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Commutative reduced filial rings

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Abstract. A ring R is filial when for every I, J, if I is an ideal of J and J is an ideal of R then I is an ideal of R. Several characterizations and results on structure of commutative reduced filial rings are obtained.

Introduction

All rings in this paper are associative but we do not assume that each ring has an identity element. By $\mathbb{Z}$ we denote the ring of integers and by $\mathbb{N}$ the set of positive integers. Moreover, by $\mathbb{P}$ we denote the set of all prime integers.

We say that a ring R is *filial* (*left filial*) when for every I, J, if I is an ideal (left ideal) of J and J is an ideal (left ideal) of R then I is an ideal (left ideal) of R.

Filial rings appeared independently in some several papers. Systematic investigations of them were begun by Ehrlich [3] (she studied there mostly commutative rings) and were continued in [2], [6], [7], [1], [5]. Systematic studies of left filial rings were started in [4]. In particular a structure theorem describing semiprime left filial was obtained (see Theorem 1) there and it was shown that semiprime left filial rings are filial. In [1] the complete classification and the method of construction of commutative filial domains was given. The classification was proceed by considering the set $\Pi(R) = \{p \in \mathbb{P} : p \text{ is not a unit in } R\}$. It was shown that for an arbitrary subset Π of the set of prime numbers, a ring R is a filial integral domain of characteristic 0 with $\Pi(R) = \Pi$ if and only if R

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is isomorphic to a subring of $\mathbb{Q}_\Pi = \prod\{\mathbb{Q}_p : p \in \Pi\}$ of the form $K \cap \mathbb{Z}_\Pi$, where $\mathbb{Z}_\Pi = \prod\{\mathbb{Z}_p : p \in \Pi\}$, K is a subfield of $\mathbb{Q}_\Pi$ such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$ we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$ and $\mathbb{Q}_p$ is the quotient field of the p-adic integers $\mathbb{Z}_p$.

In this paper we generalize methods and results obtained in [1] for reduced commutative rings.

To denote that I is an ideal of a ring R, we write $I \lhd R$. Given a ring R, we denote by R^+ the additive group of R.

The class of filial rings is closed under taking homomorphic images and ideals. Obviously, $\mathbb{Z}$ is a filial ring. However, as it was noted in [2], the ring $\mathbb{Z} \oplus \mathbb{Z}$ is not filial. Hence the class of filial rings is not closed under direct sums and extensions.

1. General properties of CRF-rings

Proposition 1. *A commutative ring R is filial if and only if for every $a \in R$, $Ra = Ra^2 + \mathbb{Z}a^2 + Ra \cap \mathbb{Z}a$.*

Proof. Suppose R is filial and let $a \in R$. Then by Proposition 2.1 of [1], $\mathbb{Z}a + Ra = Ra^2 + \mathbb{Z}a^2 + \mathbb{Z}a$. Thus $Ra \subseteq Ra^2 + \mathbb{Z}a^2 + \mathbb{Z}a$. Since $Ra^2 + \mathbb{Z}a^2 \subseteq Ra$, so by the modularity of the lattice of subgroups of R^+, $Ra = Ra^2 + \mathbb{Z}a^2 + (Ra \cap \mathbb{Z}a)$.

Conversely, let $a \in R$. Then $Ra = Ra^2 + \mathbb{Z}a^2 + Ra \cap \mathbb{Z}a$, so $\mathbb{Z}a + Ra = Ra^2 + \mathbb{Z}a^2 + Ra \cap \mathbb{Z}a = Ra^2 + \mathbb{Z}a^2 + \mathbb{Z}a$. Therefore R is filial by Proposition 2.1 of [1]. □

A ring R containing no non-zero nilpotent is called *reduced*, i.e. for every $a \in R$ if $a^2 = 0$ then $a = 0$. We say that R is a *CRF-ring* when R is a commutative reduced filial ring. A ring R is called *strongly regular* if for every $a \in R$, $a \in Ra^2$. Every strongly regular ring is reduced and the class $\mathbb{S}$ of all strongly regular rings is a radical class. It is easy to see that if $R \neq 0$ is a commutative domain and R is not a field then $\mathbb{S}(R) = 0$.

Every commutative strongly regular ring is a CRF-ring by Proposition 1.

Theorem 1 ([4], Theorem 3.4)**.** *The following conditions on a ring R are equivalent:*
(i) R is reduced and left filial,
(ii) R contains an ideal I such that I is strongly regular and R/I is a CRF-ring,
(iii) $R/\mathbb{S}(R)$ is a CRF-ring.

Lemma 1. *The additive group of every $\mathbb{S}$-semisimple CRF-ring R is torsion-free.*

Proof. Suppose R^+ is not torsion-free. Then there exist $p \in \mathbb{P}$ and $0 \neq a \in R$ such that $pa = 0$. Thus Ra is a non-zero algebra over a field of p-elements and $Ra \lhd R$, so Ra is filial. Therefore by Theorem 4.1 of [5], $Ra \in \mathbb{S}$, hence $Ra \subseteq \mathbb{S}(R) = 0$, and $Ra = 0$, a contradiction. □

Applying Theorem 1 and Lemma 1, one obtains the following:

Lemma 2. *Let R be a CRF-ring. Then $R/\mathbb{S}(R)$ is a torsion-free CRF-ring.*

Lemma 3. *Let R be a torsion-free CRF-ring. Then for every $0 \neq a \in R$, $Ra \cap \mathbb{Z}a \neq 0$.*

Proof. Suppose for some $0 \neq a \in R$, $Ra \cap \mathbb{Z}a = 0$. Then by Proposition 1, $Ra = Ra^2 + \mathbb{Z}a^2$. Suppose that $Ra^2 \cap \mathbb{Z}a^2 \neq 0$. Thus there exists $n \in \mathbb{N}$ such that $na^2 \in Ra^2$, so $na^2 = xa^2$ for some $x \in R$. Therefore $a(na - xa) = 0$, thus $(na - xa)^2 = 0$. Moreover R is reduced, hence $na - xa = 0$ and $na \in Ra$. Since the group R^+ is torsion-free and $a \neq 0$, we have $na \neq 0$. Thus $na \in Ra \cap \mathbb{Z}a = 0$, a contradiction. Therefore $Ra^2 \cap \mathbb{Z}a^2 = 0$ and by Proposition 1, $Ra^2 = Ra^4 + \mathbb{Z}a^4$. But $Ra^4 + \mathbb{Z}a^4 \subseteq Ra^3$, hence $Ra^2 \subseteq Ra^3 \subseteq Ra^2$, so $Ra^2 = Ra^3$. Therefore $a^3 = ya^3$ for some $y \in R$. Hence $a^2(a - ya) = 0$, so $(a - ya)^3 = 0$ and $a = ya$. Consequently $0 \neq a \in Ra \cap \mathbb{Z}a$, a contradiction. □

Theorem 2. *Let R be a CRF-ring. Then for every $a \in R$ there exists $n \in \mathbb{N}$ such that $na \in Ra^2$.*

Proof. Suppose R^+ is torsion-free. If $a = 0$ we can take $n = 1$. Let $a \neq 0$. Then, by Lemma 3, there exists $m \in \mathbb{N}$ such that $ma \in Ra$. Moreover $Ra \lhd R$, so Ra is filial. Applying Lemma 3 to the ring Ra one obtains that $n \cdot (ma) \in Ra(ma)$ for some $n \in \mathbb{N}$. Hence $(nm)a \in mRa^2$. Consequently $na \in Ra^2$.

Suppose now that R^+ is not torsion-free. Let $\bar{R} = R/\mathbb{S}(R)$. Then by Lemma 2, $\bar{R}$ is a torsion-free CRF-ring. Therefore for $\bar{a} = a + \mathbb{S}(R)$ by a first part of the proof, there exists $n \in \mathbb{N}$ such that $n \cdot \bar{a} \in \bar{R} \cdot \bar{a}^2$. Hence $na - ra^2 = s \in \mathbb{S}(R)$ for some $r \in R$, $s \in \mathbb{S}(R)$. But $\mathbb{S}(R) \in \mathbb{S}$, so $s = bs^2$ for some $b \in \mathbb{S}(R)$. Consequently $na - ra^2 = b \cdot (na - ra^2)^2 \in Ra^2$, so $na \in Ra^2$. □

For every torsion-free ring R we denote by $\Pi(R)$ the set

$$\Pi(R) = \{p \in \mathbb{P} : pR \neq R\}.$$

If R has an identity then $\Pi(R) = \{p \in \mathbb{P} : p \text{ is not a unit in } R\}$. Moreover, let $S(X)$ be the least multiplicative subset of $\mathbb{N}$ containing $X \subseteq \mathbb{N}$.

Lemma 4. *Let A be a non-zero ideal of a commutative filial domain R of characteristic 0. Then $\Pi(A) = \Pi(R)$ and there exists $t \in S(\Pi(A))$ such that $A = tR$.*

Proof. By Proposition 2.5 of [1] there exists a filial integral domain P of characteristic 0 such that $R \lhd P$. Hence $A \lhd R$ and $R \lhd P$, so $A \lhd P$ by filiality of P.

Now, by Theorem 3.3 of [1] there exist $m, n \in S(\Pi(P))$ such that $R = mP$ and $A = nP$. But $A \subseteq R$, then $n \in mP$ and $m \mid n$ in P. Therefore by Proposition 3.4 of [1], $m \mid n$ in $\mathbb{Z}$. Consequently $n = m \cdot t$ for some $t \in \mathbb{N}$, thus $t \in S(\Pi(P))$ by definition of $S(\Pi(P))$ and $A = nP = mtP = tR$. Since R^+ is torsion-free, then by definition of $\Pi(R)$ we have $\Pi(A) = \Pi(R)$. □

2. General properties of the radical class $\mathcal{T}_p$

Let p be a prime number. We denote by $\mathcal{T}_p$ the class of all rings R such that $pR^+ = R^+$. Let us observe that $\mathcal{T}_p$ is a radical class. For every ring $R \in \mathcal{T}_p$ and for every $n \in \mathbb{N}$, $p^n R = R$. Moreover, if R is torsion-free then $\mathcal{T}_p(R) = \bigcap_{n=1}^{\infty} p^n R$.

Remark 1. Let R be a torsion-free ring. For every prime p, $p \in \Pi(R)$ if and only if $\mathcal{T}_p(R) \neq R$.

Theorem 3. *Let p be a prime and let R be a torsion-free ring. Then the ring $R/\mathcal{T}_p(R)$ is torsion-free.*

Proof. Take any $x \in R$ such that $m \cdot x \in \mathcal{T}_p(R)$ for some $m \in \mathbb{N}$. Then $m = p^\alpha k$ for some $\alpha \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$ such that $p \nmid k$. Since $\mathcal{T}_p(R) = p^\alpha \mathcal{T}_p(R)$, so $p^\alpha (k \cdot x) \in p^\alpha \mathcal{T}_p(R)$, thus $k \cdot x \in \mathcal{T}_p(R)$, because R^+ is torsion-free. Let $n \in \mathbb{N}$. Then there exist integers l_n, k_n such that $p^n l_n + k \cdot k_n = 1$. Consequently $x = p^n(l_n x) + k_n \cdot (k \cdot x) \in p^n R + p^n \mathcal{T}_p(R) \subseteq p^n R$. Hence $x \in \bigcap_{n=1}^{\infty} p^n R$, so $x \in \mathcal{T}_p(R)$. □

Proposition 2. *Let R be a torsion-free CRF-ring. Then for every prime p the ring $R/\mathcal{T}_p(R)$ is reduced.*

Proof. Take any $a \in R$ such that $a^2 \in \mathcal{T}_p(R)$. By Theorem 2 there exists $n \in \mathbb{N}$ such that $na \in Ra^2 \subseteq \mathcal{T}_p(R)$. Hence by Theorem 3, $a \in \mathcal{T}_p(R)$. Therefore the ring $R/\mathcal{T}_p(R)$ is reduced. □

Theorem 4. *Let A and B are non-zero torsion-free CRF-rings such that $\mathcal{T}_p(A) = 0$ and $\mathcal{T}_p(B) = 0$ for some prime p. Then $A \oplus B$ is not filial.*

Proof. Suppose $A \oplus B$ is filial. By the assumption there exist $a \in A \setminus pA$ and $b \in B \setminus pB$. Observe that $p^2A \oplus p^2B + \mathbb{Z}(pa, pb) \lhd pA \oplus pB \lhd A \oplus B$. So by filiality of $A \oplus B$, $p^2A \oplus p^2B + \mathbb{Z}(pa, pb) \lhd A \oplus B$. In particular for any $\alpha \in A$

$$(pa\alpha, 0) = (\alpha, 0) \cdot (pa, pb) \in p^2A \oplus p^2B + \mathbb{Z}(pa, pb).$$

Hence there exist $x \in A$, $y \in B$, $k \in \mathbb{Z}$ such that $(pa\alpha, 0) = (p^2x, p^2y) + k \cdot (pa, pb)$, so $pa\alpha = p^2x + kpa$ and $0 = p^2y + kpb$. But A^+ and B^+ are torsion-free, so $a\alpha = px + ka$ and $0 = py + kb$.

If $p \nmid k$, then there exist $r, s \in \mathbb{Z}$ such that $kr + ps = 1$. Hence $b = krb + psb = r \cdot (-py) + psb = p(sb - ry) \in pB$, a contradiction. Therefore $p \mid k$ and since $a\alpha = px + ka$, we have $a\alpha \in pA$. Consequently

$$aA \subseteq pA. \tag{1}$$

Similarly, $bB \subseteq pB$.

Take any $n \in \mathbb{N}$ such that $aA \subseteq p^nA$ i $bB \subseteq p^nB$. We prove that

$$aA \subseteq p^{n+1}A \quad \text{and} \quad bB \subseteq p^{n+1}B.$$

By (1) we have that $p^{n+2}A \oplus p^{n+2}B + \mathbb{Z}(pa, pb) \lhd p^{n+1}A \oplus p^{n+1}B + \mathbb{Z}(pa, pb)$ and $p^{n+1}A \oplus p^{n+1}B + \mathbb{Z}(pa, pb) \lhd p^nA \oplus p^nB + \mathbb{Z}(pa, pb) \lhd A \oplus B$. So by filiality of $A \oplus B$:

$$p^{n+2}A \oplus p^{n+2}B + \mathbb{Z}(pa, pb) \lhd A \oplus B. \tag{2}$$

In particular for $\alpha \in A$, $(pa\alpha, 0) = (p^{n+2}x, p^{n+2}y) + k \cdot (pa, pb)$ for some $x \in A$, $y \in B$, $k \in \mathbb{Z}$. But A^+ and B^+ are torsion-free, so:

$$a\alpha = p^{n+1}x + ka \quad \text{and} \quad 0 = p^{n+1}y + kb. \tag{3}$$

If $p^{n+1} \nmid k$, then $k = p^\beta \cdot l$ for some $\beta \in \mathbb{N}_0$, $l \in \mathbb{Z}$, $p \nmid l$ and $\beta < n + 1$. Thus by (3), $l \cdot b \in pB$, so $b \in pB$, a contradiction. Therefore $p^{n+1} \mid k$ and by (3), $a\alpha \in p^{n+1}A$. Consequently $aA \subseteq p^{n+1}A$. Similarly, $bB \subseteq p^{n+1}B$.

Therefore $aA \subseteq p^mA$ and $bB \subseteq p^mB$ for every $m \in \mathbb{N}$. Hence $aA \subseteq \bigcap_{m=1}^{\infty} p^mA = \mathcal{T}_p(A) = 0$ and $bB \subseteq \bigcap_{m=1}^{\infty} p^mB = \mathcal{T}_p(B) = 0$, so $aA = 0$ and $bB = 0$. In particular $a^2 = 0$ and $b^2 = 0$, thus $a = 0$ and $b = 0$, a contradiction. □

Theorem 5. *Let R be a non-zero torsion-free CRF-ring such that $\mathcal{T}_p(R) = 0$ for some prime p. Then R is a domain.*

Proof. Suppose R is not a domain. Then there exist non-zero elements $a, b \in R$ such that $a \cdot b = 0$. Hence Ra and Rb are non-zero ideals of R and $(Ra \cap Rb)^2 \subseteq Ra \cdot Rb = 0$. So $Ra \cap Rb = 0$, because R is reduced. Moreover $\mathcal{T}_p(Ra) = 0$ and $\mathcal{T}_p(Rb) = 0$, thus by Theorem 4, $Ra + Rb = Ra \oplus Rb$ is not filial. But $Ra \oplus Rb \lhd R$, so $Ra \oplus Rb$ is filial, a contradiction. □

Theorem 6. *Let R be a non-zero torsion-free CRF-ring. Then for every prime $p \in \Pi(R)$, $\mathcal{T}_p(R)$ is a prime ideal of R.*

Proof. Denote $\bar{R} = R/\mathcal{T}_p(R)$. By Remark 1, $R \neq \mathcal{T}_p(R)$, so $\bar{R}$ is a non-zero commutative filial ring. Theorem 3 and Proposition 2 imply that $\bar{R}$ is reduced and torsion-free. Moreover $\mathcal{T}_p(\bar{R}) = 0$. So by Theorem 5, $\bar{R}$ is a domain. Consequently, $\mathcal{T}_p(R)$ is a prime ideal of R. □

3. Main results

Proposition 3. *Let R be a torsion-free commutative reduced ring. Then R is filial if and only if for every $a \in R$:*
(i) $Ra + \mathbb{Z}a = pRa + \mathbb{Z}a$ for every $p \in \mathbb{P}$ and (ii) $ma \in Ra^2$ for some $m \in \mathbb{N}$.

Proof. Suppose R is filial. Then (ii) holds by Theorem 2.5. Moreover, $Rp^2a^2 + \mathbb{Z}p^2a^2 + \mathbb{Z}pa \lhd Rpa + \mathbb{Z}pa \lhd R$, so $Rp^2a^2 + \mathbb{Z}p^2a^2 + \mathbb{Z}pa \lhd R$, by filiality of R. Hence $Rp^2a^2 + \mathbb{Z}p^2a^2 + \mathbb{Z}pa = Rpa + \mathbb{Z}pa$. But R is torsion-free, so $Ra + \mathbb{Z}a = Rpa^2 + \mathbb{Z}pa^2 + \mathbb{Z}a$. Therefore $Ra + \mathbb{Z}a \subseteq pRa + \mathbb{Z}a \subseteq Ra + \mathbb{Z}a$. Thus $Ra + \mathbb{Z}a = pRa + \mathbb{Z}a$.

Conversely, let (i) and (ii) holds. If $k, l \in \mathbb{N}$ are such that $Ra + \mathbb{Z}a = kRa + \mathbb{Z}a = lRa + \mathbb{Z}a$, then $Ra + \mathbb{Z}a = (kl)Ra + \mathbb{Z}a$. Hence, if $k_1, \dots, k_s \in \mathbb{N}$ and $Ra + \mathbb{Z}a = k_iRa + \mathbb{Z}a$ for $i = 1, \dots, s$, then $Ra + \mathbb{Z}a = (k_1 \cdot \dots \cdot k_i)Ra + \mathbb{Z}a$. Therefore by (i) we have that $Ra + \mathbb{Z}a = nRa + \mathbb{Z}a$ for every $n \in \mathbb{N}$. Since by (ii) there exist $m \in \mathbb{N}$ and $b \in R$ such that $ma = ba^2$ and moreover $Ra + \mathbb{Z}a = mRa + \mathbb{Z}a$, so $Ra + \mathbb{Z}a = Rba^2 + \mathbb{Z}a \subseteq Ra^2 + \mathbb{Z}a \subseteq Ra^2 + \mathbb{Z}a^2 + \mathbb{Z}a \subseteq Ra + \mathbb{Z}a$. Consequently $Ra + \mathbb{Z}a = Ra^2 + \mathbb{Z}a^2 + \mathbb{Z}a$ and R is filial by Proposition 2.1 of [1]. □

Applying Proposition 3, one immediately obtain the following.

Corollary 1. *Let R be a torsion-free commutative reduced ring with an identity. Then R is filial if and only if:*
(i) $R = pR + \mathbb{Z} \cdot 1$ for every $p \in \mathbb{P}$ and (ii) for every $a \in R$ there exists $m \in \mathbb{N}$ such that $ma \in Ra^2$.

Lemma 5. *Let A be a non-zero commutative filial domain of characteristic 0 which is not a field. Then there exist $a \in A$ and $k \in \mathbb{N}$ such that $ax = kx$ for every $x \in A$. Moreover $A = pA + \mathbb{Z}a$ and $|A/pA| = p$ for every $p \in \Pi(A)$.*

Proof. From Proposition 2.5 of [1] there exists a filial domain P of characteristic 0 such that $A \lhd P$. Thus P is not a field and by Theorem 3.1 of [1], $\Pi(P) \neq \emptyset$. Therefore by Lemma 4, $\Pi(A) = \Pi(P)$ and there exists $k \in S(\Pi(A))$ such that $A = kP$. So $a = k \cdot 1 \in A$, which means that $ax = kx$ for every $x \in A$. Take any $p \in \Pi(A)$. Corollary 1 implies that $P = pP + \mathbb{Z} \cdot 1$. Hence $A = pA + \mathbb{Z} \cdot (k \cdot 1) = pA + \mathbb{Z}a$. Thus $a \notin pA$. Consequently, $|A/pA| = p$. □

Lemma 6. *Let A be a non-zero commutative domain of characteristic 0. If $|A/pA| = p$ for every $p \in \Pi(A)$ and for every $x \in A$ there exists $m \in \mathbb{N}$ such that $mx \in Ax^2$, then A is filial.*

Proof. If $\Pi(A) = \emptyset$ then A is a $\mathbb{Q}$-algebra, so $x \in Ax^2$ for every $x \in A$. Hence A is a strongly regular ring and A is filial. Now, let $\Pi(A) \neq \emptyset$. Take any $0 \neq a \in A$. Then there exists $m \in \mathbb{N}$ such that $ma \in Aa^2$. Thus $ma = ba^2$ for some $b \in A$. Hence $m = ba$, which means that there exists a minimal natural number $n \in A$. Suppose that $n \in pA$ for some $p \in \Pi(A)$. Then there exists $c \in A$ such that $n = pc$. If $n = pk$ for some $k \in \mathbb{N}$, then $k \in A$, a contradiction. So, $(p, n) = 1$ and there exist $u, v \in \mathbb{Z}$ such that $nu + pv = 1$. Therefore for $x \in A$, $x = (nu)x + p(vx) = p(ucx) + p(vx) \in pA$ and $A = pA$, a contradiction. Consequently, $n \notin pA$ for every $p \in \Pi(A)$. Hence $A = pA + \mathbb{Z}m$ for every $p \in \Pi(A)$. Thus $Aa + \mathbb{Z}a = pAa + \mathbb{Z}a$ and A is filial by Proposition 3. □

Theorem 7. *Let R be a torsion-free commutative reduced ring such that $\Pi(R) \neq \emptyset$. Then R is filial if and only if $|R/pR| = p$ for every $p \in \Pi(R)$ and for every $a \in R$ there exists $m \in \mathbb{N}$ such that $ma \in Ra^2$.*

Proof. Suppose R is filial. By Proposition 3 it suffices to prove that $|R/pR| = p$ for every $p \in \Pi(R)$. So, take any $p \in \Pi(R)$. Then by Theorems 3 and 6 we have that $\bar{R} = R/\mathcal{T}_p(R)$ is a non-zero commutative filial domain of characteristic 0. Since $\mathcal{T}_p(R) \subseteq pR$ and $p\bar{R} \neq \bar{R}$, thus $R/pR \cong \bar{R}/p\bar{R}$ and $|R/pR| = p$ by Lemma 5.

Conversely, suppose $|R/pR| = p$ for every $p \in \Pi(R)$ and for every $a \in R$ there exists $m \in \mathbb{N}$ such that $ma \in Ra^2$. By Proposition 3 it suffices to prove that $Ra + \mathbb{Z}a = pRa + \mathbb{Z}a$ for all $a \in R$ and $p \in \Pi(R)$. Take any $p \in \Pi(R)$. Let $\bar{R} = R/\mathcal{T}_p(R)$. Then by Theorem 3, $\bar{R}$ is torsion-free. Moreover $p\bar{R} \neq \bar{R}$, so $p \in \Pi(\bar{R})$. It is easy check that $q\bar{R} = \bar{R}$ for every $q \in \Pi(\bar{R})$. Take any $x \in R$ such that $x^2 \in \mathcal{T}_p(R)$. Then $mx \in Rx^2$

for some $m \in \mathbb{N}$. Thus $mx \in \mathcal{T}_p(R)$. Consequently, $x \in \mathcal{T}_p(R)$ and $\bar{R}$ is reduced. Take any $a, b \in R \setminus \mathcal{T}_p(R)$. Suppose that $ab \in \mathcal{T}_p(R)$. Let n, m be the minimal non-negative integers such that $a \in p^n R$ and $b \in p^m R$, respectively. Therefore $a = p^n x$ and $b = p^m y$ for some $x, y \in R \setminus pR$. Obviously, $xy \in \mathcal{T}_p(R)$. Moreover, $|R/pR| = p$, hence $R = pR + \mathbb{Z}x$. By an easy induction, $R = p^k R + \mathbb{Z}x$ for every $k \in \mathbb{N}$. Take any $k \in \mathbb{N}$. Then $y = p^k r + lx$ for some $r \in R$ and $l \in \mathbb{Z}$. Since $xy \in \mathcal{T}_p(R)$, we have that $y^2 \in p^k R$. Consequently, $y^2 \in \mathcal{T}_p(R)$, so $y \in \mathcal{T}_p(R)$, a contradiction. Therefore $\bar{R}$ is a domain. Lemma 6 implies that $\bar{R}$ is filial. Now, by Lemma 5 there exist $c \in R$ and $k \in \mathbb{N}$ such that $cx - kx \in \mathcal{T}_p(R)$ for every $x \in R$ and $R = pR + \mathbb{Z}c$. Take any $a \in R$. Then $s = ac - ka \in \mathcal{T}_p(R)$ and $ls = zs^2$ for some $l \in \mathbb{N}$ and $z \in R$. Thus $ls \in \mathcal{T}_p(R)s$. Which gives that there exists $l_0 \in \mathbb{N}$ such that $(p, l_0) = 1$ and $l_0 s \in \mathcal{T}_p(R)s$. Hence $l_0 s \in pRs$, so $l_0 s \in pRa$ and $l_0 ac \in pRa + \mathbb{Z}a$. Moreover, $pac \in pRa + \mathbb{Z}a$. Consequently, $ac \in pRa + \mathbb{Z}a$. Since $R = pR + \mathbb{Z}c$, which gives that $Ra \subseteq pRa + \mathbb{Z}a$. Therefore, $Ra + \mathbb{Z}a = pRa + \mathbb{Z}a$, and the proof is completed. □

Theorem 8. *Let I be an ideal of a commutative ring R. Let I and R/I be torsion-free CRF-rings. If $\Pi(I) \cap \Pi(R/I) = \emptyset$ then R is a torsion-free CRF-ring and $\Pi(R) = \Pi(I) \cup \Pi(R/I)$.*

Proof. Obviously, R is a torsion-free commutative reduced ring and for every $a \in R$ there exists $m \in \mathbb{N}$ such that $ma \in Ra^2$. Take any $p \in \Pi(R)$. Suppose $p \notin \Pi(R/I)$. Then $R = pR + I$. Since $R \neq pR$, hence $I \neq pI$ and $p \in \Pi(I)$. Therefore $\Pi(R) \subseteq \Pi(I) \cup \Pi(R/I)$. Take any $p \in \Pi(I)$. Suppose $p \notin \Pi(R)$. Then $R = pR$ and $I \subseteq pR$. Moreover R/I is torsion-free, so $I = pI$, a contradiction. Hence $\Pi(I) \subseteq \Pi(R)$. Take any $p \in \Pi(R/I)$. Then $p(R/I) \neq R/I$, so $pR \neq R$. Which implies that $\Pi(R/I) \subseteq \Pi(R)$. Consequently $\Pi(R) = \Pi(I) \cup \Pi(R/I)$.

Now, take any $p \in \Pi(I)$. Then by assumptions, $p \notin \Pi(R/I)$, so $pR + I = R$. Moreover $pI \neq I$ and R/I is torsion-free. Which means that $I \cap pR = pI$. Consequently, $R/pR \cong I/pI$ and $|R/pR| = p$.

Finally, take any $p \in \Pi(R/I)$. Then $pR + I \neq R$ and, by assumptions, $pI = I$. Hence $I \subseteq pR$ and $R/pR \cong (R/I)/p(R/I)$. Consequently, $|R/pR| = p$.

Therefore R is filial, by Theorem 7, and the proof is completed. □

Corollary 2. *Let I be an ideal of a torsion-free CRF-ring R. If R/I is torsion-free then $\Pi(I) \cap \Pi(R/I) = \emptyset$.*

Proof. By assumptions, I and R/I are torsion-free CRF-rings. Suppose there exists $p \in \Pi(I) \cap \Pi(R/I)$. Then $pI \neq I$ and $pR + I \neq R$. Therefore

$pR \neq R$ and $p \in \Pi(R)$. So, $|R/pR| = p$, by Theorem 7. Consequently, pR is a maximal ideal of R. Moreover, R/I is torsion-free, so $I \not\subseteq pR$. Therefore $R = pR + I$, a contradiction. □

As an immediate consequence of Theorem 7 one obtains the following.

Corollary 3. *Let T be a non-empty subset of $\mathbb{N}$ such that for every $t \in T$ there exists a torsion-free CRF-ring R_t such that $\Pi(R_t) \neq \emptyset$. If for every distinct $t, s \in T$, $\Pi(R_t) \cap \Pi(R_s) = \emptyset$, then $R = \bigoplus_{t \in T} R_t$ is a torsion-free CRF-ring and $\Pi(R) = \bigcup_{t \in T} \Pi(R_t)$.*

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