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RESEARCH ARTICLE

Self-similar groups and finite Gelfand pairs

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Dedicated to V.I. Sushchansky on the occasion of his 60th birthday

Abstract. We study the Basilica group B, the iterated monodromy group I of the complex polynomial $z^2 + i$ and the Hanoi Towers group $H^{(3)}$. The first two groups act on the binary rooted tree, the third one on the ternary rooted tree. We prove that the action of B, I and $H^{(3)}$ on each level is 2-points homogeneous with respect to the ultrametric distance. This gives rise to symmetric Gelfand pairs: we then compute the corresponding spherical functions. In the case of B and $H^{(3)}$ this result can also be obtained by using the strong property that the rigid stabilizers of the vertices of the first level of the tree act spherically transitively on the respective subtrees. On the other hand, this property does not hold in the case of I.

1. Introduction

We study three self-similar groups from the theory of finite Gelfand pairs viewpoint. All of them can be obtained as iterated monodromy groups (see, for the definitions, [Nek1] and [Nek2]).

The first two groups that we consider are groups of automorphisms of the infinite rooted binary tree.

The Basilica group B was introduced by R.I. Grigorchuk and A. Żuk in [GŻ] as a group generated by a three-state automaton. It is a remarkable fact due to Nekrashevych [Nek2] that this group can be described as the iterated monodromy group $IMG(z^2 - 1)$ of the complex polynomial

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$z^2 - 1$. Moreover, B is the first example of an amenable group (a highly non–trivial and deep result of Bartholdi and Virág [BV]) not belonging to the class SG of subexponentially amenable groups, which is the smallest class containing all groups of subexponential growth and closed after taking subgroups, quotients, extensions and direct unions.

The group $I = IMG(z^2 + i)$ has been introduced in [BGN] and later studied by Bux and Pĕrez ([BP]) who proved that it has intermediate growth and so it is amenable.

We show that the action of B and I on each level of the infinite rooted binary tree T is 2-points homogeneous. Indeed, for all $n \geq 1$, the parabolic subgroup K_n (which stabilizes a fixed vertex in the $n-$th level) acts transitively on each sphere defined with respect to the ultrametric distance. This parallels the analogous result for the Grigorchuk group by Grigorchuk and Bartholdi [BG] (see also [BHG]).

However, we prove that the Basilica group has the following strong property: the rigid vertex stabilizers of the first level, namely $Rist_B(0)$ and $Rist_B(1)$, act spherically transitively on the respective subtrees T_0 and T_1. This gives a second proof of the fact that the action on each level of T is 2-points homogeneous (these are, in fact, the classical arguments used to prove the analogous statement for $Aut(T)$, the whole group of automorphisms of T). On the other hand, the group I does not share this property with B.

The last section of the paper is devoted to the study of the Hanoi Towers group $H^{(3)}$, acting on the infinite rooted ternary tree. The definition of Hanoi Towers group $H^{(q)}$ can be given more generally for every $q \geq 3$ as a group of automorphisms acting on the rooted $q-$ary tree (see [GŠ1]). It derives its name from the fact that it models the well known Hanoi Towers game on q pegs. The case of $H^{(3)}$ is particularly interesting since this group can be also obtained as the iterated monodromy group defined by the map $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ given by $f(z) = \overline{z}^2 + \frac{16}{27\,\overline{z}}$. It is known (see [GŠ2]) that $H^{(3)}$ is amenable but it does not belong to the class EG of elementary groups, which is the smallest class containing all finite and abelian groups and closed after taking subgroups, quotients, extensions and direct unions. For the group $H^{(3)}$ we have proven both the results that hold for B. In the rest of the paper we will denote the group $H^{(3)}$ by H.

Finally, for B, I and H the decomposition into irreducible submodules of the corresponding permutation representation together with the associated spherical functions are given.

2. Groups acting on trees

Consider the infinite $q-$ary rooted tree, i.e. the rooted tree in which each vertex has q children. We will denote this tree by T. If $X = \{0, 1, \dots, q-1\}$ is an alphabet of q elements, X^* is the set of all finite words in X. Moreover, we can identify the set of infinite words in X with the elements of the boundary of T. Each vertex in the n-th level L_n of T will be identified with a word of length n in the alphabet X.

The set L_n can be endowed with an ultrametric distance d, defined in the following way: if $x = x_0 \dots x_{n-1}$ and $y = y_0 \dots y_{n-1}$, then

$$d(x, y) = n - \max\{i : x_k = y_k, \ \ \forall k \leq i\}.$$

We observe that $d = d'/2$, where d' denotes the usual geodesic distance.

In this way (L_n, d) becomes an ultrametric space, in particular a metric space, on which the automorphisms group $Aut(T)$ acts isometrically. Note that the diameter of (L_n, d) is exactly n.

To indicate the action of an automorphism $g \in Aut(T)$ on a vertex x, we will use the notation $g(x)$ or x^g.

We recall that a group G acting on T is *self-similar* if for every $g \in G$, $x \in X$, there exist $g_x \in G$, $x' \in X$ such that $g(xw) = x' g_x(w)$ for all $w \in X^*$. Moreover, a self-similar group G can be embedded into the wreath product $G \wr X = (G^q) \rtimes S_q$, where S_q is the symmetric group on q elements.

Every automorphism $g \in Aut(T)$ can be represented by its *labelling*. The labelling of $g \in Aut(T)$ is realized as follows: given a vertex $x = x_0 \dots x_{n-1} \in T$, we associate with x a permutation $g_x \in S_q$ giving the action of g on the children of x. Formally, the action of g on the vertex labelled with the word $x = x_0 \dots x_{n-1}$ is

$$x^g = x_0^{g_\emptyset} x_1^{g_{x_0}} \dots x_{n-1}^{g_{x_0 \dots x_{n-2}}}.$$

It will be used also $g(x)$ to indicate the element x^g.

We recall now that, for an automorphisms group $G \leq Aut(T)$, the *stabilizer* of the vertex $x \in T$ is the subgroup of G defined as $Stab_G(x) = \{g \in G : g(x) = x\}$ and the stabilizer of the n-th level is $Stab_G(n) = \bigcap_{x \in L_n} Stab_G(x)$. Observe that $Stab_G(n)$ is a normal subgroup of G of finite index for all $n \geq 1$. In particular, an automorphism $g \in Stab_G(1)$ can be identified with the elements $g_i, i = 0, 1, \dots, q-1$ that describe the action of g on the respective subtrees T_i rooted at the vertex i of the first level, which are clearly isomorphic to the entire tree T. So we get the following embedding

$$\varphi : Stab_G(1) \longrightarrow \underbrace{Aut(T) \times Aut(T) \times \dots \times Aut(T)}_{q \ \ \text{times}}$$

that associates with g the $q-$ple $(g_0, g_1, \ldots, g_{q-1})$. G is said to be *fractal* if the map

$$\varphi : Stab_G(1) \longrightarrow G \times G \times \cdots \times G$$

is a subdirect embedding, that is it is surjective on each factor.

In what follows we will often use the notion of rigid stabilizer. For a group G acting on T and a vertex $x \in T$, the *rigid vertex stabilizer* $Rist_G(x)$ is the subgroup of $Stab_G(x)$ consisting of the automorphisms acting trivially on the complement of the subtree T_x rooted at x. Equivalently, they have a trivial labelling at each vertex outside T_x. The rigid stabilizer of the n-th level is defined as $Rist_G(n) = \prod_{x\in L_n} Rist_G(x)$. In contrast to the level stabilizers, the rigid level stabilizers may have infinite index and may even be trivial. We observe that if the action of G on T is spherically transitive, then the subgroups $Stab_G(x)$, $x \in L_n$ are all conjugate, as well as the subgroups $Rist_G(x)$, $x \in L_n$.

We recall the following definitions for spherically transitive groups (see, for more details, [BGŠ]).

- G is *regular weakly branch* on K if there exists a normal subgroup $K \neq \{1\}$ in G, with $K \leq Stab_G(1)$, such that $\varphi(K) > K \times K \times \cdots \times K$. In particular G is *regular branch* on K if it is regular weakly branch on K and K has finite index in G.

We observe that this property for the subgroup K is stronger than fractalness, since the map φ is surjective on the whole product $K \times K \times \cdots \times K$.

- G is *weakly branch* if $Rist_G(x) \neq \{1\}$, for every $x \in T$ (this automatically implies $|Rist_G(x)| = \infty$ for every x). In particular, G is *branch* if $[G : Rist_G(n)] < \infty$ for every $n \geq 1$.

3. Gelfand pairs

In this section we consider the groups $B_n = B/Stab_B(n), I_n = I/Stab_I(n)$ and $H_n = H/Stab_H(n)$ and the associated action on the level L_n. Let x_0 be a fixed leaf in L_n and denote by K_n the corresponding parabolic subgroup. We can apply the theory of finite Gelfand pairs to the pairs $(B_n, K_n), (I_n, K_n)$ and (H_n, K_n) for all n.

3.1. Basic results of finite Gelfand pairs theory.

We present now some basic elements of the theory of finite Gelfand pairs (see [CST]).

Let G be a finite group and let $K \leq G$, denote $X = G/K = \{gK : g \in G\}$ the associated homogeneous space.

(G, K) is a *Gelfand pair* if the algebra $L(K\backslash G/K)$ of bi-K-invariant functions on G is commutative. The following are equivalent:

1. (G, K) is a Gelfand pair;
2. the decomposition of the space $L(X)$ into irreducible submodules under the action of G is multiplicity-free, i.e. each irreducible submodule occurs with multiplicity 1;

A particular example of a Gelfand pair is given by the *symmetric Gelfand pairs*: for every $g \in G$ one has $g^{-1} \in KgK$.

This is the case if G acts on a finite metric space (X, d) isometrically and with a 2-points homogeneous action, i.e. in such a way that for all $x, y, x', y' \in X$ such that $d(x, y) = d(x', y')$ there exists $g \in G$ such that $gx = x'$ e $gy = y'$.

We can observe that the K-orbits under this action are the spheres of center x_0 and radius j, with $j = 0, \ldots, n$. Hence, a function $f \in L(X)$ is K-invariant if and only if it is constant on these spheres.

Lemma 3.1. Let G act spherically transitively on T. Denote by G_n the quotient group $G/Stab_G(n)$ and by K_n the stabilizer in G_n of a fixed leaf $x_0 \in L_n$. Then the action on L_n is 2-points homogeneous if and only if K_n acts transitively on each sphere of L_n.

Proof. Suppose that K_n acts transitively on each sphere of L_n and consider the elements x, y, x' and y' such that $d(x, y) = d(x', y')$. Since the action of G_n is transitive, there exists an automorphism $g \in G_n$ such that $g(x) = x'$. Now $d(x', g(y)) = d(x', y')$ and so $g(y)$ and y' are in the same sphere of center x' and radius $d(x', y')$. But K_n is conjugate with $Stab_{G_n}(x')$ and so there exists an automorphism $g' \in Stab_{G_n}(x')$ carrying $g(y)$ to y'. The composition of g and g' is the required automorphism.

Suppose now that the action of G_n on L_n is 2-points homogeneous and consider two elements x and y in the sphere of center x_0 and radius i. Then $d(x_0, x) = d(x_0, y) = i$. So there exists an automorphism $g \in Stab_{G_n}(x_0)$ such that $g(x) = y$. This completes the proof. □

If (G, K) is a Gelfand pair and $L(X) = \bigoplus_{i=0}^{n} V_i$ is a decomposition of $L(X)$ into irreducible submodules, then for each $i = 0, 1, \ldots, n$ there exists a unique (up to normalization) bi-K-invariant function ϕ_i whose G-translates generate the V_i's. In particular, we will require that these functions take value exactly 1 on the element $x_0 \in X$ stabilized by K. The functions ϕ_i, $i = 0, 1, \ldots, n$ are called *spherical functions.*

As an example, the function $\phi_0 \equiv 1$ is a spherical function, what corresponds to the fact that the trivial representation always occurs in the decomposition of the space $L(X)$ into irreducible submodules.

We now give a Lemma (see [CST]) that will be very useful later.

Lemma 3.2. Let G, K and $X = G/K$ the associated homogeneous space. If we have a decomposition $L(X) = \bigoplus_{t=0}^{h} Z_t$ into submodules such that the number of K-orbits of X is exactly $h+1$, then the submodules Z_t's are irreducible and (G, K) is a Gelfand pair.

3.2. The case of the full automorphisms group $Aut(T)$.

If one considers the action of the full automorphisms group of the $q-$ary rooted tree

$$Aut(T)_n = Aut(T)/Stab_{Aut(T)}(n)$$

on L_n one gets, for every n, a 2-points homogeneous action, giving rise to the symmetric Gelfand pair $(Aut(T)_n, K_n)$, with $K_n = Stab_{Aut(T)_n}(0^n)$.

Theorem 3.3. The action of $Aut(T)_n$ on (L_n, d) is 2-points homogeneous.

Proof. We use induction on the depth n of the tree T.

$n = 1$. The assertion follows from the 2-transitivity of the group S_q.

$n > 1$. Let (x, y) and (x', y') be pairs of vertices in L_n with $d(x, y) = d(x', y')$. If $d(x, y) < n$, then vertices x and y belong to the same subtree of T and so $x_1 = y_1$. Analogously for x' and y'. Applying, if necessary, the transposition $(x_1 x'_1) \in S_q$, we can suppose $x_1 = y_1 = x'_1 = y'_1$, so that x, x', y and y' belong to the same subtree of depth less or equal to $n-1$, and then induction works.

Finally, consider the case $d(x, y) = d(x', y') = n$. Consider the automorphism $g \in Aut(T)$ such that $g(x_1) = x'_1$ and $g(y_1) = y'_1$ and which acts trivially on the other vertices of L_1. Now we have that x and x' belong to the same subtree T'. Analogously y and y' belong to the same subtree T'', with $T' \neq T''$. The restriction of $Aut(T)_n$ to T' and T'' respectively acts transitively on each level. So there is an automorphism g' of T' carrying x to x' and acting trivially on T'' and analogously there is an automorphism g'' of T'' carrying y to y' and trivial on T'. The assertion is proved. $\square$

The decomposition of the space $L(L_n)$ under the action of $Aut(T)_n$ is known.

Denote $W_0 \cong \mathbb{C}$ the trivial representation and for every $j = 1, \ldots, n$, define the following subspace

$$W_j = \left\{ f \in L(L_n) : \ f = f(x_1, \ldots, x_j), \ \sum_{x=0}^{q-1} f(x_1, x_2, \ldots, x_{j-1}, x) \equiv 0 \right\}$$

of dimension $q^{j-1}(q-1)$. One can verify that the W_j's are $Aut(T)_n$-invariant, pairwise orthogonal and that the following decomposition holds

$$L(L_n) = \bigoplus_{j=0}^{n} W_j.$$

Since the spheres centered at $x_0 = 0^n$ (and so the K_n-orbits) are exactly $n+1$, we have from Lemma 3.2 that the subspaces W_j's are irreducible.

There exists a complete description of the corresponding spherical functions. For every $j = 0, \ldots, n$ we get

$$\phi_j(x) = \begin{cases} 1, & d(x, x_0) < n - j + 1; \\ \frac{1}{1-q}, & d(x, x_0) = n - j + 1; \\ 0, & d(x, x_0) > n - j + 1. \end{cases}$$

If we consider a countable subgroup of $Aut(T)$ and the relative action on L_n, we can ask if it is possible to find the same results about Gelfand pairs obtained for the full automorphisms group. In some cases the answer is positive. In what follows we will investigate this problem for the groups B, I and H.

3.3. The case of the Basilica group

The Basilica group is generated by the automorphisms a and b having the following self-similar form:

$$a = (b, 1), \quad b = (a, 1)\varepsilon$$

where ε denotes the nontrivial permutation of the group S_2. In the following figure the labelling of the automorphisms a and b are presented. Observe that the labelling of each vertex not contained in the leftmost branch of the tree is trivial.

It can be easily proved that the Basilica group is a fractal group. In fact, the stabilizer of the first level is

$$Stab_B(1) = \langle a, a^b, b^2 \rangle,$$

with $a = (b, 1)$, $a^b = (1, b^a)$ and $b^2 = (a, a)$.

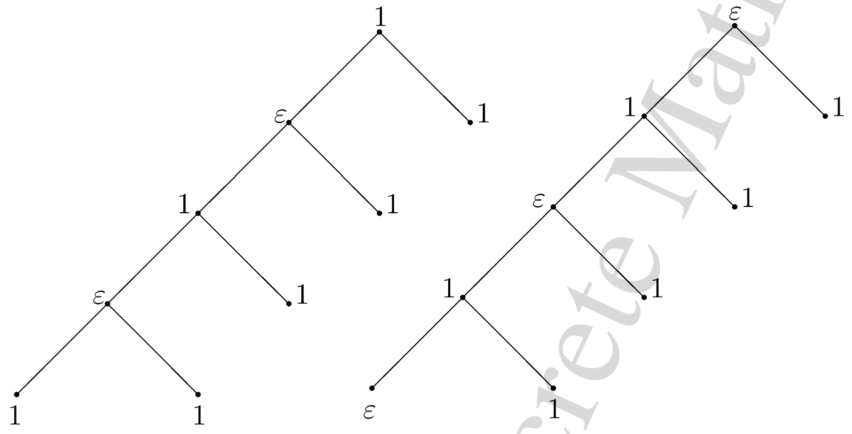

Figure 1: Labelling of the generators a and b.

It is obvious that the action of the Basilica group on the first level of T is transitive. Since this group is fractal, it easily follows that the action is also spherically transitive, i.e. transitive on each level of the tree. Moreover, it is known (see [GŻ]) that the Basilica group is weakly regular branch over its commutator subgroup B'.

Theorem 3.4. The action of the Basilica group B on L_n is 2-points homogeneous for all n.

Proof. From Lemma 3.1 it suffices to show that the action of the parabolic subgroup $K_n = Stab_{B_n}(0^n)$ is transitive on each sphere.

Denote by u_j the vertex $0^{j-1}1$ for every $j = 1, \ldots, n$. Observe that the automorphisms

$$(b^2)^a = a^{-1}b^2a = (b^{-1}, 1)(a, a)(b, 1) = (a^b, a) = ((1, b^a), a)$$

and

$$b^ab^{-1}a = (b^{-1}, 1)(a, 1)\varepsilon(b, 1)(1, a^{-1})\varepsilon(b, 1) = (1, b)$$

belong to K_n for each n. Moreover, using the fractalness of B, it is possible to find elements $g_j \in K_n$ such that the restriction $g_j|T_{0^{j-1}}$ is $(b^2)^a = ((1, b^a), a)$ or $b^ab^{-1}a = (1, b)$. So, the action of such automorphisms on the subtree T_{u_j} corresponds to the action of the whole group $B = \langle a, b\rangle$ on T. We can regard this action as the action of K_n on the spheres of center $x_0 = 0^n$, and so we get that K_n acts transitively on these spheres. This implies that the action of B is 2-points homogeneous on L_n. □

Corollary 3.5. For every $n \geq 1$, (B_n, K_n) is a symmetric Gelfand pair.

The number of K_n-orbits is exactly the number of the irreducible submodules occurring in the decomposition of $L(L_n)$ under the action of B_n. Since the submodules W_j's described in the previous section are $n+1$ as the K_n-orbits, it follows that the Basilica group admits the same decomposition into irreducible submodules and the same spherical functions that we get for $Aut(T)_n$.

Remark 3.6 (The Grigorchuk group).

A similar argument can be used in the case of the Grigorchuk group. It is a fractal, regular branch group, generated by the automorphisms

$$a = (1,1)\varepsilon, \quad b = (a,c), \quad c = (a,d), \quad d = (1,b).$$

The action of this group on the binary rooted tree is 2-points homogeneous on the level L_n for all n. As a consequence, the decomposition of $L(L_n)$ under the action of this group into irreducible submodules is still $L(L_n) = \bigoplus_{j=0}^{n} W_j$, where the W_j's are the subspaces defined above. See [Gri] and [BHG] for more details.

We can observe that in the proof of the Theorem 3.3 the fundamental fact is that the automorphisms g' and g'' act transitively on the subtrees T' and T'', respectively, and trivially elsewhere. Moreover, the only fractalness does not guarantee that the action is 2-points homogeneous, as one can easily verify in the case of the Adding Machine (see, for the definition, [BGN]), for which one gets symmetric Gelfand pairs only for $n = 1, 2$. On the other hand, if a fractal group G acts 2-transitively on L_1 and if it has the property that the rigid stabilizers of the vertices of the first level $Rist_G(i), i = 0, 1, \ldots, q-1$ are spherically transitive for each i, the proof of the Theorem 3.3 works again by taking the automorphisms g' and g'' in the rigid vertex stabilizers. But this is not a necessary condition, as the example of the Grigorchuk group shows.

In fact, one can verify (see [BG]) that, in this case, $Rist_G(0) = \langle d^a, d^{ac} \rangle$, with $d^a = (b,1)$ and $d^{ac} = (b^a, 1)$. So $Rist_G(0)$ fixes the vertices 00 and 01, and then it does not act transitively on the subtree T_0. This shows, for instance, that a fractal regular branch group could not have this property, which appears to be very strong.

On the other hand, a direct computation shows that Basilica group has this property, what gives another proof that the action on each level L_n is 2-points homogeneous.

Theorem 3.7. Let B be the Basilica group. Then the rigid vertex stabilizers $Rist_B(i)$, $i = 0, 1$, act spherically transitively on the corresponding subtrees T_i.

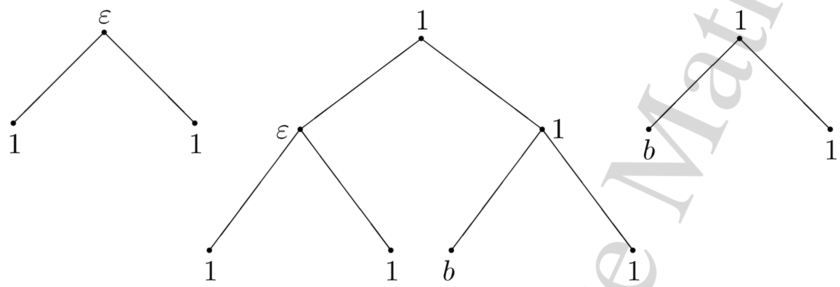

Figure 2: Labelling of the generators a, b and c.

Proof. Since B is spherically transitive and so $Rist_B(0) \simeq Rist_B(1)$, it suffices to prove the assertion only for $Rist_B(0)$. Consider the automorphisms $a = (b, 1)$ and $a^{b^2} = (b^a, 1)$ in $Rist_B(0)$. We want to show that the subgroup $\langle a, a^{b^2}\rangle$ is spherically transitive on T_0, equivalently we will prove that the group $\langle b, b^a\rangle$ is spherically transitive on T.

The latter is clearly transitive on the first level. To complete it suffices to prove its fractalness. We have

$$b^{-1}b^a = (1, a^{-1})\varepsilon(b^{-1}, 1)(a, 1)\varepsilon(b, 1) = (1, a^{-1}b^{-1})\varepsilon(a, b)\varepsilon = (b, (b^{-1})^a)$$

and

$$(b^{-1}b^a)^{b^2} = (a^{-1}, a^{-1})(b, (b^{-1})^a)(a, a) = (b^a, (b^{-1})^{a^2}),$$

and so the projection on the first factor gives both the generators b and b^a. The elements

$$(b^{-1}b^a)^{-1} = (b^{-1}, b^a), \quad ((b^{-1}b^a)^{-1})^{b^{-2}} = ((b^{-1})^{a^{-1}}, b)$$

fulfill the requirements for the projection on the second factor and this completes the proof. □

3.4. The case of $IMG(z^2 + i)$.

Consider now the group $I = IMG(z^2 + i)$, i.e. the iterated monodromy group defined by the map $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ given by $f(z) = z^2 + i$. The generators of this group have the following self-similar form:

$$a = (1, 1)\varepsilon, \quad b = (a, c), \quad c = (b, 1),$$

where ε denotes, as usual, the nontrivial permutation in S_2. In the following figure we present the corresponding labellings.

One can easily prove the following relations:

$$a^2 = b^2 = c^2 = (ac)^4 = (ab)^8 = (bc)^8 = 1.$$

Moreover, the stabilizer of the first level is $Stab_I(1) = \langle b, c, b^a, c^a\rangle$. In particular, since

$$b^a = (c, a), \quad c^a = (1, b),$$

I is a fractal group. It is obvious that I acts transitively on the first level of the rooted binary tree. Since this group is fractal, it follows that this action is also spherically transitive.

Moreover, it is known (see [GSŠ]) that I is a regular branch group over its subgroup N defined by

$$N = \langle [a, b], [b, c]\rangle^I.$$

Also for the group I it is possible to prove the same result proven for the Basilica group in Theorem 3.4. So consider the $n-$th level L_n of the tree and the group $I_n = I/Stab_I(n)$. In order to get an easy computation, we choose the vertex $x_0 = 1^n \in L_n$ and we set $K_n = Stab_{I_n}(1^n)$. In the following theorem we will prove that the action of the parabolic subgroup K_n is transitive on each sphere.

Theorem 3.8. The action of the group I on L_n is 2-points homogeneous for all n.

Proof. Denote by u_j the vertex $1^{j-1}0$ for every $j = 1, \dots, n$. Using the fractalness of I, it is possible to find an element $g_j \in K_n$ such that the restriction $g_j|T_{1^{j-1}}$ is b and an element $h_j \in K_n$ such that the restriction $h_j|T_{1^{j-1}}$ is c. Consider now the automorphism $b^a b b^a = (c, a)(a, c)(c, a) = (a^c, c^a)$. By fractalness it is possible to find an element $k_j \in K_n$ such that the restriction $k_j|T_{1^{j-1}}$ is $b^a b b^a$. The action of the subgroup generated by the automorphisms g_j, h_j, k_j on the subtree T_{u_j} corresponds to the action of the subgroup $H = \langle a, b, a^c\rangle$ on T. It is easy to prove that this action is spherically transitive. In fact it is obvious that H acts transitively on the first level, so it suffices to show that H is fractal. To show this consider, for instance, the elements

$$b = (a, c), \quad a^c a = (b, b), \quad b^a b b^a = (a^c, c^a)$$

and

$$b^a = (c, a), \quad a^c a = (b, b), \quad b b^a b = (c^a, a^c).$$

Now, the action of H on T_{u_j} can be regarded as the action of K_n on the spheres of center x_0, and so we get that K_n acts transitively on these spheres. This implies that the action of I on L_n is 2-points homogeneous, as required. □

Corollary 3.9. For every $n \geq 1$, (I_n, K_n) is a symmetric Gelfand pair.

As in the case of the Basilica group, it follows that the group I_n admits the same decomposition into irreducible submodules and the same spherical functions that we get for $Aut(T)_n$.

It is possible to show that the rigid stabilizers of the vertices of the first level of T do not act spherically transitively on the corresponding subtrees T_0 and T_1. In fact, the rigid stabilizer of the first level is $Rist_I(1) = \langle c\rangle^G$, so every automorphism in $Rist_I(1)$ is the product of elements of the form c^g, where $g = w(a, b, c)$ is a word in a, b and c, and of their inverses. Set $\varphi(c^g) = (g_0, g_1)$. We want to show, by induction on the length of the word $w(a, b, c)$, that we suppose reduced, that in both g_0 and g_1 the number of occurrences of a is even. This will imply that the action of $Rist_I(1)$ on the first level of the subtrees T_0 and T_1 cannot be transitive and will prove the assertion.

If $|w(a, b, c)| = 0$, then $c^g = c = (b, 1)$. If $|w(a, b, c)| = 1$, then we can have $c^a = (1, b)$, $c^b = (b^a, 1)$ or $c^c = c = (b, 1)$. Let us suppose the result to be true for $|w'(a, b, c)| = n - 1$. Then we have $c^{w(a,b,c)} = c^{w'(a,b,c)x}$, with $x \in \{a, b, c\}$ and $c^{w'(a,b,c)} = (g'_0, g'_1)$ such that in both g'_0 and g'_1 the number of occurrences of a is even. If $x = a$, we get $c^{w(a,b,c)} = (g'_1, g'_0)$, if $x = b$, we get $c^{w(a,b,c)} = ((g'_0)^a, (g'_1)^b)$ and if $x = c$ then we get $c^{w(a,b,c)} = ((g'_0)^b, g'_1)$. In all cases, we get a pair (g_0, g_1) satisfying the condition that in both g_0 and g_1 the number of occurrences of a is even, as required.

3.5. The case of the Hanoi Towers group H

The Hanoi Towers group H is a group of automorphisms of the rooted ternary tree. For the rooted ternary tree all the definitions of level stabilizer, rigid level stabilizer, fractalness, spherically transitive action, given in the binary case, hold.

The generators of H have the following self-similar form:

$$a = (1, 1, a)(01), \quad b = (1, b, 1)(02), \quad c = (c, 1, 1)(12),$$

where $(01), (02)$ and (12) are transpositions in S_3. In the following figures we present the corresponding labellings.

From the definition it easily follows that $a^2 = b^2 = c^2 = 1$. Considering the following elements belonging to $Stab_H(1)$

$$acab = (a, cb, a), \quad bcba = (b, b, ca), \quad cacb = (c, ab, c),$$

$$caba = (cb, a, a), \quad (ac)^2ba = (ab, c, c), \quad cbab = (ca, b, b),$$

one can deduce that H is a fractal group. It is obvious that H acts transitively on the first level of the rooted ternary tree. Since this group is fractal, it follows that this action is also spherically transitive.

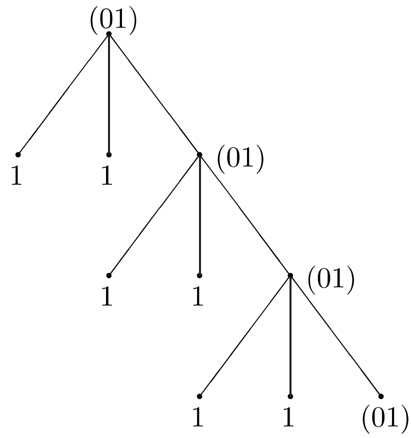

Figure 3: Labelling of the generators a and b.

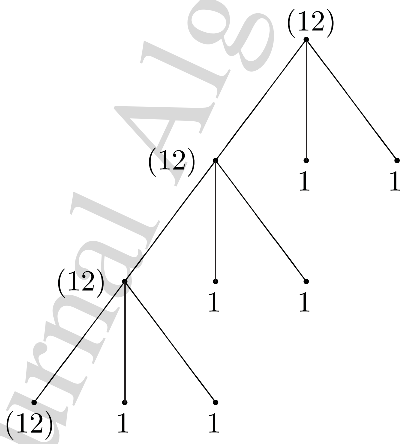

Figure 4: Labelling of the generator c.

Moreover, it is known (see [GŠ1]) that H is a regular branch group over its commutator subgroup H'. We observe that we have not the inclusion $H' \leq Stab_H(1)$ that we have in the case of the Basilica group and in the case of $IMG(z^2 + i)$.

Also for the group H it is possible to prove that its action on L_n, $n \geq 1$, gives rise to symmetric Gelfand pairs as it has been proven for B and I. So consider the $n-$th level L_n of the tree and the group $H_n = H/Stab_H(n)$. Fix the vertex $x_0 = 0^n \in L_n$ and set $K_n = Stab_{H_n}(x_0)$. In the following theorem we will prove that the action of the parabolic subgroup K_n is transitive on each sphere.

Theorem 3.10. The action of the group H on L_n is 2-points homogeneous for all n.

Proof. Denote by u_j the vertex $0^{j-1}1$ and by v_j the vertex $0^{j-1}2$, for every $j = 1, \dots, n$. Consider the element

$$acb = (1, c, ab)(12).$$

Using the fractalness of H, it is possible to find an element $g_j \in K_n$ such that the restriction $g_j|T_{0^{j-1}}$ is acb. Since H is regular branch over H', there exists a subgroup H_j of K_n such that $H_j|_{T_{u_j}} = H'$ and which fixes any vertex of the tree whose u_j is not an ancestor. Let us prove that the action of H' on the whole tree is spherically transitive. Considering, for example, the element $[c, b] = cbcb = (cb, c, b)(012)$, one gets that this action is transitive on the first level. Since $H' \geq H' \times H' \times H'$, the action is transitive on each level of the tree. So the action of the subgroup $K = \langle H_j, g_j \rangle$ on the subtree $T_{0^{j-1}}$ is transitive on the vertices of L_n belonging to the subtrees T_{u_j} and T_{v_j}. This action can be regarded as the action of K_n on the spheres of center x_0, and so we get that K_n acts transitively on these spheres. This implies that the action of H is 2-points homogeneous on L_n, as required. □

Corollary 3.11. For every $n \geq 1$, (H_n, K_n) is a symmetric Gelfand pair.

As in the case of the Basilica group and of $IMG(z^2 + i)$, the group H_n admits the same decomposition into irreducible submodules and the same spherical functions that we get for $Aut(T)_n$.

Now we want to prove that the action of the rigid vertex stabilizers $Rist_H(0), Rist_H(1)$ and $Rist_H(2)$ is spherically transitive on the subtrees T_0, T_1 and T_2, respectively. Since these subgroups are conjugate, is suffices to prove the result for $Rist_H(0)$. We use again the fact that H is regular branch over its commutator subgroup H'. So there exists a subgroup $L \leq H'$ such that $L|_{T_0} = H'$ and $L|_{T_1} = L|_{T_2} = 1$. In particular,

L is a subgroup of $Rist_H(0)$. Since H' is spherically transitive on T, it follows that $Rist_H(0)$ is spherically transitive on T_0, as required.

This property of the rigid vertex stabilizers, together with the fractalness of H and with the fact that the action of H on the first level is 2-transitive, gives a second proof of the fact that the action of H_n on L_n is 2-points homogeneous, following the same idea that we used for the Basilica group.

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