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Investigations of Mealy automata growth at iterations

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Dedicated to V.I. Sushchansky on the occasion of his 60th birthday

Abstract. The problem of the research of Mealy automata growth at iterations is considered in the paper. We describe the application of the mathematical modelling method to this problem, and consider properties of growth of Mealy automata. We show several equivalence relations and automaton sequences that are used in these investigations.

1. Introduction

The notion of growth was introduced in the 50th of the last century independently by Švarc [27] and Milnor [15]. Mainly, the growth of geometrical objects was studied, but later growth functions were defined for various algebraic objects, too [1, 29]. The growth function is a positively defined function of a natural argument, that characterizes the properties of original object such as "complexity" or "asymptotical behavior". For some objects (for ex., semigroups) different growth functions are considered simultaneously.

One of the most studied characteristics of growth functions is growth order that characterizes its asymptotic behavior. Depending on objects that define growth functions, possible growth orders form different sets. Growth functions of (semi)groups may have polynomial, intermediate, and exponential growth. Historically, the most attention is attracted to

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the the objects of intermediate growth. The question on the existence of groups of intermediate growth was set up by Milnor in 1968 [16], and was solved by Grigorchuk in 1984 [7] (see also [8]). On the other hand, the first example of the finitely generated semigroup of intermediate growth was constructed by Govorov in 1972 [6]. Growth functions of finitely generated semigroups and the set of their growth orders are characterized by Trofimov in [28].

The growth function of a Mealy automaton was introduced in 1988 by Grigorchuk [9], and he shown that the growth functions of an invertible automaton and automaton transformation group, defined by it, have the same growth order. These interrelations allow to apply results that concern the growth of groups and semigroups to the growth of Mealy automata (see, for ex., [10]). Therefore various examples of Mealy automata with polynomial, intermediate and exponential growth were constructed.

In 90th of the last century attention of researchers was attracted to investigations of the growth of invertible Mealy automata (see, for ex., [4, 11, 10, 12]), because in this case automaton transformation group is considered. But growth properties of groups and semigroups are distinct in kind, and investigations of the growth of arbitrary Mealy automata show principal distinctions between the cases of invertible and arbitrary Mealy automata (see, for ex., [22, 19, 20]).

Despite the fact that the growth of Mealy automata has been actively studied since it was introduced, at the end of 20th century the growth of individual invertible automata and some sets were investigated only (see the list of open problems in the survey [10]). That is why in 1997 Grigorchuk set up the problem "to investigate the growth functions of all 2-state Mealy automata over a 2-symbol alphabet". Denote this set of automata by the symbol $A_{n\times m}$. The first step in these investigations is the list of groups, defined by invertible automata from $A_{2\times 2}$, that is shown in the theorem of [10].

The automaton transformation semigroups defined by all automata from $A_{2\times 2}$ were described, and the growth functions of these automata and semigroups were calculated in 2002 [22]. The interesting automata in $A_{2\times 2}$ are found (see, for ex., [18, 2]), but this set demonstrates the valuable properties as integral object, too. By considering automata with larger number of states over larger alphabets, new interesting properties such as non-monotonic growth functions and new growth orders are found. Many questions concerning the growth of Mealy automata are still open and seems to be very interesting (see [10, 12]). Therefore the research of all Mealy automata growth is a natural continuation of these investigations.

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2. Research scope

2.1. Definitions

We will use definitions from [25]. Here we fix necessary notations.

By $\mathbb{N}$ we mean the set of positive integers $\mathbb{N} = \{1, 2, \ldots\}$. For $m \geqslant 2$ the symbol X_m denotes the m-symbol alphabet $\{x_0, x_1, \ldots, x_{m-1}\}$. We denote the set of all finite words over X_m, including the empty word ε, by the symbol X_m^*, and denote the set of all infinite (to right) words by X_m^ω.

Definition 2.1. [14] A *non-initial Mealy automaton* is quadruple $A = (X_m, Q_n, \pi, \lambda)$, where $Q_n = \{q_0, q_1, \ldots, q_{n-1}\}$ is the finite set of states; X_m is input and output alphabet; and $\pi : X_m \times Q_n \to Q_n$ and $\lambda : X_m \times Q_n \to X_m$ are its transition and output functions, respectively.

We denote the set of all n-state Mealy automata over a m-symbol alphabet by the symbol $A_{n\times m}$. The set of all Mealy automata is denoted by $\mathfrak{A}$.

The *automaton transformation* f_i defined by A at the state $q_i \in Q_n$, $0 \leqslant i < n$, is defined by the equality $f(u) = \lambda(u, q_i)$, $u \in X_m^\omega$.

The Mealy automaton $A = (X_m, Q_n, \pi, \lambda)$ defines the set

$$F_A = \{f_0, f_1, \ldots, f_{n-1}\}$$

of automaton transformations over X_m^ω. The Mealy automaton A is called *invertible* if all transformations from the set F_A are bijections. It is easy to show that A is invertible iff the transformation $\lambda(\cdot, \mathsf{f})$ is a permutation of X_m for each state $\mathsf{f} \in Q_n$.

Definition 2.2 ([5])**.** The Mealy automata $A_i = (X_m, Q_{n_i}, \pi_i, \lambda_i)$ for $i = 1, 2$, are called *equivalent* if $F_{A_1} = F_{A_2}$.

Definition 2.3 ([5])**.** The Mealy automata $A_i = (X_m, Q_n, \pi_i, \lambda_i)$ for $i = 1, 2$ are called *Q-isomorphic* if there exists the permutation $\theta \in Sym(Q_n)$ such that

$$\theta\pi_1(\mathsf{x}, \mathsf{f}) = \pi_2(\mathsf{x}, \theta\mathsf{f}) \qquad \text{and} \qquad \lambda_1(\mathsf{x}, \mathsf{f}) = \lambda_2(\mathsf{x}, \theta\mathsf{f})$$

for all $\mathsf{x} \in X_m$ and $\mathsf{f} \in Q_n$.

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Proposition 2.4 ([5]). *Each class of equivalent Mealy automata over the alphabet X_m contains, up to Q-isomorphism, a unique automaton that is minimal with respect to the number of states (such an automaton is called reduced).*

The minimal automaton can be found using the standard algorithm of minimization.

The product of automata corresponds to superposition of automaton transformations, or corresponds to sequential applying of automata-factors. We apply the automaton transformations in right to left order, that is for arbitrary automaton transformations f, g and for all $u \in X_m^\omega$ the equality $f \cdot g(u) = f(g(u))$ holds.

The power A^n is defined for any automaton A and any positive integer n. Let us denote the minimal Mealy automaton [5] equivalent to A^n by the symbol $A^{(n)}$.

Definition 2.5 ([9]). The function γ_A of a natural argument, defined by the equality

$$\gamma_A(n) = \left| Q_{A^{(n)}} \right|,$$

where $n \in \mathbb{N}$, is called the *growth function* of the Mealy automaton A.

Definition 2.6. Let $A = (X_m, Q_n, \pi, \lambda)$ be a Mealy automaton. A semigroup

$$S_A = sg\left(f_0, f_1, \ldots, f_{n-1}\right)$$

is called the *automaton transformation semigroup defined by A*.

Let S be an arbitrary semigroup with the finite set of generators U. The *growth function γ_S of S relative to the system U of generators* is defined [13] by the following equality

$$\gamma_S(n) = \left| \left\{ \ s \in S \ \middle| \ \ell(s) \leqslant n \ \right\} \right|, \, n \in \mathbb{N}.$$

Proposition 2.7 ([9]). *For any $n \in \mathbb{N}$ the value $\gamma_A(n)$ equals the number of those elements of S_A that can be presented as a product of length n of the generators $\{f_0, f_1, \ldots, f_{n-1}\}$.*

2.2. Growth of Mealy automaton set

The growth of the set $\mathfrak{A}$ of all Mealy automata is investigated. An arbitrary Mealy automaton A unambiguously defines the growth function γ_A and the automaton transformation semigroup S_A with the growth function γ_S. We consider the set of automaton growth functions $\Gamma_{\mathfrak{A}}$ and the set of automaton transformation semigroups $\mathfrak{S}$, defined by all automata

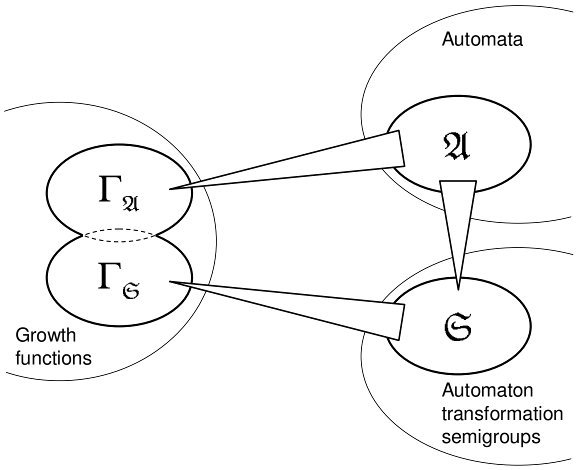

Figure 1: Considered objects

from the set $\mathfrak{A}$, and the set of semigroup growth functions $\Gamma_{\mathfrak{S}}$, defined by all semigroups from $\mathfrak{S}$ (see Fig. 1). Obtained sets $\mathfrak{A}$, $\mathfrak{S}$, $\Gamma_{\mathfrak{A}}$ and $\Gamma_{\mathfrak{S}}$ are close related each other, and it's useful to consider them together. The natural mappings between them are not bijective, and they define the mappings from $\Gamma_{\mathfrak{A}}$ and $\Gamma_{\mathfrak{S}}$ to $\mathfrak{B}(\mathfrak{A})$, $\mathfrak{B}(\mathfrak{S})$, respectively, and from $\mathfrak{S}$ to $\mathfrak{B}(\mathfrak{A})$, where $\mathfrak{B}(Y)$ denotes the boolean of the set Y. In the sequel text we will refer on the last mappings as "backward" mappings.

We are interested in regularities of the sets $\Gamma_{\mathfrak{A}}$ and $\Gamma_{\mathfrak{S}}$, and try answer the question why certain automaton A and corresponding semigroup S_A show one or another growth properties. In addition, we study properties of "backward" mappings, that is the question how Mealy automaton sets, that define the same semigroups or growth function, are constructed. Sometimes, the special subsets of $\mathfrak{A}$ are separated and investigated as independent cases.

According to the main objects, considered questions can be sort into the following categories:

$\mathfrak{A}$: studying how relations between Mealy automata influence on the properties of their growth functions, and investigating structure of the subsets of $\mathfrak{A}$ with predefined growth function properties.

$\mathfrak{S}$: characterizing semigroups from $\mathfrak{S}$, structure of their defining relation sets and the properties (semigroup identities, ideals, Green relations and subsemigroups, etc.) depending on growth functions.

$\Gamma_{\mathfrak{A}}$, $\Gamma_{\mathfrak{S}}$: investigating the construction of growth functions from $\Gamma_{\mathfrak{A}}$ and $\Gamma_{\mathfrak{S}}$, their possible growth orders, asymptotical behavior and vari-

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ous numerical properties, describing the statistical distribution in these sets by various characteristics such that degree ration, growth orders.

$\mathfrak{A}$ and $\mathfrak{S}$: determining structure of automaton sets that define the same semigroups, researching relations between the properties of such automata and properties of the corresponding semigroup (structure of the defining relation set, normal form, etc.).

$\mathfrak{A}$ and $\Gamma_{\mathfrak{A}}$, $\mathfrak{S}$ and $\Gamma_{\mathfrak{S}}$: studying relations between the various properties of growth functions (that are mentioned above) and the properties of corresponding automata (semigroups).

$\Gamma_{\mathfrak{A}}$ and $\Gamma_{\mathfrak{S}}$: investigating correlation between these sets, determining functions that belong to $\Gamma_{\mathfrak{A}}$, but not to $\Gamma_{\mathfrak{S}}$, and vice versa.

Moreover, the research includes questions how properties of growth interrelated with properties of close objects such as graphs, acceptor automata and so on.

2.3. Useful properties of automaton set

In this section we considered several properties of a Mealy automaton, that are used by investigations of a certain automaton. Let A be an arbitrary automaton, S_A be the automaton transformation semigroup, defined by A. Here we use notions and definitions of semigroup objects from [13].

Let $A = (X_m, Q_n, \pi, \lambda)$ be an arbitrary Mealy automaton. Let $k \geqslant 1$, and for arbitrary k-tuple of indexes $(i_1, i_2, \ldots, i_k)$, where $0 \leqslant i_j < n$, the following objects can be considered:

- the state $\mathsf{f} = (q_{i_1}, q_{i_2}, \ldots, q_{i_k})$ of the automaton A^k;
- the automaton transformation $\mathrm{f} = f_{i_1} \cdot f_{i_2} \cdot \ldots \cdot f_{i_k}$ over the set X_m^ω defined by the automaton A^k;
- the semigroup word $\mathsf{s} = f_{i_1} f_{i_2} \ldots f_{i_k}$ in the semigroup S_A.

It follows from the definition of automaton transformation semigroup, defined by a Mealy automaton, that the transformation f is defined by A at the state f, and this transformation is defined by the semigroup word s. These objects are close related, and in the sequel text we freely interchange them.

Let the set of all semigroup words is ordered by lexicographical homogeneous order, i.e. words are ordered by length, and then ordered in

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lexicographical order. Two semigroups words s_1 and s_2 are equal in S_A iff they define the same automaton transformation over X_m^ω (or X_m^*). On the other hand, it means that corresponding automaton states are equivalent. The algorithm that determine equivalent states is considered in [5]. Therefore each class of equivalent semigroup words contains a unique element that is minimal in the mentioned above order.

Proposition 2.8. *Each semigroup word $\mathsf{s} \in S_A$ can be reduced to the normal form.*

Proof. Let $\mathsf{s} \in S_A$ be an arbitrary semigroup word

$$\mathsf{s} = f_{i_1} f_{i_2} \dots f_{i_t},$$

where $t \geqslant 1$, $0 \leqslant i_j < n$. The corresponding to s automaton transformation f belongs to A^t. Therefore it's enough to apply the minimization algorithm to the automaton, that equals the direct sum [5] of the automata $A^1, A^2, \dots, A^t$. It provides the automaton transformation f′ defined by A^k at the state f' for some $1 \leqslant k \leqslant t$, that is the equivalent transformation to $f_{i_1} f_{i_2} \dots f_{i_t}$. Assume that $\mathsf{f}' = q_{j_1} \dots q_{j_k}$. Then the semigroup word $\mathsf{s}' = f_{j_1} f_{j_2} \dots f_{j_k}$ corresponds to the minimal semigroup element that is equal to s. Thus, s' is the normal form of s. □

Theorem 2.9. *For any automaton A there exists the rewriting system in the semigroup S_A.*

Proof. Describe the following approach to the investigations of the automaton A.

Let us consider the sequential degrees A^p, $p = 1, 2, \dots$. We form the set of relations, and relations are ordered by applying the lexicographical homogeneous order first to left, and then to right parts. We start from $p = 1$ and the empty set of relations R. Each iteration provides the minimized automaton $A^{(p)}$.

Let p be fixed, and consider the automaton B_p that equals the product of $A^{(p)}$ and A. Any automaton transformation $f = f_{i_1} f_{i_2} \dots f_{i_p}$ of length p is checked against the set R. Relations are considered as they ordered, and if there exists the relation $u = w$ such that semigroup word u contains in the semigroup word s, defined by f, then u is replaced by v. It gives the semigroup element s' that is equivalent to s, but is lesser. If s has length p, then it is checked against the set R once again. Because each applying of relations from R gives lesser element than previous one, then this process will end. At the end, it gives the semigroup words s'' that is equivalent to s, but any relations from R can not be applied to s''.

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After all states of B_p of length p are reduced by applying the relations from R, the resulting automaton is minimized by the minimization algorithm. It provides the set of additional relations that are added to the set R.

Now, let us show that the relations set R is the terminating confluent rewriting system. This system is terminating, because each applying gives the smaller semigroup word, and all words are ordered. Below we show that the applying relations in two different ways till any relations can not be applied provide the same result. It follows from the algorithm, described above, that there exists the sequence of applying relations from R, that reduce the element s to its normal form. Thus, the rewriting system R is confluent.

Let s be an arbitrary semigroup word, $\mathsf{s} = f_{i_1} f_{i_2} \dots f_{i_k}$, and let it is reduced to the semigroup words s_1 and s_2 by applying relations from R in different ways. Assume that $\mathsf{s}_1 = f_{i_1} f_{i_2} \dots f_{i_{k_1}}$ and $\mathsf{s}_2 = f_{j_1} f_{j_2} \dots f_{j_{k_2}}$, $1 \leqslant k_1, k_2 \leqslant k$, $0 \leqslant i_l, j_l < n$. The automaton transformation f_i, defined by s_i, is included in k_i-th power of A. At k_i iteration of the algorithm it is checked against the set R, respectively. If any of relations from R can not be applied to s_1 and s_2, that is after the applying the minimization algorithm both elements were determined as irreducible. Because both of them equal s, and the normal form is a unique, then $\mathsf{s}_1 = \mathsf{s}_2$.

Thus, the set R is the terminating confluent rewriting system. Note, that this rewriting system contains the set of defining relations. □

It is possible to reduce number of words that are used for testing semigroup elements for equivalence by using notions of dual automata.

Definition 2.10 ([5])**.** Let $A = (X_m, Q_n, \pi, \lambda)$ be an arbitrary Mealy automaton. The automaton $\overline{A} = \left(Q_n, X_m, \overline{\pi}, \overline{\lambda}\right)$ such that

$$\overline{\pi}(\mathsf{f}, \mathsf{x}) = \lambda(\mathsf{x}, \mathsf{f}) \qquad \text{and} \qquad \overline{\lambda}(\mathsf{f}, \mathsf{x}) = \pi(\mathsf{x}, \mathsf{f})$$

for any $\mathsf{x} \in X_m$, $\mathsf{f} \in Q_n$, is called *the dual automaton to A*.

Definition 2.11. We say that two words $u_1, u_2 \in X_m^*$ are *equivalent* for the automaton A, if the equality $u_1 = u_2$ holds in the semigroup $S_{\overline{A}}$, defined by the dual automaton $\overline{A}$.

Proposition 2.12. *Let u_1, u_2 be equivalent words for A. Then for any semigroup element $\mathsf{s} \in S_A$ the equality $\pi(u_1, \mathsf{s}) = \pi(u_2, \mathsf{s})$ holds.*

Proof. Let $\mathsf{s} \in S_A$ be an arbitrary semigroup element. It can be considered as a word over the alphabet Q_n. It follows from Definition 2.11, that the equality

$$\overline{\lambda}(\mathsf{s}, u_1) = \overline{\lambda}(\mathsf{s}, u_2)$$

holds. As $\overline{\lambda}(\mathsf{s}, u_1) = \pi(u_1, \mathsf{s})$, then the equality $\pi(u_1, \mathsf{s}) = \pi(u_2, \mathsf{s})$ holds, that proves the proposition. $\square$

Theorem 2.13. *Two semigroup words* s_1 *and* s_2 *are equal in the semigroup* S_A *iff for all elements* $u \in S_{\overline{A}}$ *and all symbols* $x \in X_m$ *the following equality holds:*

$$\mathsf{s}_1(ux) = \mathsf{s}_2(ux). \tag{2.1}$$

Proof. Let $\mathsf{s}_1, \mathsf{s}_2$ be arbitrary semigroup words such that the equality $\mathsf{s}_1 = \mathsf{s}_2$ holds in S_A. Then they define the same automaton transformation, and for any word $v \in X_m^*$ the equality $\mathsf{s}_1(v) = \mathsf{s}_2(v)$ holds.

As X_m is the set of generators of $S_{\overline{A}}$, then this semigroup can be considered as the set of words over the alphabet X_m, whence $S_{\overline{A}} \times X_m \subseteq X_m^*$. Therefore for any $u \in S_{\overline{A}}$ and $\mathsf{x} \in X_m$ the equality (2.1) follows from $\mathsf{s}_1(w) = \mathsf{s}_2(w)$ for $w = u\mathsf{x}$.

Now, let the equality (2.1) holds, and assume by contradiction that s_1 and s_2 are non-equivalent. Then there exists the word $v = w\mathsf{x} \in X_m^*$, $w \in X_m^*$, $\mathsf{x} \in X_m$, of minimal length such that

$$\mathsf{s}_1(w) = \mathsf{s}_2(w) \qquad \text{and} \qquad \mathsf{s}_1(v) \neq \mathsf{s}_2(v),$$

whence

$$\lambda(\mathsf{x}, \pi(w, \mathsf{s}_1)) \neq \lambda(\mathsf{x}, \pi(w, \mathsf{s}_2)).$$

The word w can be considered as the semigroup word, and there exists the equivalent element $u \in S_{\overline{A}}$, $u = w$. It follows from Proposition 2.12 that for all elements $\mathsf{s} \in S_A$ the equality

$$\pi(u, \mathsf{s}) = \pi(w, \mathsf{s})$$

holds. Then

$$\mathsf{s}_i(u\mathsf{x}) = \lambda(u, \mathsf{s}_i) \cdot \lambda(\mathsf{x}, \pi(u, \mathsf{s}_i)) = \lambda(u, \mathsf{s}_i) \cdot \lambda(\mathsf{x}, \pi(w, \mathsf{s}_i)),$$

where $i = 1, 2$, and therefore the outputs $\mathsf{s}_1(u\mathsf{x})$ and $\mathsf{s}_2(u\mathsf{x})$ differ at least in the end symbol. Hence, the inequality $\mathsf{s}_1(u\mathsf{x}) \neq \mathsf{s}_2(u\mathsf{x})$ holds that contradicts the conditions of the theorem. $\square$

In addition, the following proposition show that semigroup of origin and dual automaton are infinite simultaneously.

Proposition 2.14. *Let* A *be an arbitrary automaton. Then the semigroup* S_A *is infinite if and only if the semigroup* $S_{\overline{A}}$ *is infinite.*

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3. Method and techniques of the research

Let O be some object, and we denote a model of O by the symbol $M(O)$.

3.1. Application of mathematical modelling method

The investigated sets are countable infinite sets of discrete objects. On the one hand, it is unknown how to find automata that demonstrates interesting properties. On the other hand, the studying of particular automata can't show patterns of relationship in the set of all Mealy automata. Therefore we propose to use the mathematical modelling method in the research of growth of the set $\mathfrak{A}$. The mathematical modelling method requires to construct a model of the growth of $\mathfrak{A}$, that have lesser complexity but preserve investigated properties [17]. As the sets $\mathfrak{S}$, $\Gamma_{\mathfrak{A}}$, $\Gamma_{\mathfrak{S}}$ is defined by $\mathfrak{A}$, then the model $M(\mathfrak{A})$ defines in the same way the models $M(\mathfrak{S})$, $M(\Gamma_{\mathfrak{A}})$, $M(\Gamma_{\mathfrak{S}})$ of the corresponding objects.

The research starts from studying some Mealy automata. Investigating examples, we study properties of these automata, and it leads us to hypotheses that concern the properties of the growth of the set $\mathfrak{A}$. Considering the growth of all Mealy automata, we check up these hypotheses, and it may produce new examples of automata and corresponding semigroups with interesting properties of growth. Therefore, the investigations of the growth of certain automata and the growth of the set $\mathfrak{A}$ should be carried out simultaneously and exchange by results and hypotheses.

But, many questions should be considered and answered until the start of modelling:

- what is the accuracy of adequate model;
- how the automata that demonstrate the most "typical" properties should be selected;
- what properties (excepting tabulated values of functions) should be simulated;
- how the data should be analyzed;
- how hypotheses can be constructed and proved.

It follows from the notes above, that two parallel modelling are carrying out: it's necessary to construct the model $M(\mathfrak{A})$ of the set $\mathfrak{A}$, and the model of the growth of $M(\mathfrak{A})$. The last model will be correct, if $M(\mathfrak{A})$ contains elements with the most "typical" growth properties.

The set $\mathfrak{A}$ can be presented as the infinite join of the finite sets

$$\mathfrak{A} = \bigcup_{n\geqslant 1} \bigcup_{m\geqslant 1} A_{n\times m}.$$

But the set $\mathfrak{A}$ is not just a join of the sets $A_{n\times m}$. Many properties is appeared only if the set $\mathfrak{A}$ (or the model $M(\mathfrak{A})$) is considered as a whole object. For example, the investigations of $A_{2\times 2}$ discover several "basic" properties of growth [22]. We mark out some of these properties:

- $A_{2\times 2}$ includes automata of all "main" (polynomial, intermediate, and exponential) growth;
- all growth functions of automata from $A_{2\times 2}$, excluding the growth function of intermediate growth, can be described by finite recurrent relations;
- the growth function of the automaton I_2 of intermediate growth has the growth order $[\exp\sqrt{n}]$ [23, 2].

Therefore, the model $M(\mathfrak{A})$ can be constructed as a join of the models $M(A_{n\times m})$. Obviously, more accurate models $M(A_{n\times m})$ provides more adequate model for $\mathfrak{A}$.

In order to construct the model of $A_{n\times m}$, we should consider automata from this set. The set $A_{n\times m}$ contains finite count of automata, but it's not necessary consider all of them. In Section 4 we describe equivalence relations, that allows choose representatives in $A_{n\times m}$. Each of this representatives should be modelled in order to obtain its growth function and semigroup. We propose to model the automaton by considering sequentially its degrees. Depending of the number considered powers and the complexity of the automaton, the model with some degree of certainty describes the properties of the automaton.

Note, that each set $A_{n\times m}$ can be modelled separately, but automata from sets with different parameters may have interrelation. Therefore the modelling of a particular automaton may use previously obtained information.

3.2. Modelling of automata and automaton set

The properties and data that describe investigated objects are separated into three categories: the initial data, the structural properties and the asymptotical behavior.

The initial data includes data that describe an object, and is used as a basis for the studying of particular object. An arbitrary Mealy

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automaton is fully described by the number of states, the cardinality of the alphabet, and the Moore diagram. The transition and output functions can be used instead of Moore diagram. Investigated sets of Mealy automata are described by the fixed initial set of automata, and the construction algorithm that produce the next automaton.

An arbitrary semigroup is defined by the set of generators and defining relations, and a growth function is fully defined by close formulae or by the growth series. Let's note, that when the researcher starts to consider some automaton A, then the set of defining relations of the corresponding semigroup and close formulae for growth functions are objectives of the investigations. That is why the initial data of S_A is the set of generators, compounded from all automaton transformations of A, and initial data of the growth function γ_A is some number of tabulated values.

The structural properties describe inner structure of an object, if it is considered independently of close related objects. For example, automaton transformation semigroups are considered as abstract semigroups, and growth functions are considered as positively defined functions of a natural argument. These properties are used for the studying correlations between the properties of related objects. Mainly, we consider those, that can be used in the studying of asymptotical behavior and the growth properties.

The structural properties of an semigroup include the normal form of semigroup elements, the reduction algorithm, ideal structure of semigroup, semigroup identities. The structural properties of a Mealy automaton include the properties of its automaton transformations, that are already included into the structural properties of the corresponding semigroup, and the properties of the Moore diagram when it is considered as a labelled oriented graph.

The structural properties of growth functions includes different classifications of functions: monotonic or non-monotonic, composite or non-composite, bounded or unbounded, and so on. Depending on these properties, various numerical properties of functions are considered. One of the most interesting questions are the questions when the close formulae can be constructed, which partitions of positive integers are used for it, when the growth function can be described by recurrent relations.

The properties of asymptotical behavior include the asymptotical behavior of particular object and the asymptotical properties of indexed sets of objects as its index tends to infinity. As we study growth, then primary asymptotics properties for all investigated objects are its growth orders. The asymptotical behavior of growth functions also include asymptotics, growth series. For the set $\Gamma_{\mathfrak{A}}$ of automaton growth functions the statical distribution by degree ratio and growth orders is investigated. The

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asymptotical properties of semigroups include Hausdorff dimensions, and the properties of semigroup word actions over the boundary of X_m^*. The notion of n-state automata sequences (see Section 5) allow consider the limit of sequences of semigroups and growth functions.

3.3. Implementation of the method

We propose the following implementation of mathematical modelling method to the investigations of the growth of $\mathfrak{A}$. Results of the modelling of automata, semigroups and growth functions are collect in data warehouse. The researcher analyze obtained data, and set up various hypotheses concerning the properties of particular automata and the automaton sets. Each hypothesis is checked against the set of already considered automata, and, if necessary, additional automata are considered. Investigations of these automata are added to the main investigation. Also, basing on the proved facts, the process of modelling may be changed.

During last years information technologies are successfully applied in various branches of mathematics, and various specialized programming system (for ex., GAP for groups, Singular for algebraic geometry) or general-purpose programming system (for ex., MathLab, Mathematica) are widely used. But such approach were not applied to the investigations of Mealy automata growth. Therefore the key point of the research is the using of specialized programming system. But the programming complex is just a tool that implements the method of investigations.

Thus, the method applying contains three stages: collecting data; analyzing obtained data and setting up various hypotheses; and proving hypotheses or constructing contrary.

The first stage can be implemented by the special interactive programming complex. Possible structure of the programming complex are considered in [22, 24]. It models Mealy automata, starting from the initial data on automata, semigroup and growth functions, forms aggregative data, and made predictions. All these data are stored in databases. In addition, sometimes the researcher don't know exactly what kind of automata he/she is looking for. That is why such programming complex should be constructed as the interactive programming system, that quickly responds on user's challenges, allow easy change modelled sets, set or remove restrictions of the research.

Preliminary analysis of data, predicting, and hypothesis formulating can be implemented by mathematician who used intellectual software with adaptive and self-trained algorithms. Mainly, the researcher studies the structural and asymptotics properties, basing on the aggregate data, obtained by the complex. He/she looks over these data, considers various

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diagrams and graphs, and then sets up hypotheses concerning the current questions. The programming complex may check these hypothesis over the set of modelled automata and semigroups, but it can't prove them.

The last stage is not formalized and therefore is the most complex. The mathematician should prove these hypotheses by mathematical methods. Checks by the programming systems at the most may produce the contrary. The rest of proving is made by the researcher.

4. Equivalence relations over the set $\mathfrak{A}$, defined by growth

In this section we introduce some equivalences that are used in the growth investigations.

4.1. Definitions

Recall, that two Mealy automata A_1 and A_2 is called equivalent if $F_{A_1} = F_{A_2}$.

Definition 4.1. The Mealy automata $A_i = (X_m, Q_n, \pi_i, \lambda_i)$ for $i = 1, 2$ are called *similar* if there exist permutations $\xi \in Sym(X_m)$ and $\theta \in Sym(Q_n)$ such that

$$\theta\pi_1(\mathsf{x}, \mathsf{f}) = \pi_2(\xi\mathsf{x}, \theta\mathsf{f}), \qquad \xi\lambda_1(\mathsf{x}, f) = \lambda_2(\xi\mathsf{x}, \theta\mathsf{f})$$

for all $\mathsf{x} \in X_m$ and $\mathsf{f} \in Q_n$.

Proposition 4.2. *[18] Let $A_i = (X_m, Q_n, \pi_i, \lambda_i)$, $i = 1, 2$, be two similar Mealy automata. Then these automata define the isomorphic automatic transformation semigroups and have the same growth function.*

Using technique from [26], we prove the formula of the number of classes of the similarity.

Proposition 4.3. *The maximal number $E(n, m)$ of pairwise non-similar automata from the set $A_{n\times m}$ is defined by the following equality:*

$$E(n,m) = \frac{1}{n!\cdot m!}\cdot \sum_{\substack{\xi\in Sym(X_m)\\ \theta\in Sym(Q_n)}} \prod_{p=1}^{m}\prod_{q=1}^{n}\left(\sum_{s,t\,|\,[p,q]} s\,t\,j_s(\xi^{-1})j_t(\theta^{-1})\right)^{j_p(\xi)j_q(\theta)\langle p,q\rangle}$$

where $[p, q]$ is the minimal common multiple of the numbers p and q, $\langle p, q\rangle$ is the maximal common divider, and $j_s(\sigma)$ — the number of loops of length s of the permutation σ.

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4.2. Equivalences

We introduce several equivalence relations over the set of Mealy automata that are defined by growth functions. Let A_1 and A_2 be arbitrary Mealy automata. Let S_i be automaton transformation semigroup, defined by A_i, and the symbol $A_i{}'$ denotes the result of applying the minimization algorithm to A_i, $i = 1, 2$.

The following table contains the list of the relation notations and conditions where the pair (A_1, A_2) belongs to the particular equivalence relation.

$\mathbb{A}_{\mathbb{I}}$: the automata A_1 and A_2 are Q-isomorphic;

$\mathbb{A}_{\mathbb{S}}$: the automata A_1 and A_2 are similar;

$\mathbb{A}_{\mathbb{E}}$: the minimized automata $A_1{}'$ and $A_2{}'$ are similar;

$\mathbb{S}_{\mathbb{I}}$: the semigroups S_1 and S_2 are isomorphic, and the identity is included to the system of generators if S_1, S_2 are monoids;

$\mathbb{G}_{\mathbb{E}}$: the growth functions γ_{A_1} and γ_{A_2} of the automata A_1 and A_2, respectively, are coincide for all values $n \geqslant 1$;

$\mathbb{G}_{\mathbb{C}}$: there exists $N \in \mathbb{N}$ such that the growth functions γ_{A_1} and γ_{A_1} are coincide for all values $n \geqslant N$;

$\mathbb{G}_{\mathbb{O}}$: the growth orders of automata coincide, that is the equality $[\gamma_{A_1}] = [\gamma_{A_2}]$ holds.

These relations concern the main questions that are studied in the research of growth. The first three equivalence are close related each other. The most used of them is the equivalence $\mathbb{A}_{\mathbb{S}}$, and the equivalence $\mathbb{A}_{\mathbb{E}}$ obtained as the expanding of $\mathbb{A}_{\mathbb{S}}$ from $A_{n\times m}$ to the set $\mathfrak{A}$. Below we consider these interrelations in details. It follows from Proposition 4.2 that similar automata have the same growth functions, and therefore pairwise non-similar automata can be selected as representatives of the set $A_{n\times m}$ for the research of the growth of $\mathfrak{A}$. Proposition 4.3 provide the formulae for count of pairwise non-similar automata. It gives $E(2, 2) = 76$, $E(2, 3) = E(3, 2) = 4\,003$, $E(3, 3) = 10\,766\,772$, and so on. It much lesser than the cardinality $|A_{n\times m}| = n!\cdot m!$. Unfortunately, the numerical characteristics of other relations from this list are unknown.

The structure of the relation $\mathbb{S}_{\mathbb{I}}$ shows the relations between the sets $\mathfrak{A}$ and $\mathfrak{S}$. Empirical results provide several hypotheses that describe these relations. One of them is that ratio of the count of classes of relations $\mathbb{A}_{\mathbb{S}}$ and $\mathbb{S}_{\mathbb{I}}$ that contains only automata from the set $A_{n\times m}$ tends to one

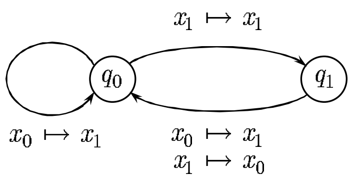

Figure 2: The automaton B_1

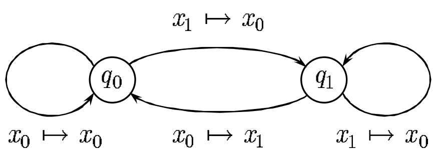

Figure 3: The automaton B_2

as n, m tends to the infinity. It means that for large enough n, m almost all non-similar automata define non-isomorphic semigroups.

The investigations of the relation $\mathbb{G}_{\mathbb{E}}$ is a objective of the research of $\Gamma_{\mathfrak{A}}$, because the classes of $\mathbb{G}_{\mathbb{E}}$ is in one-to-one correspondence with the functions from $\Gamma_{\mathfrak{A}}$. But construction of these classes isn't well investigated.

The classes of the equivalence $\mathbb{G}_{\mathbb{O}}$ discover the behavior of growth function at initial values of arguments. Experiments provide many examples of growth functions that differ for N first values, and then coincide for all $n \geqslant N$. We call this number N by *stable period*. This definition is substantiated by the observation that such functions are describe by close formulae exactly for all $n \geqslant N$.

One of the first question concerning the set $\Gamma_{\mathfrak{A}}$ is what growth orders are demonstrated by growth functions. Classes of the equivalence $\mathbb{G}_{\mathbb{O}}$ corresponds to the set of all possible growth orders of Mealy automata. Also, construction of these classes answers the questions such as what is the maximal power of polynomial growth functions depending on initial parameters, what intermediate growth orders may have automata from some set $A_{n\times m}$, others.

4.3. Examples

In the section we consider several automata that show interrelations between introduced equivalences.

Let B_1, B_2 and B_3 be Mealy automata, shown on Fig. 2, 3 and 4 respectively, and let S_{B_i} be the automaton transformation semigroup, defined by B_i, $i = 1, 2, 3$. Let Φ_n denote the Fibonacci numbers, defined

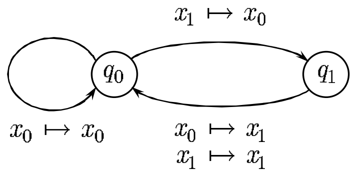

Figure 4: The automaton B_3

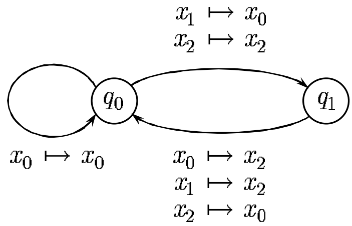

Figure 5: The automaton C_1

by $\Phi_n = \Phi_{n-1} + \Phi_{n-2}$, $\Phi_1 = \Phi_2 = 1$. The following theorem holds.

Theorem 4.4. *1. The semigroups S_{B_1} and S_{B_2} coincide, and has the following presentation:*

$$S_B = \langle\ f_0, f_1\ \mid\ f_1^2 f_0 = f_0 f_1 f_0\ \rangle.$$

2. The semigroups S_{B_3} has the following presentation:

$$S_{B_3} = \langle\ f_0, f_1\ \mid\ f_1^2 f_0 = f_1 f_0^2\ \rangle.$$

3. The automata B_1, B_2 and B_3 define the same growth function γ_B of exponential growth, that is defined by the following equality:

$$\gamma_B(n) = \Phi_{n+3} - 1,\ n \geqslant 1.$$

Obviously, all automata B_1, B_2 and B_3 are minimized, and not similar. But it follows from Theorem 4.4 that these automata define just two different semigroup. That is why the pair (B_1, B_2) belongs to $\mathbb{S}_\mathbb{I}$, but doesn't belong to $\mathbb{A}_\mathbb{E}$. In addition, the pair (B_2, B_3) belongs to $\mathbb{G}_\mathbb{E}$, but doesn't belong to $\mathbb{S}_\mathbb{I}$.

Let C_1 and C_2 be Mealy automata, shown on Fig. 5 and 6 respectively, and let S_{C_i} be the automaton transformation semigroup, defined by C_i, $i = 1, 2$. The following theorems describe properties of these automata.

Theorem 4.5. *1. The semigroup S_{C_1} has the following presentation:*

$$S_{C_1} = \langle\ f_0, f_1\ \mid\ f_i^2 f_j = f_j,\ i, j = 0, 1\ \rangle.$$

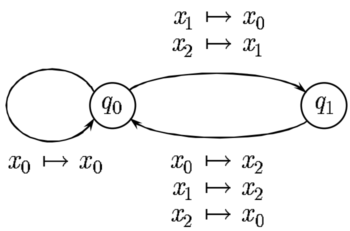

Figure 6: The automaton C_2

2. *The growth function γ_{C_1} of C_1 have the linear growth, and is defined by the equality:*

$$\gamma_{C_1}(n) = 2n, \ n \geqslant 1.$$

Theorem 4.6. *1. The semigroup S_{C_2} has the following presentation:*

$$S_{C_2} = \left\langle \ f_0, f_1 \ \middle| \ \begin{array}{l} f_i f_j^2 f_{1-j} = f_i f_0^3, \ f_i^2 f_{1-i} f_j = f_0^4, \\ f_i f_0^4 = f_i f_0^3, \ f_i f_1^5 = f_i f_1^3, \ i, j \in \{0, 1\} \end{array} \right\rangle.$$

2. *The growth function γ_{C_2} of C_2 have the linear growth, and is defined by the equality:*

$$\gamma_{C_2}(n) = 2^n, \ n = 1, 2, 3 \qquad \textit{and} \qquad \gamma_{C_2}(n) = 2n, \ n \geqslant 4.$$

The growth functions of C_1 and C_2 coincide for all $n \geqslant 4$, but differs for value of argument $n = 3$. Therefore the pair (C_1, C_2) belongs to the equivalence $\mathbb{G}_{\mathbb{C}}$, but doesn't belong to $\mathbb{G}_{\mathbb{E}}$.

4.4. The properties of relations

The main result of this section is the theorem of inclusions between introduced equivalences.

Theorem 4.7. *The following inequalities hold*

$$\mathbb{A}_{\mathbb{I}} \subsetneq \mathbb{A}_{\mathbb{S}} \subsetneq \mathbb{A}_{\mathbb{E}} \subsetneq \mathbb{S}_{\mathbb{I}} \subsetneq \mathbb{G}_{\mathbb{E}} \subsetneq \mathbb{G}_{\mathbb{C}} \subsetneq \mathbb{G}_{\mathbb{O}}.$$

At first, consider the properties of similar automata:

Proposition 4.8. *Let $A_i = (X_m, Q_n, \pi_i, \lambda_i)$, $i = 1, 2$, be two similar Mealy automata such that*

$$\theta\pi_1(\mathsf{x}, \mathsf{f}) = \pi_2(\xi\mathsf{x}, \theta\mathsf{f}) \qquad \textit{and} \qquad \xi\lambda_1(\mathsf{x}, f) = \lambda_2(\xi\mathsf{x}, \theta\mathsf{f})$$

for permutations $\xi \in Sym(X_m)$ and $\theta \in Sym(Q_n)$. Then the following equality

$$f_{\mathsf{f},A_1}(u) = \xi^{-1} f_{\theta\mathsf{f},A_2}(\xi u) \tag{4.2}$$

holds for any $\mathsf{f} \in Q_n$ and $u \in X_m^*$.

Proof. The transition and the output functions of A_1 and A_2 satisfy the following equalities:

$$\begin{aligned} \pi_1(\mathsf{x},\mathsf{f}) &= \theta^{-1}\pi_2(\xi\mathsf{x},\theta\mathsf{f}), & \lambda_1(\mathsf{x},\mathsf{f}) &= \xi^{-1}\lambda_2(\xi\mathsf{x},\theta\mathsf{f}), \\ \pi_2(\mathsf{x},\mathsf{f}) &= \theta\pi_1(\xi^{-1}\mathsf{x},\theta^{-1}\mathsf{f}), & \lambda_2(\mathsf{x},\mathsf{f}) &= \xi\lambda_1(\xi^{-1}\mathsf{x},\theta^{-1}\mathsf{f})), \end{aligned} \tag{4.3}$$

for all $\mathsf{x} \in X_m$, $\mathsf{f} \in Q_n$.

Let $\mathsf{f} \in Q_n$ be an arbitrary state and $u \in X_m^*$ be an arbitrary word, $u = u_0u_1\dots u_{k-1}$. Using an induction on the length of the word u, let's prove that the equality (4.2) holds. For $k = 1$ from (4.3) immediately follows

$$f_{\mathsf{f},A_1}(u_0) = \lambda_1(u_0,\mathsf{f}) = \xi^{-1}\lambda_2(\xi u_0,\theta\mathsf{f}) = \xi^{-1}f_{\theta\mathsf{f},A_2}(\xi u_0),$$

and (4.2) holds. Let $k > 1$ and denote $u_0u_1\dots u_{k-2}$ by the symbol v. It follows from the induction hypothesis and (4.3) that the equalities

$$\begin{aligned} f_{\mathsf{f},A_1}(u) &= \lambda_1(u_0u_1\dots u_{k-2}u_{k-1},\mathsf{f}) = \lambda_1(vu_{k-1},\mathsf{f}) \\ &= \lambda_1(v,\mathsf{f})\cdot\lambda_1(u_{k-1},\pi_1(v,\mathsf{f})) \\ &= \xi^{-1}\lambda_2(\xi v,\theta\mathsf{f})\cdot\xi^{-1}\lambda_2(\xi u_{k-1},\theta\theta^{-1}\pi_2(\xi v,\theta\mathsf{f})) = \\ &= \xi^{-1}\lambda_2(\xi(vu_{k-1}),\theta\mathsf{f}) = \xi^{-1}f_{\theta\mathsf{f},A_2}(\xi u), \end{aligned}$$

holds, whence (4.2) is true. □

Let's note, that statement of the proposition can be inverted, whence the following equality holds

$$f_{\mathsf{f},A_2}(u) = \xi f_{\theta^{-1}\mathsf{f},A_1}(\xi^{-1}u)$$

for all $u \in X_m^*$.

Proof of Theorem 4.7.

It follows from Definitions 2.3 and 4.1 that Q-isomorphic automata are similar for $\xi = e$, where e is the identical permutation, whence $\mathbb{A}_{\mathbb{I}} \subsetneq \mathbb{A}_{\mathbb{S}}$.

By definition, similar automata contains the same number of states. Let q_i and q_j be the equivalent states of A_1. Then for any $u \in X_m^*$ the equality $f_{q_i,A_1}(u) = f_{q_j,A_1}(u)$ holds. Therefore the equality

$$\xi f_{\theta^{-1}q_i,A_1}(\xi^{-1}u) = \xi f_{\theta^{-1}q_j,A_1}(\xi^{-1}u)$$

holds, and it follows from Proposition 4.8 that the equality $f_{\theta q_i,A_2}(u) = f_{\theta q_j,A_2}(u)$ holds. Thus, the states θq_i and θq_j are equivalent states of A_2, and the equivalence $\mathbb{A}_{\mathbb{S}}$ is included in $\mathbb{A}_{\mathbb{E}}$. As two automata that

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belongs to the equivalence $\mathbb{A}_{\mathbb{E}}$ may have different number of states, then the inequality $\mathbb{A}_{\mathbb{S}} \subsetneq \mathbb{A}_{\mathbb{E}}$ is true.

The equivalent automata define the same semigroup, and it follows from Proposition 4.2 that similar automata define isomorphic semigroup. Therefore if A_1 and A_2 such that (A_1', A_2') belongs to $\mathbb{A}_{\mathbb{S}}$, then they define isomorphic semigroups. Let us consider automata B_1 and B_2, shown on Fig 2 and 3, respectively. The inequality $\mathbb{A}_{\mathbb{E}} \neq \mathbb{S}_{\mathbb{I}}$ follows from the notes after Theorem 4.4. Thus, the inequality $\mathbb{A}_{\mathbb{E}} \subsetneq \mathbb{S}_{\mathbb{I}}$ is proved.

Let us denote the count of semigroup words of the semigroup S, that can be presented as a product of n generators, by the symbol $\delta_S(n)$. It follows from Proposition 2.7 that $\gamma_A(n) = \delta_{S_A}(n)$. Let us assume, that A_1 and A_2 define the isomorphic monoid. Then any semigroup word of length k, $k \leqslant n$, can be expand to the word of length n by the product of $(n-k)$ identities. Therefore $\gamma_{S_{A_i}}(n) = \delta_{S_{A_i}}(n)$ for all $n \geqslant 1$. As by definition isomorphic semigroups define the same growth functions, then $\delta_{S_{A_1}}(n) = \delta_{S_{A_2}}(n)$ for all $n \geqslant 1$, and the growth functions of A_1 and A_2 are equal.

Now let S_{A_i}, $i = 1, 2$ be isomorphic semigroups without identity. Then there exists the isomorphism $\phi : S_{A_1} \to S_{A_2}$, $\phi(g_1 g_2 \ldots g_n) = \phi(g_1)\phi(g_2)\ldots\phi(g_n)$, where $n \geqslant 1$ and g_i are generators of S. The isomorphism ϕ maps generators of S_{A_1} to generators of S_{A_2}. Therefore ϕ sets a bijection between the sets of semigroup elements of the both semigroup that can be presented as a product of n generators. Hence, $\delta_{S_{A_1}}(n) = \delta_{S_{A_2}}(n)$ for all $n \geqslant 1$. The inclusion $\mathbb{S}_{\mathbb{I}} \subseteq \mathbb{G}_{\mathbb{E}}$ is proved. Above we consider the automata B_2 and B_3. It follows from Theorem 4.4 that these automata have the same growth function, but don't define the isomorphic semigroups.

If present the growth functions as arbitrary positively defined functions of a natural argument, then the inclusions $\mathbb{G}_{\mathbb{E}} \subseteq \mathbb{G}_{\mathbb{C}} \subseteq \mathbb{G}_{\mathbb{O}}$ are obvious. It follows from Theorems 4.5 and 4.6 that $\mathbb{G}_{\mathbb{E}} \neq \mathbb{G}_{\mathbb{C}}$. The growth order doesn't depend on particular values of functions, and, for example, all functions of exponential growth order has the growth order $[\exp^n]$. Therefore stated inequalities $\mathbb{G}_{\mathbb{E}} \subsetneq \mathbb{G}_{\mathbb{C}} \subsetneq \mathbb{G}_{\mathbb{O}}$ are correct.

The theorem is completely proved. □

5. The Mealy automaton sequences

5.1. Definitions

Let $n \geqslant 2$ and $k \geqslant 2$ are fixed. We call the sequence $\mathfrak{A} = \{A_m, m \geqslant k\}$ of Mealy automata such that the automaton A_m belongs to $A_{n\times m}$ by *the n-state Mealy automaton sequence*. In the sequel text, we use notion "an

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automaton sequence".

The transition and output functions of each A_m are discrete functions, and we may consider their pointwise limits as m tends to infinity. It allows define the pointwise limit of automaton sequence as the automaton over the infinite alphabet.

Definition 5.1. [21] Let $\mathbf{A} = \{A_m = (X_m, Q, \pi_m, \lambda_m), m \geqslant k\}$, $k \geqslant 2$, be an arbitrary n-state Mealy automaton sequence. The automaton $A_\infty = (X, Q, \pi, \lambda)$ is called *the limit automaton* of the sequence $\mathfrak{A}$, if for any state $q \in Q$ and any symbol $x \in X$ there exists the number $M \geqslant k$ such that the equalities

$$\pi_m(q, x) = \pi(q, x) \qquad \text{and} \qquad \lambda_m(q, x) = \lambda(q, x)$$

hold for all $m \geqslant M$.

Each automaton A_m defines the automaton transformation semigroup S_{A_m}, where we fix the natural set of generators, and the growth function γ_{A_m}. Hence the automaton sequence $\mathbf{A}$ defines the sequence of the semigroups $\mathbf{S} = \{S_{A_m}, m \geqslant k\}$ and the sequence of the growth functions $\mathbf{G} = \{\gamma_{A_m}, m \geqslant k\}$. The limit function $\gamma_{\mathbf{A}}$ of $\mathbf{G}$ is defined as the pointwise limit of γ_{A_m} as $m \to \infty$, if it exists. Otherwise, we say that the limit of $\mathbf{G}$ doesn't exist.

Similarly, we define the limit of the semigroup sequence $\mathbf{S}$, if semigroups compose increasing (S_{A_m} is a factor-semigroup of $S_{A_{m+1}}$) or decreasing ($S_{A_{m+1}}$ is a factor-semigroup of S_{A_m}) sequence. Let R_i be the set of relations of the semigroup S_{A_i}, $i \geqslant k$. Then the following relations

$$R_k \supseteq R_{k+1} \supseteq \ldots \supseteq R_m \supseteq \ldots$$

or

$$R_k \subseteq R_{k+1} \subseteq \ldots \subseteq R_m \subseteq \ldots$$

hold, and the semigroup $S_{\mathbf{A}}$ is defined as the semigroup with the set of defining relations equals the join or the intersection of semigroups from $\mathbf{S}$ respectively.

Studied examples of automaton sequences provide finite automata, that define the same semigroup as the limit automaton. Therefore let us introduce notion of the finite limit automaton.

Definition 5.2. [21] Let $\mathbf{A} = \{A_m, m \geqslant k\}$, $k \geqslant 2$, be an arbitrary n-state Mealy automaton sequence. We say that the n-state automaton B over the finite alphabet is *the finite limit automaton* of the sequence $\mathbf{A}$, if the equalities $\gamma_B = \gamma_{\mathbf{A}}$ and $S_B \cong S_{\mathbf{A}}$ hold.

Note, that in general case the finite limit automata form some set. The question is appeared how define those of them that are related to the sequence $\mathbf{A}$.

5.2. Expanding sequences

Let $A = (X_m, Q, \pi, \lambda) \in A_{n\times m}$ be an arbitrary automaton. The n-state automaton $A_1 = (X_{m+1}, Q, \pi_1, \lambda_1)$ over the $(m+1)$-symbol alphabet such that the equalities

$$\pi_1(\mathsf{x}, \mathsf{f}) = \pi(\mathsf{x}, \mathsf{f}) \qquad \text{and} \qquad \lambda_1(\mathsf{x}, \mathsf{f}) = \lambda(\mathsf{x}, \mathsf{f})$$

hold for all $\mathsf{x} \in X_m$, $\mathsf{f} \in Q$, is called *the extension* of A [3]. Note that for all $\mathsf{f} \in Q$ and $u \in X_m^\omega$ the equality holds

$$f_{\mathsf{f},A_1}(u) = f_{\mathsf{f},A}(u).$$

Definition 5.3. We say that the n-state Mealy automaton sequence is *expanding* if the automaton A_{m+1} is an extension of the automaton A_m for all $m \geqslant k$.

Let $\mathbf{A}$ be an expanding automaton sequence. The following proposition show some useful properties of expanding sequences.

Proposition 5.4. *1. The semigroup S_{A_m} is a factor-semigroup of the semigroup $S_{A_{m+1}}$ for all $m \geqslant k$.*

2. There exists a unique finite or infinite sequence

$$k = m_1 < m_2 < m_3 < \dots$$

such that for all $i \geqslant 1$ the automata A_{m_i} and $A_{m_{i+1}}$ have different growth functions, and all automata A_m for $m_i \leqslant m < m_{i+1}$, have the same growth function and define the same semigroup.

This Proposition allows describe the limit of $\mathbf{S}$ and $\mathbf{G}$.

Proposition 5.5. *For any expanding automaton sequence exist the limits of the sequences of semigroups $\mathbf{S}$ and growth functions $\mathbf{G}$. In addition,*

1. There exists the sequence of integers

$$1 \leqslant N_{m_1} < N_{m_2} < N_{m_3} < \dots$$

such that the pointwise limit of $\mathbf{G}$ is defined by the equality

$$\gamma_{\mathbf{A}}(n) = \begin{cases} \gamma_{A_{m_1}}(n), & \text{if } 1 \leqslant n < N_{m_1}; \\ \gamma_{A_{m_2}}(n), & \text{if } N_{m_1} \leqslant n < N_{m_2}; \\ \gamma_{A_{m_3}}(n), & \text{if } N_{m_2} \leqslant n < N_{m_3}; \\ \dots, & \dots \end{cases}$$

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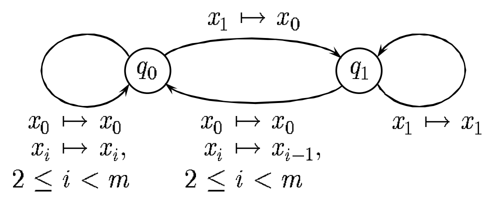

Figure 7: Automata P_m

2. For $i \geqslant 1$ there exists the set R_i of defining relations in the semigroup $S_{A_{m_i}}$ such that

$$R_1 \supseteq R_2 \supseteq R_3 \supseteq \dots,$$

whence the defining relation set of the semigroup $S_{\mathbf{A}}$ is defined by the equality

$$R_{S_{\mathbf{A}}} = \bigcap_{i \geqslant 1} R_i.$$

Below we considered several examples. First two examples are expanding automaton sequences **P** and **E** of automata of polynomial and exponential growth orders. The third example demonstrates the non-expanding automaton sequence **I** of automata of intermediate growth order. In the last case the sequence of growth functions is monotonic decreasing growth functions that tends to polynomial function.

5.3. Examples

1. Let $\mathbf{P} = \{P_m, m \geqslant 3\}$ be automaton sequence such that the automaton P_m is shown on Fig. 7.

Theorem 5.6 ([19]). *For any $m \geqslant 3$ the semigroup S_{P_m}, defined by P_m, has the following presentation:*

$$S_{P_m} = \left\langle \ f_0, f_1 \ \middle| \ \begin{array}{l} f_1 f_0^{p_1} f_1 \prod\limits_{i=2}^{m-2} (f_0^{p_i} f_1) = f_0^{p_1+1} f_1 \prod\limits_{i=2}^{m-2} (f_0^{p_i} f_1) \\ p_1 = 1, 2; p_2, p_3, \dots, p_{m-2} \geqslant 0 \end{array} \right\rangle .$$

All semigroups S_{P_m} for $m \geqslant 4$ are infinitely presented.

Theorem 5.7 ([19]). *1. For $m \geqslant 3$ the growth function γ_{P_m} is defined by the following equality:*

$$\gamma_{P_m}(n) = \sum_{i=0}^{m-1} C_n^i, \ n \geqslant 1,$$

where C_n^i is a binomial coefficient.

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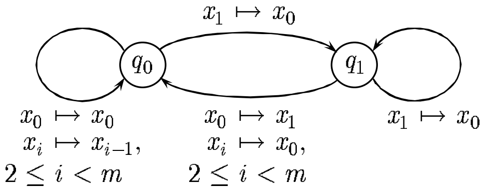

Figure 8: Automata E_m

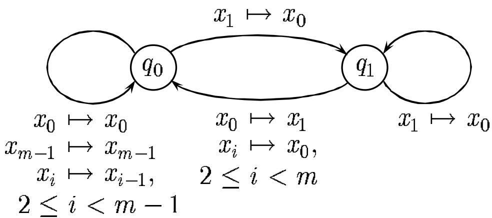

Figure 9: Automata L_m

2. *The pointwise limit of the sequence $\{\gamma_{P_m}, m \geqslant 3\}$ of polynomial growth functions is the exponential function 2^n, that is for any positive integer $n \geqslant 1$ the equality holds*

$$\lim_{m\to\infty} \gamma_{P_m}(n) = 2^n.$$

The sequence $\mathbf{P}$ is the expanding sequence of automata of polynomial growth order. But it follows from Theorem 5.7 that the pointwise limit of increasing sequence of polynomial growth orders is not polynomial growth order. Also, there exist plenty of 2-state automata that define free semigroup, and it is not easy to separate those that can be considered as finite limit of $\mathbf{P}$, and relate to the sequence. Some of 2-state automata that define free semigroup are consider in [10], [18], etc.

2. Let L_m for $m \geqslant 3$ be the automaton, shown on Fig. 9, and $\mathbf{E} = \{E_m, m \geqslant 3\}$ be automaton sequence such that the automaton E_m is shown on Fig. 8.

For $p > 1$ the symbol r_p denotes semigroup relation $r_p : f_1^2 f_0^p f_1 = f_0 f_1 f_0^p f_1$. Recall that Φ_n denote the Fibonacci numbers, defined by $\Phi_n = \Phi_{n-1} + \Phi_{n-2}$, $\Phi_1 = \Phi_2 = 1$. The following theorems follow from results of [21].

Theorem 5.8. *1. The automaton E_m for any $m \geqslant 3$ defines the automaton transformation semigroup*

$$S_{E_m} = \left\langle\ f_0, f_1 \ \middle|\ r_p,\ 1 \leqslant p \leqslant m-2;\ f_1^2 f_0^{m-1} = f_0 f_1 f_0^{m-1}\ \right\rangle.$$

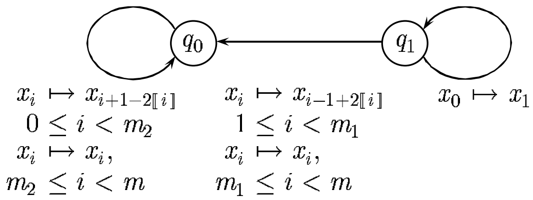

Figure 10: Automata I_m

2. The automaton L_m for any $m \geqslant 3$ defines the semigroup

$$S_L = \langle\ f_0, f_1 \ \mid\ r_p,\, p \geqslant 1\ \rangle.$$

Theorem 5.9. *1. The growth function γ_{E_m} of E_m is defined by the equality*

$$\gamma_{E_m}(n) = \Phi_{n+4} - \begin{cases} (n+2), & \text{if } 1 \leqslant n \leqslant m; \\ \Phi_{n+4-m} + (m-1), & \text{if } n > m. \end{cases}$$

2. The growth function γ_{L_m} of L_m is defined by the equality

$$\gamma_L(n) = \Phi_{n+4} - (n+2),\ n \geqslant 1.$$

3. The functions γ_L is the pointwise limit of the functions γ_{E_m} as m tends to the infinity, that is for each $n \geqslant 1$ the equality hold

$$\gamma_{E_m}(n) \xrightarrow[m\to\infty]{} \gamma_L(n).$$

According to Definition 5.3, $\{E_m, m \geqslant 3\}$ is the expanding sequence of automata of exponential growth orders. The pointwise limit of corresponding growth functions is the function of exponential growth order, and the semigroup S_L may be considered as the limit of the sequence $\{S_{E_m}, m \geqslant 3\}$. It follows from Theorem 5.9 that any automaton L_m is the finite limit automaton for the sequence $\{E_m, m \geqslant 3\}$.

3. Let $\mathbf{I} = \{I_m, m \geqslant 2\}$ be automaton sequence such that the automaton I_m is shown on Fig. 10. Here $[\![p]\!]$ denotes the parity of nonnegative integer p, $m_1 = m - [\![m-1]\!]$, and $m_2 = m - [\![m]\!]$. Let L be automaton shown on Fig. 11.

Theorem 5.10. *The automaton I_m, $m \geqslant 2$ has intermediate growth such that $[\gamma_{I_m}] \leqslant [\exp(\sqrt{n})]$.*

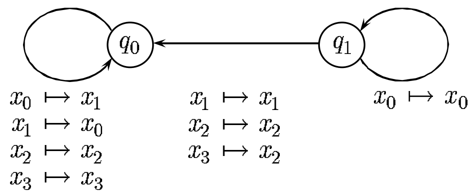

Figure 11: Automaton L

Theorem 5.11. *1. The automaton L defines the following semigroup*

$$S_L = \left\langle f_0, f_1 \;\middle|\; \begin{array}{l} f_1 (f_0 f_1)^{p_1} (f_1 f_0)^{p_1} f_1^2 = f_1 (f_0 f_1)^{p_1} (f_1 f_0)^{p_1}, \\ (f_1 (f_0 f_1)^{p_1} (f_1 f_0)^{p_2})^2 = ((f_0 f_1)^{p_2} (f_1 f_0)^{p_1} f_1)^2, \\ f_0^2 = 1;\ p_1 \geqslant 0, p_2 > 0 \end{array} \right\rangle$$

2. The growth function γ_L of the automaton L is defined by the equality

$$\gamma_L(n) = \frac{1}{96} \left(n^3 + 21n^2 + 92n + \begin{cases} 96, & n = 4l, \\ 15n + 63, & n = 4l + 1, \\ 108, & n = 4l + 2, \\ 3n + 75, & n = 4l + 3. \end{cases} \right).$$

for all $n \in \mathbb{N}$.

3. The functions γ_L is the pointwise limit of the functions γ_{I_m} as m tends to infinity, that is for each $n \geqslant 1$ the equality hold

$$\gamma_{I_m}(n) \xrightarrow[m \to \infty]{} \gamma_L(n).$$

The sequence **I** is not expanding. Empirically shown, that the sequence of the growth functions is the monotonic decreasing sequence, that pointwisely tends to the polynomial growth function γ_L. In addition, the semigroup S_L equals the limit of the semigroup sequence $\{S_{I_m}, m \geqslant 2\}$.

6. Conclusion

The problem of the investigations of Mealy automata growth at iterations is considered in the paper. We mark out the main studied objects, consider the interrelations between them. Moreover we formulate the basic questions and problems that are researched, and consider those properties that are studied by investigations.

The studied objects are infinite countable sets, and their investigations have the large complexity. In the paper we consider the applying of the method of mathematical simulation to these investigations. It includes formalized parts, that can be covered by specialized software, and non-formalized part that requires speculations of mathematicians. In order to reduce the complexity of the research, there were considered several equivalence relations that allow to choose representatives of studied sets, and the using of automaton sequences. But there are many open questions concerning these approaches.

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