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Automorphisms of kaleidoscopical graphs

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Dedicated to V.I. Sushchansky on the occasion of his 60th birthday

Abstract. A regular connected graph Γ of degree s is called kaleidoscopical if there is a $(s+1)$-coloring of the set of its vertices such that every unit ball in Γ has no distinct monochrome points. The kaleidoscopical graphs can be considered as a graph counterpart of the Hamming codes. We describe the groups of automorphisms of kaleidoscopical trees and Hamming graphs. We show also that every finitely generated group can be realized as the group of automorphisms of some kaleidoscopical graphs.

1. Introduction

Let $\Gamma(V,E)$ be a connected graph with the set of vertices V and the set of edges E. Given any $u,v \in V$ and $r \in \mathbb{N}$, we denote by $d(u,v)$ the length of the shortest path between u and v, and put

$$B(v,r) = \{x \in V \ : \ d(x,v) \leqslant r\}.$$

Let $s > 1$ be a natural number. A graph $\Gamma(V,E)$ is said to be *regular of degree s* if $|B(v,1)| = s+1$ for every $v \in V$.

A regular graph $\Gamma(V,E)$ of degree s is called *kaleidoscopical* if there exists a coloring $\chi : V \longrightarrow \{0,1,\dots,s\}$ such that χ is a bijection on every ball $B(v,1)$, $v \in V$. For motivation to study kaleidoscopical graphs as a graph generalization of the Hamming codes see [1, Chapter 6].

Let $\Gamma(V,E)$ and $\Gamma'(V',E')$ be kaleidoscopical graphs of degree s and let $\chi : V \longrightarrow \{0,1,\dots,s\}$, $\chi' : V' \longrightarrow \{0,1,\dots,s\}$ be its kaleidoscopical colorings. A mapping $f : V \longrightarrow V'$ is called a *homomorphism* if

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(i) $\chi(x) = \chi'(f(x))$ for every $x \in V$;

(ii) if $\{x, y\} \in E$ then $\{f(x), f(y)\} \in E'$.

In the case $V = V'$, $\chi = \chi'$, a bijective homomorphism $f : V \longrightarrow V$ is called an *automorphism*. We denote by $Aut(\Gamma(V, E))$ the group of all automorphisms of a kaleidoscopical graph $\Gamma(V, E)$.

We prove that

- for a kaleidoscopical tree T of degree s, $Aut(T)$ is a free group of rank $\frac{s(s-1)}{2}$;
- for a Hamming graph Γ determined by a Hamming pair (X, Y) in a group G, $Aut(\Gamma) = Y$;
- for every finitely generated group H, there exists a kaleidoscopical graph Γ such that $Aut(\Gamma) = H$.

2. Kaleidoscopical and Hamming pairs

Let G be a group with the identity e and let X, Y be subsets of G. We say that (X, Y) is a *kaleidoscopical pair* in G provided that X is finite and the following conditions hold

(i) $e \in X$, $X = X^{-1}$, $G = \langle X \rangle$ where $\langle X \rangle$ is a subgroup generated by X;

(ii) $e \in Y$, $G = XY$;

(iii) $XX \bigcap Y^{-1}Y = XX \bigcap YY^{-1} = \{e\}$.

By this definition, every element $g \in G$ has the unique representation $g = xy$, $x \in X$, $y \in Y$. We define the *standard coloring* $\chi : G \longrightarrow X$ by the rule $\chi(g) = x$. By [1, Theorem 6.2], $Cay(G, X)$ is a kaleidoscopical graph. We remind that the Cayley graph $Cay(G, X)$ is a graph with the set of vertices G and the set of edges

$$\{\{x, y\} \ : \ x, y \in G, x \neq y, xy^{-1} \in X\}.$$

A kaleidoscopical pair (X, Y) is called a *Hamming pair* if Y is a subgroup of X. In this case, the kaleidoscopical graph $Cay(G, X)$ is called a *Hamming graph*.

Theorem 1. Let (X, Y) be a kaleidoscopical pair in a group G, χ be the standard coloring of G. Then the following statement are equivalent

(i) (X, Y) is a Hamming pair;

(ii) if $g_1, g_2 \in G$, $x \in X$ and $\chi(g_1) = \chi(g_2)$ then $\chi(xg_1) = \chi(xg_2)$.

Proof. [1, Theorem 6.3]. □

Let (X, Y) be a Hamming pair in a group G. Since X is finite, G is finitely generated. Since $G = XY$ then Y is a subgroup of finite index. Every subgroup of finite index of a finitely generated group is also of finite index, so Y is finitely generated.

Theorem 2. For every finitely generated group Y, there exist a group G and a finite subset $X \subseteq G$ such that Y is a subgroup of G and (X, Y) is a Hamming pair in G.

Proof. Let S be a finite system of generators of Y such that $S = S^{-1}$, $e \in S$. For every $s \in S$, we fix some cyclic group $\langle a_s \rangle$ of order 8, and put

$$G = A \times Y, \quad A = \oplus_{s \in S} \langle a_s \rangle,$$

$$X = \{a_s^2 s, a_s^{-2} s^{-1} \ : \ s \in S \setminus \{e\}\} \cup \{A \setminus \{a_s^2, a_s^{-2} \ : \ s \in S \setminus \{e\}\}\}$$

Then we have

$$X = X^{-1}, \quad e \in X, \quad G = XY, \quad XX \cap Y = \{e\},$$

so (X, Y) is a Hamming pair in G. □

3. Kaleidoscopical semigroups

Let $s > 1$ be a natural number and $X = \{0, 1, \ldots, s\}$. The *kaleidoscopical semigroup* $KS(X)$ in the alphabet X is a semigroup determined by the relations $xx = x$, $xyx = x$ for all $x, y \in X$. For our purposes it is convenient to identify $KS(X)$ with the set of all non-empty words in X without the factors xx, xyx where $x, y \in X$. We show that the semigroup $KS(X)$ acts transitively on the set V of vertices of every kaleidoscopical graph of degree s with a kaleidoscopical coloring $\chi : V \longrightarrow \{0, 1, \ldots, s\}$.

Let $x \in X$ and $v \in V$. Pick $u \in B(v, 1)$ such that $\chi(u) = x$ and put $x(v) = u$. Then we extend inductively the action from X onto $KS(X)$ by the following rule. If $w \in KS(X)$, $w = xw_1$, $w_1 \in KS(X)$ and $v \in V$, we put $w(v) = x(w_1(v))$. We observe that the sequence of colors of the vertices on the shortest path from a vertex $v_1 \in V$ to a vertex $v_2 \in V$ determines a word $w \in KS(X)$ such that $w(v_1) = v_2$. It follows that the described action is transitive.

For every $x \in X$, the set $KG(X, x)$ of all words from $KS(X)$ with the first and the last letter x is a subgroup (with the identity x) of the semigroup $KS(X)$. To obtain the inverse element to the word $w \in KG(X, x)$ it suffices to write w in the reverse order. The group $KG(X, x)$ is called the kaleidoscopical group in the alphabet X with the identity x.

For proof of the following statements see [1, Chapter 6].

- The only idempotents of the semigroup $KS(X)$ are the words x, xy where $x, y \in X$, $x \neq y$.
- The kaleidoscopical group $KG(X, x)$ is a free group of rank $\frac{s(s-1)}{2}$ with the set of free generators

$$\{xyzx \ : \ y, z \in X \backslash \{x\}, y < z\}.$$

- The kaleidoscopical semigroup $KS(X)$ is isomorphic to the sandwich product $L(x) \times KG(X, x) \times R(x)$ with the multiplication

$$(l_1, g_1, r_1)(l_2, g_2, r_2) = (l_1, g_1 r_1 l_2 g_2, r_2),$$

where $L(x) = \{yx \ : \ y \in X\}$, $\quad R(x) = \{xy \ : \ y \in X\}$.

4. Automorphisms

Lemma. Let $\Gamma(V, E)$, $\Gamma'(V', E')$ be kaleidoscopical graphs of degree $s > 1$ with kaleidoscopical colorings $\chi : V \longrightarrow \{0, 1, \ldots, s\}$, $\chi' : V' \longrightarrow \{0, 1, \ldots, s\}$. Let $f_1 : V \longrightarrow V'$, $f_2 : V \longrightarrow V'$ be homomorphisms from $\Gamma(V, E)$ to $\Gamma'(V', E')$. If there exists a vertex $v \in V$ such that $f_1(v) = f_2(v)$ then $f_1 \equiv f_2$.

Proof. Since every kaleidoscopical graph is connected, it suffices to show that $f_1(u) = f_2(u)$ for every $u \in B(v, 1)$. We put $w = f_1(v)$ and note that $\chi'(w) = \chi(v)$. Since f_1, f_2 are homomorphisms then $f_1(u), f_2(u) \in B(w, 1)$. Moreover, $\chi'(f_1(u)) = \chi(u)$, $\chi'(f_2(u)) = \chi(u)$ but $B(w, 1)$ has only one point of color $\chi(u)$ so $f_1(u) = f_2(u)$. $\square$

A regular tree T of degree s with fixed kaleidoscopical s-coloring of the set of its vertices is called a *kaleidoscopical tree* of degree s.

By [1, Theorem 6.4], every kaleidoscopical graph of degree s is a homomorphic image of T.

Theorem 3. Let T be a kaleidoscopical tree of degree $s > 1$. Then $Aut(T)$ is a free group of rank $\frac{s(s-1)}{2}$.

Proof. In view of Section 3, it suffices to show that $Aut(T)$ is isomorphic to the kaleidoscopical group $KG(X, x)$ where $X = \{0, 1, \ldots, s\}$, $x \in X$. We fix an arbitrary vertex v of T of color x. Given an arbitrary $f \in Aut(T)$, we choose $w_f \in KG(X, x)$ such that $w_f(v) = f(v)$. Thus, we have defined the mapping

$$\varphi : Aut(T) \longrightarrow KG(X, x), \quad \varphi(f) = w_f.$$

By Lemma, φ is injective. Since $Aut(T)$ acts transitively on the set of vertices of color x, then φ is surjective. Hence, f is a bijection. Given any $f, g \in Aut(T)$, we have

$$\varphi(f, g) = w_{fg} = w_f w_g = \varphi(f)\varphi(g),$$

so φ is an isomorphism. □

Theorem 4. Let G be a group with a Hamming pair (X, Y), $\Gamma = Cay(G, X)$. Then $Aut(\Gamma)$ is isomorphic to Y.

Proof. Let $\chi : G \longrightarrow \{0, 1, \ldots, s\}$ be the standard coloring (see Section 2) of Γ. Clearly, χ is stable on every left coset of G by Y. Hence, every element $y \in Y$ determines an automorphism of Γ by the rule $f_y(g) = gy^{-1}$. On the other hand, let h be an arbitrary automorphism of Γ. Then $h(e) = y^{-1}$ for some $y \in Y$. By Lemma, $h = f_y$. □

Theorem 5. Every finitely generated group H is isomorphic to the group $Aut(\Gamma)$ for some kaleidoscopical graph Γ.

Theorem 6. By Theorem 2, there exists a group G with a Hamming pair (X, Y) such that Y is isomorphic H. Then we apply Theorem 4.

References

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