenience $\overline{R} = R/N$, $\overline{A} = (A + N)/N$, $\overline{B} = (B + N)/N$. Then $\overline{R} = \overline{A} + \overline{B}$ and $\overline{B}$ contains a finite dimensional ideal $\overline{J} = (J + N)/N$. By the previous step $\overline{R}$ contains a finite dimensional subalgebra $\overline{AJA} = (AJA + N)/N$. Consider the right annihilator $Ann^r_{\overline{R}}(\overline{AJ})$. Since $\overline{J} = (J + N)/N$ is a finite dimensional ideal of the algebra $\overline{B}$ then $\overline{AJ}$ is annihilated on the right by some subalgebra $\overline{B}_0$ of the algebra $\overline{B}$ of finite codimension in $\overline{B}$. Further, the subalgebra $\overline{AJA} = \overline{AJ} \cdot \overline{A}$ is finite dimensional and therefore for any element $\overline{g} \in \overline{AJ}$ by Lemma 2 there exists a subspace $\overline{A}_g$ of finite codimension in $\overline{A}$ such that $\overline{g}\overline{A}_g = 0$. But then the right annihilator of the element $\overline{g}$ is of finite codimension in $\overline{R}$ (because it contains the sum $\overline{B}_0 + \overline{A}_g$) and hence by Remark 1 the element $\overline{g}$ belongs to a finite dimensional right ideal of the algebra $\overline{R}$.

It follows easily from this that the finite dimensional subalgebra $\overline{AJA}$ is contained in some finite dimensional right ideal of the algebra $\overline{R}$. But then the subalgebra $\overline{AJA}$ is contained by Lemma 3 in some nilpotent-by-finite ideal $\overline{T} = T/N$ of the algebra $\overline{R} = R/N$. After passing on to the algebra R one can easily see that the subalgebra AJA is contained in the nilpotent-by-finite ideal T of the algebra R (because ideal N is nilpotent).

Consider now the quotient algebra $\widetilde{R} = R/T$. This algebra is obviously a sum of two its subalgebras $\widetilde{A} = (A+T)/T$ and $\widetilde{B} = (B+T)/T$ and it holds by above $\widetilde{A}\widetilde{J}\widetilde{A} = 0$ in $\widetilde{R}$. As $\widetilde{J}$ is a finite dimensional ideal of $\widetilde{B}$ then $\tilde{A}\tilde{J}$ has evidently the right annihilator in $\widetilde{R}$ of finite codimension in $\widetilde{R}$. By Lemma 4 the product $\widetilde{A}\tilde{J}$ is contained in some nilpotent-by-finite ideal $\widetilde{U}$ of the algebra $\widetilde{R}$. Denote by U the complete preimage of $\widetilde{U}$ in R. Then this ideal nilpotent-by-finite by Lemma 5 and therefore the left annihilator of the subalgebra $(J+U)/U$ of the quotient algebra R/U is of finite codimension in R/U. Then by Lemma 4 J is contained in some nilpotent-by-finite ideal of the algebra R. □

Theorem 2. *Let R be an associative algebra over an arbitrary field which is a sum $R = A + B$ of a subalgebra A with zero square and an ideally finite subalgebra B. Then R possesses a locally nilpotent ideal I such that the subalgebra $\overline{B} = (B+I)/I$ of the quotient algebra $\overline{R} = R/I$ is contained in the ideal $F(\overline{R})$ which is generated by all finite dimensional ideals of $\overline{R}$.*

Proof. Since B is ideally finite then for a set Λ of indices it holds $B = \sum_{\lambda\in\Lambda} B_\lambda$ where B_λ are finite dimensional ideals of the subalgebra B. For any ideal B_λ denote by I_λ a nilpotent ideal of the algebra R such that $(B_\lambda + I_\lambda)/I_\lambda$ is contained in some finite dimensional ideal $\overline{R}_\lambda = R_\lambda/I_\lambda$ of the quotient algebra $\overline{R} = R/I_\lambda$ (such an ideal does exist by Theorem 1). Denote by I the sum $I = \sum_{\lambda\in\Lambda} I_\lambda$. It is easily shown that I is a locally nilpotent ideal of the algebra R and every subalgebra $(B_\lambda + I)/I$ of the quotient algebra R/I is contained in a finite dimensional ideal of the algebra R/I. Therefore $(B+I)/I$ is contained in the ideal $F(R/I)$ of the algebra R/I which is generated by all finite dimensional ideals of R/I. □

Corollary 3. *If the algebra R in the Theorem 2 is semiprime then the subalgebra B is contained in the ideal $F(R)$ which is generated by all finite dimensional ideals of the algebra R.*

Really the ideal I in the proof of the Theorem 2 is zero because all ideals I_λ are zero by the semiprimity of R.

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