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Open problems in Radical theory (ICOR-2006)

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1. The Zhevlakov radical J of the free alternative ring F on a countably infinite set of generators is the set of nilpotent elements (and $\neq 0$). This result was obtained independently by Shestakov and Slater, and an account can be found on pp.268-274 of K.A.Zhevlakov *et al.*: Rings that are nearly associative (trans H.F.Smith), Academic Press, New York etc., 1982. Thus J is a $T-$ideal and defines a non-trivial variety. What exactly is this variety and does it have any relevance for radical theory?

2. For a subring S of a ring A (associative this time, though the problem can be generalized) the *idealizer* $I(S)$ of S is $\{a \in A : aS + Sa \subseteq S\}$. This is the largest subring of A which has S as an ideal. Now let $\mathcal{R}$ be a radical class, $S \in \mathcal{R}$.

(i) When is $S = \mathcal{R}(I(S))$?

(ii) For which radical classes $\mathcal{R}$ is it true that for every $\mathcal{R}$-subring T of every ring A, there is an $\mathcal{R}$-subring S such that $S = \mathcal{R}(I(S))$ and $T \subseteq S$?.

It has been observed by Szász (On the idealizer of a subring, Monatshefte Math. 75(1971), 65-68) that maximality of S as an $\mathcal{R}$-subring is sufficient in (i), though it is not necessary. If $\mathcal{R}$ is *strict*, then in (ii) we always have $T \subseteq \mathcal{R}(A) = \mathcal{R}(I(\mathcal{R}(A))$.

3. All necessary information on (associative) rings, modules and radicals considered here one can find for example in [1].

A ring R is *right U-primitive* if there exists a right, faithful, uniform, prime R-module. Left U-primitive rings one can define in an analogous way.

Clearly, right (left) primitive rings are right (left) U-primitive. Commutative domains and prime rings with no nonzero prime ideals are left and right U-primitive. The class of all right (left) U-primitive rings is a special class of rings. It can be proved that the upper radical defined by this class coincides with the lower nil-radical.

Journal Algebra Discrete Math.

Question 1. *Is every prime ring right U-primitive? The case of rings with ACC and/or DCC condition on prime ideals seems to be of special interest.*

Question 2. *Is every right U-primitive ring left U-primitive?*

Agata Smoktunowicz proved in [2] that over every countable field F there exists a simple nil-algebra.

Question 3. *Can the above result be extended to the case of an arbitrary field?*

Further a ring R will be called *totally nil* if for every $n \geq 1$ the polynomial ring $R[t_1, \ldots, t_n]$ is a nil-ring. Totally nil rings form an important radical class, contained strictly between locally nilpotent radical and upper nil-radical. Algebras over uncountable fields are known to be totally nil.

Question 4. *Let F be any field. Does there exist a simple algebra over F being totally nil?*

3. A (Kurosh–Amitsur) radical γ is said to be hereditary, if $I \triangleleft A \in \gamma$ implies $I \in \gamma$ for every ring A and ideal I of A. A radical γ has the *Amitsur property*, if

$$\gamma(A[x]) = (\gamma(A[x]) \cap A)[x]$$

for every ring A and polynomial ring $A[x]$.

If a radical γ has the Amitsur property, then its semisimple class $\mathcal{S}\gamma = \{A \mid \gamma(A) = 0\}$ is *polynomially extensible*, that is, $A \in \mathcal{S}\gamma$ implies $A[x] \in \mathcal{S}\gamma$.

Problem: *Does there exist a (hereditary) radical γ with polynomially extensible semisimple class $\mathcal{S}\gamma$ such that γ does not have the Amitsur property?*

References

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