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RESEARCH ARTICLE

Groups associated with modules over nearrings

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Dedicated to V.I. Sushchansky on the occasion of his 60th birthday

Abstract. We construct a group $D(I,T)$ associated with the pair (I,T), where I is a nontrivial distributive submodule of a left N-module G, T is a nontrivial subgroup of the unit group $U(N)$ of a right nearring N with an identity element, and find criteria for $D(I,T)$ to be a Frobenius group.

0. Let N be a right nearring under two operations "+" and "·" with the identity element 1, i.e. $(N,+)$ is a group with the zero 0, multiplication "·" is associative and $(y+z)\cdot x = y\cdot x + z\cdot x$ for all $x,y,z \in N$. As usual, an additive group $(G,+)$ with the zero e is called a left N-module if $(x+y)g = xg + yg$ and $x(yg) = (xy)g$ for any $g \in G$ and $x,y \in N$. A subgroup H of G is called an N-submodule (or an N-subgroup) of G if $HN \subseteq H$. Recall that an N-module G is abelian if the additive group $(G,+)$ is abelian and $x(g+h) = xg + xh$ for all $g,h \in G$, $x \in N$. A submodule I of an N-module G will be called distributive with respect to subset T of N if $x(g+h) = xg + xh$ for all $g,h \in G$ and $x \in T$. Moreover, G is unitary if $1g = g$ for any $g \in G$.

As it is well known $U(N) = \{d \in N \mid d \text{ is invertible in } N\}$ is a group under "·" (which is called the unit group of N).

In this paper we construct a group $D(I,T)$, where I is a nontrivial distributive submodule of a left N-module G, T is a nontrivial subgroup of the unit group $U(N)$ of a right nearring N with an identity element, and find criteria for $D(I,T)$ to be a Frobenius group.

Throughout this paper, all nearrings are right with an identity element and all modules are left. If H is a group, F its subgroup and $x,y \in H$, then $[x,y] = x^{-1}y^{-1}xy$ is the commutator of x and y, $y^x = x^{-1}yx$ and $F^x = x^{-1}Fx = \{y^x \mid y \in F\}$.

Other general notations and conventions in this paper follow closely those used in [1] and [2].

1. Let N be a right nearring and G be a left N-group. If T is a subgroup of $U(N)$ and I is an N-subgroup of G, then on the set of pairs

$$D(I,T) = \{(a,b) \mid a \in I, b \in T\}$$

we define the algebraic operation by the rule

$$(a,b)(u,v) = (bu + a, bv). \tag{1}$$

Lemma 1. Let N be a right nearring with the identity element 1, G an unitary left N-module with the zero e. If T is a subgroup of the unit group $U(N)$ of N, I is an N-subgroup which is distributive with respect to T of G (in particular, I is an abelian N-submodule of G), then $D(I,T)$ is a group with the identity element $(e,1)$ under the operation given by the rule (1) and, moreover,

$$D(I,T) = E \rtimes F,$$

where a subgroup $E = \{(a,1) \mid a \in I\}$ is isomorphic to the additive group I^+ of I and $F = \{(0,b) \mid b \in T\}$ is isomorphic to T.

Proof. The proof is immediate. We remark only that $(a,b)^{-1} = (-b^{-1}a, b^{-1})$ for any $a \in I$ and $b \in T$. □

Remember that a semidirect product $H = E \rtimes F$ of groups E and F is called a Frobenius group with a kernel E and a complement F if

$$F \cap F^g = 1$$

for all $g \in H \backslash F$ and

$$E \backslash \{1\} = H \backslash \bigcup_{h \in H} F^h.$$

The following result extends Theorem 2.3 from [3].

Theorem 2. Let N be a right nearring with the identity element 1 and the zero 0, G an unitary left N-module with the zero e, T a nontrivial subgroup of the unit group $U(N)$, I is a nontrivial N-subgroup of G which is distributive with respect to T. Then

$$H = D(I,T) = E \rtimes F$$

is a Frobenius group with a kernel E and a complement F, where E is isomorphic to the additive group I^+ of I and F is isomorphic to T, if and only if the following conditions hold:

1. $ann_{(T-1)}(i) = \{t-1 \in (T-1) \mid (t-1)i = e\} = \{0\}$ for any nontrivial element $i \in I$;

2. $I = (b-1)I$ for every nontrivial element b of T.

Proof. ($\Rightarrow$) Let $H = E \rtimes F$ be a Frobenius group with a kernel $E \cong I^+$ and a complement $F \cong T$. By Lemma 1.1 of [3] for every element $a \in I$ and every element $t \in T$ there exists $a_1 \in I$ such that

$$(a,1) = [(a_1,1),(e,t)].$$

But then

$$(a,1) = (-a_1,1)(e,t^{-1})(a_1,1)(e,t) = (1e - a_1, 1t^{-1})(1e + a_1, 1t) = \\ (-a_1,t^{-1})(a_1,t) = (t^{-1}a_1 - a_1, t^{-1}t) = (t^{-1}a_1 - a_1, 1).$$

This means that

$$a = t^{-1}a_1 - a_1 = (t^{-1}-1)a_1 \in (t^{-1}-1)I.$$

As a consequence $I = (t^{-1}-1)I$ for each nontrivial element $t \in T$.

Let a be any nontrivial element of I. Suppose that $(b-1)a = e$ for some element $b \in T$. Then

$$(e,b) = ((b-1)a,b) = (ba-a,b) = (-a,b)(a,1) = (1e-a,1b)(a,1) = \\ = (-a,1)(e,b)(a,1) = (a,1)^{-1}(e,b)(a,1) \in F^{(a,1)} \bigcap F.$$

Since

$$F^{(u,v)} \cap F = \langle (e,1) \rangle$$

for each $(u,v) \in H \setminus F$, we conclude that $b-1=0$.

($\Leftarrow$) Suppose that the conditions (1) and (2) are true for nontrivial elements $b \in T$ and $a \in I$. Since

$$a = (b-1)a_1$$

for some elements $a_1 \in I$, we deduce that

$$[(a_1,1),(e,b^{-1})] = (-a_1,1)(-be,b)(a_1,1)(e,b^{-1}) = \\ (-a_1,1)(e,b)(a_1,1)(e,b^{-1}) = (1e-a_1,1\cdot b)(1e+a_1,1b^{-1}) = \\ (-a_1,b)(a_1,b^{-1}) = (ba_1 - a_1, bb^{-1}) = (a,1).$$

This yields that $E = [E,(e,b)]$ for any nontrivial element $(e,b) \in F$.

Let $x \in I$ and $t, y \in T$, where $t \neq 1$. Then

$$(e,t)^{(x,y)} = ((y^{-1}t - y^{-1})x, y^{-1}ty) \notin E.$$

Suppose that $(c,t) \in H \setminus \bigcup_{(u,v)\in H} F^{(u,v)}$. Inasmuch as $(t-1)I = I$, there exists an element $x \in I$ such that $c = (t-1)y^{-1}x$ and so $(c,t) = (e, yty^{-1})^{(x,y)}$. Hence

$$E \setminus \{(e,1)\} = H \setminus \bigcup_{(x,y)\in H} F^{(x,y)}.$$

Now if $(u,v) \in H \setminus F$ and $(e,b) \in F \cap F^{(u,v)}$, then there is an element $(e,w) \in F$ such that

$$(e,b) = (u,v)^{-1}(e,w)(u,v)$$

and therefore

$$(e,b) = (-v^{-1}u, v^{-1})(e,w)(u,v) = (v^{-1}e - v^{-1}u, v^{-1}w)(u,v) = \\ (-v^{-1}u, v^{-1}w)(u,v) = (v^{-1}wu - v^{-1}u, v^{-1}wv).$$

Since $u = vi$ for some $i \in I$, we conclude that $e = v^{-1}wu - v^{-1}u = v^{-1}wvi - v^{-1}vi = (v^{-1}wv - 1)i$ and so, in view of (1), $v^{-1}wv - 1 = 0$. But then $b = 1$. Hence $F \cap F^{(u,v)} = \langle (e,1) \rangle$. □

Corollary 3. If P is a skew-field and T is a nontrivial subgroup of the multiplicative group P^*, then $D(P^+, T)$ is a Frobenius group, where P^+ is the additive group of P. □

Corollary 4. If G is a nontrivial abelian unitary left module over a right nearfield N, then $D(G,T)$ is a Frobenius group for every nontrivial subgroup T of the multiplicative group N^*. □

As in [2, Definition 1.6.34], a nearring N is called subcommutative if $aN = Na$ for each $a \in N$. Recall [2, Definition 1.9.6] that a left N-module G is called strongly monogenic if $G = Ng$ for some $g \in G$ and for all $h \in G$ it is either $Nh = G$ or $Nh = \{e\}$. Moreover, G is faithful if $nG \neq \{e\}$ for any nonzero $n \in N$.

Proposition 5. Let G be a nontrivial faithful abelian strongly monogenic unitary left N-module, N a subcommutative right nearring N with the identity element 1. If T is a nontrivial subgroup of the unit group $U(N)$, then $D(G,T) = E \rtimes F$ is a Frobenius group with a kernel $E \cong G^+$ and a complement $F \cong T$.

Proof. Let us $t \in T$. If $(e,t) \in F \cap F^{(g,1)}$ for some nonzero $g \in G$, then

$$(e,t) = (g,1)^{-1}(e,v)(g,1)$$

for some element $v \in T$ and from this

$$(e,t) = (-g,1)(e,v)(g,1) = (-g,v)(g,1) = ((v-1)g,v).$$

This gives that $(v-1)g = e$ and $v = t$. Since $G = Ng$ and

$$(v-1)G = (v-1)Ng = N(v-1)g = Ne = \{e\},$$

we obtain by the faithfulness of G that $t = 1$. Hence $F \cap F^{(g,1)} = \langle (e,1) \rangle$.

Now if h is a nonzero element of G and t is a nontrivial element of T, then

$$(t-1)G = (t-1)Nh = N(t-1)h = G$$

and therefore

$$E \setminus \{(e,1)\} = H \setminus \bigcup_{(u,v)\in H} F^{(u,v)}.$$

□

A zero-symmetric right nearring N is local if ${}_NL = \{k \in N \mid Nk \neq N\}$ is an N-subgroup [4].

Proposition 6. Let G be a nontrivial abelian monogenic unitary left N-module, where N is a local right nearring with the identity element 1 and the zero $0 \neq 1$. If $D(G, U(N))$ is a Frobenius group, then N is a nearfield.

Proof. By the monogenity $G = Ng$ for some nonzero element $g \in G$. Let j be a nontrivial element of ${}_NL$. Since $D(G, U(N))$ is Frobenius, we deduce that $G = (1-(1-j))G = jG$ and so $g = jng$ for some $n \in N$. But then $(1-jn)g = e$. In view of Corollary 2.6 and Lemma 2.4 from [4] there exists some $t \in N$ such that $t(1-jn) = 1$ and therefore $g = t(1-jn)g = te = e$, a contradiction. This means that ${}_NL = \{0\}$, as desired. □

Example 7. Let $(G,+)$ be a group with the zero e. The set $M(G) = \{f : G \to G \mid f \text{ is a mapping}\}$ is a right nearring with the identity element i_G under two operations "+" and "∘" defined by the rules

$$(f_1+f_2)(x) = f_1(x) + f_2(x) \text{ and } (f_1 \circ f_2)(x) = f_1(f_2(x))$$

for all elements $x \in G$ and $f_1, f_2 \in M(G)$, where $f(x)$ means an image of x with respect to $f \in M(G)$. Hence G is an unitary left $M(G)$-module.

1) Let G be a torsion-free divisible abelian group. If $s : G \to G$ is a mapping defined by the rule $s(g) = 2g$ for all $g \in G$, then $s^n(g) = 2^n g$ for all $n \in \mathbb{Z}$ and $(i_G - s^n)(h) = (1 - 2^n)h \neq e$ for each nonzero element $h \in G$. Moreover, $(i_G - s^n)(G) = G$ for any nonzero $n \in \mathbb{Z}$. By Theorem 2 $D(G, \langle s \rangle)$ is a torsion-free Frobenius group.

2) If $f : G \to G$ is a regular automorphism of G and $G = \{g - f^n(g) \mid g \in G\}$ for any nonzero $n \in \mathbb{Z}$, then $D(G, \langle f \rangle)$ is a Frobenius group.

3) Let G be a torsion-free abelian group, $2G = G$ anf $t : G \to G$ is a mapping defined by the rule $t(g) = -g$ for each $g \in G$. Then $t^2 = i_G$, $(1 - t)(h) = h + h \neq e$ for any nontrivial $h \in G$ and $(1 - t)(G) = 2G = G$. This means that $D(G, \langle t \rangle)$ is a Frobenius group.

4) Let N be a distributive nearring with the identity element 1 and P be a subfield of N with the identity element 1. Suppose that $G = N^+$ is the additive group of N and a is a fixed element from $P \setminus \{0, 1\}$. Then a mapping $\phi : G \to G$ given by $\phi(u) = ua$ $(u \in N)$ is an automorphism of G and $\phi^n(u) = ua^n$ for any $n \in \mathbb{Z}$.

If $a^n \neq 1$ for any nonzero $n \in \mathbb{Z}$, then $(i_G - \phi^n)(h) = h(1 - a^n) = 0$ if and only if $h = 0$. Since $1 - a^n \in P^*$, we deduce that $(i_G - \phi^n)(G) = G$. Hence $D(N^+, \langle \phi \rangle)$ is a Frobenius group.

Now suppose that $a^n = 1$ and n is the smallest positive integer with this property. Then $(i_G - \phi^s)(h) = h(1 - a^s) \neq 0$ for all nonzero $h \in G$ and integers s such that $1 \leq s \leq n - 1$. Moreover, $(i_G - \phi^s)(G) = G(1 - a^s) = G$. From this it follows that $D(N^+, \langle \phi \rangle)$ is a Frobenius group.

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