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RESEARCH ARTICLE

Theory of paradeterminants and its applications

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Abstract. We consider elements of linear algebra based on triangular tables with entries in some number field and their functions, analogical to the classical notions of a matrix, determinant and permanent. Some properties are investigated and applications in various areas of mathematics are given.

Introduction

In the past decades the process of implementing of the notions and methods of linear algebra into combinatorial analysis has been intensified. In particular, there is a well-known monograph by Babai and Frankl [1], and also monographs by V.E. Tarakanov [16] and I.V.Protasov, O.M.Khromulyak [2], which discuss this topic.

The present paper is a continuation of this process. The functions of triangular tables analogical to classical functions of determinant and permanent are considered. While the idea of a determinant is mainly based on the notions of a permutation and a transversal (tuple of elements of a square matrix taken one at a time from each row and each column), the idea of paradeterminant is based on the notions of an ordered partition of a positive integer [6] and a monotransversal (tuple of elements of a triangular matrix taken by one from each column). Mainly due to the implementation of the notion of ordered partition in the construction of functions of triangular tables, the latter gained many applications in combinatorial analysis (see [20]–[22], [24]).

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The author was led to the idea of paradeterminants by the combinatorial problem of computing the number $P(n_1, n_2, \ldots, n_r)$ of shortest paths in Ferrer graphs. Originally the following formula was obtained

$$P(n_1, n_2, \ldots, n_r) =$$

$$= \sum_{\{s_1^{k_1},\ldots,s_l^{k_l}\}\in\Xi(r)} (-1)^{r-(\lambda_1+\ldots+\lambda_p)} \frac{\Pi_{j=1}^{l}\Pi_{i=0}^{k_j-1}(n_{s_j} - k_j + i + 2)}{\Pi_{i=1}^{l} k_i!},$$

where $\Xi(r)$ is the set of multisets $A = \{s_1^{k_1}, \ldots, s_l^{k_l}\}$ defined on page 112, and $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ is a secondary specification [17] of a multiset A. This result was obtained in 1985. Later on, around 2000, while simplifying this formula author came to the idea of some functions of triangular tables of elements, which are called here paradeterminants and parapermanents of triangular matrices, after a suggestion by A.G. Ganyushkin.

The main notions of the theory of paradeterminants were exposed by the author in 2002 (see [21],[19],[18]). Later, A.G. Ganyushkin [25] proved theorem 2, which happened to be useful in applications of paradeterminants to partition polynomials and formal power series. Since paradeterminants have found their applications in several areas of mathematics, the problem of construction of convenient algorithms for computing these functions of triangular matrices naturally arose. This problem was formulated by I.V. Protasov and successfully solved (see theorem 4) by I.I. Lischinsky [25]. In 2003 I.I. Lishchinsky found the connection between paradeterminants and some class of determinants (see theorem 18). Approximately at the same time, during discussion related to some problem about F–determinants, it was noticed by the author and N.M.Dyakiv [26] that F–determinants [21] are a special case of paradeterminants.

Since 2002, all the efforts of the author have been devoted to finding applications of paradeterminants and parapermanents. As a result, their applications were found in number theory and the theory of continued fractions. They appeared to be useful in solving linear recurrent equations, for partition polynomials, operations with formal power series and also in solving combinatorial and some other problems.

It should be noted here that the theory of paradeterminants is being developed under a constant communication between the author and the two Ukrainian mathematicians R.I. Grigorchuk and A.G. Ganyushkin.

1. Definition of a paradeterminant and parapermanent

Let K be a fixed number field.

Definition 1. A triangular table of numbers from some field K

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n \tag{1.1}$$

is called a triangular matrix, and the number n is called its order.

Note that a triangular matrix in the definition is not a matrix in the usual sense because it is triangular rather than rectangular table of numbers.

Definition 2. A matrix of the form

$$A = \begin{pmatrix} M_1 & & & \\ 0 & M_2 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & M_s \end{pmatrix}_n , \tag{1.2}$$

where $M_i,\ i = 1, \ldots, s$ are some triangular matrices and 0's denote some rectangular zero matrices, is called a triangular block–diagonal matrix

To every element a_{ij} of the matrix (1.1) we correspond the $(i-j+1)$ elements $a_{ik},\ k = j, \ldots, i$, which are called the *derived elements* of the matrix generated by the *key element a_{ij}*.

The product of all derived elements generated by the element a_{ij} is denoted by $\{a_{ij}\}$ and called the *factorial product of the key element a_{ij}*, i.e.

$$\{a_{ij}\} = \prod_{k=j}^{i} a_{ik}.$$

Definition 3. A tuple of key elements of the matrix (1.1) is called a normal tuple of this matrix if the derived elements of these key elements form a monotransversal, i.e. they form a set of elements of cardinality n, no two of which belong to the same column in the matrix.

Let $\mathbb{P}(n)$ be the set of all ordered partitions (compositions) (see [5], [6], p. 67) of a positive integer n into positive integer summands. It is known that

$$|\mathbb{P}(n)| = \sum_{r=1}^{n} \binom{n-1}{r-1} = 2^{n-1}. \tag{1.3}$$

It is easy to see that there is a $1-1$ correspondence between normal tuples of key elements of matrix (1.1) and compositions of a positive integer n.

We associate a sign $(-1)^{\varepsilon(a)}$ to every normal tuple a of key elements, where $\varepsilon(a)$ is the sum of all the indices of the key elements of this tuple.

Definition 4. The paradeterminant of a triangular matrix (1.1) is the number

$$ddet(A) = \left\langle \begin{matrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right\rangle =$$

$$= \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_r)\in\mathbb{P}(n)} (-1)^{\varepsilon(a)} \prod_{s=1}^{r} \{a_{i(s),j(s)}\},$$

where $a_{i(s),j(s)}$ is the key element corresponding to the s-th component of the partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$, and the symbol $\varepsilon(a)$ is the sign of the normal tuple a of key elements.

In analogy to the notion of paradeterminant of a matrix (1.1) we introduce the notion of a parapermanent of a triangular matrix.

Definition 5. The paradeterminant of a triangular matrix (1.1) is the number

$$pper(A) = \left[\begin{matrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right] = \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_r)\in\mathbb{P}(n)} \prod_{s=1}^{r} \{a_{i(s),j(s)}\},$$

where $a_{i(s),j(s)}$ is the key element corresponding to the s-th component of the partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$.

Remark 1. According to (1.3) the paradeterminant and the parapermanent of a matrix of order n consist of 2^{n-1} summands.

Example 1. The parapermanent of a 3-rd order matrix is equal to:

$$\left\langle \begin{matrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{matrix} \right\rangle = a_{11}a_{22}a_{33} - a_{21}a_{22}a_{33} - a_{11}a_{32}a_{33} + a_{31}a_{32}a_{33}.$$

Definition 6. *(see [13], [3])* A multiset A is any unordered tuple of elements of some set $[A]$, which is called a basis of this multiset.

The number k of times an element a of a set $[A]$ occurs in a multiset A is called the multiplicity of a in A and is denoted as $a^k \in A$. Multisets are mostly written in their *canonical form* as $A = \{a_1^{k_1}, \ldots, a_n^{k_n}\}$.

Definition 7. $\Xi(n)$-set is the set of all ordered multisets

$$\xi = \{\xi(1), \xi(2), \ldots, \xi(n)\}$$

satisfying the following conditions*:*

1) $\xi(j)$ satisfies inequality $j \leq \xi(j) \leq n,\ j = 1, 2, \ldots, n$;

*2)*for each $j = 1, 2, \ldots, n$ the following equality holds: $\xi(j) = \xi(j+1) = \ldots = \xi(\xi(j))$.

Remark 2. If $j = n$ then inequality 1) implies $\xi(n) = n$.

Proposition 1. *(i) The set* $\Xi(n)$ *contains* 2^{n-1} *elements, and* $\Xi(1) = \{\{1\}\}$.
(ii) If $\Xi(k)$ *is already constructed, then the elements of* $\Xi(k+1)$ *can be obtained in the following way:* 2^{k-1} *elements are obtained by adding* $k+1$ *at the* $(k+1)$*-st place in each* k*–multiset from* $\Xi(k)$*, and the other* 2^{k-1} *elements are obtained by replacing* k *with* $k+1$ *and adding* $k+1$ *at the* $(k+1)$*-st place in each* k*-multiset from* $\Xi(k)$*.*

Theorem 1. If A is a triangular matrix (1.1) then the following equalities hold:

$$ddet(A) = \sum_{\xi \in \Xi(n)} (-1)^{n-r} \cdot a_{\xi(1),1} a_{\xi(2),2} \cdot \ldots \cdot a_{\xi(n),n},$$

$$pper(A) = \sum_{\xi \in \Xi(n)} a_{\xi(1),1} a_{\xi(2),2} \cdot \ldots \cdot a_{\xi(n),n},$$

where r is the number of elements in the basis of the multiset ξ, i.e. number of its distinct elements.

Theorem 2. *(Ganyushkin O.G.)* If A is a triangular matrix (1.1), then the following equalities hold:

$$\mathrm{ddet(A)} = \sum_{r=1}^{n} \sum_{\alpha_1+\ldots+\alpha_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{\alpha_1+\ldots+\alpha_s, \alpha_1+\ldots+\alpha_{s-1}+1}\}, \quad (1.4)$$

$$\mathrm{pper(A)} = \sum_{r=1}^{n} \sum_{\alpha_1+\ldots+\alpha_r=n} \prod_{s=1}^{r} \{a_{\alpha_1+\ldots+\alpha_s, \alpha_1+\ldots+\alpha_{s-1}+1}\}, \quad (1.5)$$

where the summation is over the set of positive integer solutions of the equation $\alpha_1 + \ldots + \alpha_r = n$.

Let us define product of a vector $(b_1, b_2, \ldots, b_n)$ with a matrix paradeterminant (1.1), using equality (1.4) by

$$(b_1, b_2, \ldots, b_n) \cdot ddet(A) \stackrel{def}{=} \quad (1.6)$$

$$= \sum_{r=1}^{n} b_r \cdot \sum_{\alpha_1+\ldots+\alpha_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{\alpha_1+\ldots+\alpha_s, \alpha_1+\ldots+\alpha_{s-1}+1}\}.$$

Analogously define product of a vector $(b_1, b_2, \ldots, b_n)$ with a matrix parapermanent using equality (1.5) by

$$(b_1, b_2, \ldots, b_n) \cdot pper(A) \stackrel{def}{=} \tag{1.7}$$

$$= \sum_{r=1}^{n} b_r \cdot \sum_{\alpha_1+\ldots+\alpha_r=n} \prod_{s=1}^{r} \{a_{\alpha_1+\ldots+\alpha_s, \alpha_1+\ldots+\alpha_{s-1}+1}\}.$$

To each element a_{ij} of a triangular matrix (1.1) we associate the triangular table of elements of this matrix that has a_{ij} in the bottom left corner. We call this table a *corner* of the matrix and denote it by R_{ij}. Obviously, the corner R_{ij} is a triangular matrix of order $(i - j + 1)$, and it contains only elements a_{rs} of matrix (1.1) whose indices satisfy the inequalities $j \le s \le r \le i$.

In the sequel we will assume that

$$ddet(R_{01}) = pper(R_{01}) = ddet(R_{n,n+1}) = pper(R_{n,n+1}) = 1.$$

Definition 8. A rectangular table, denoted $T(i)$, $i = 1, 2, \ldots, n$, of elements of a triangular matrix (1.1) is inscribed in this matrix if one of its vertices coincides with the element a_{n1}, and the opposite one coincides with the element a_{ii}. We will denote this table by $T(i)$.

Remark 3. According to definition 8, $T(1)$ is the fist column and $T(n)$ is the last row of the matrix.

In order to compute paradeterminants and parapermanents it is convenient to use *algebraic complements.*

Definition 9. The numbers

$$D_{ij} = (-1)^{i+j} \cdot ddet(R_{j-1,1}) \cdot ddet(R_{n,i+1}),$$

$$P_{ij} = pper(R_{j-1,1}) \cdot pper(R_{n,i+1}),$$

where $R_{j-1,1}$ and $R_{n,i+1}$ are corners, are called the algebraic complements to the factorial product $\{a_{ij}\}$ of the key element a_{ij} of the triangular matrix (1.1).

More detailed information about the notions in the first section can be found in [18].

2. Properties of paradeterminants and parapermanents of triangular matrices

Although the definitions of paradeterminants and parapermanents significantly differ from the classical definitions of determinants and permanents, their properties are similar in many ways.

Proposition 2. *[18]* Let each elements $a_{ri},\ r = i, i+1, \ldots, n$ of the i-th column of the triangular matrix (1.1) be a sum of two elements $b_{ri} + c_{ri}$. Then the paradeterminant of this matrix equals the sum of two paradeterminants corresponding to matrices whose elements are equal to the elements in A, except for the elements in the i-th column, which are equal to b_{ri} and $c_{ri}, r = i, i+1, \ldots, n$, respectively.

Analogous statement is true for parapermanents.

Proposition 3. *[18]* For a block-diagonal triangular matrix (1.2) the following is true:

$$ddet(A) = ddet(M_1) \cdot ddet(M_2) \cdot \ldots \cdot ddet(M_s)$$

$$pper(A) = pper(M_1) \cdot pper(M_2) \cdot \ldots \cdot pper(M_s)$$

A proposition on differentiation of paradeterminants and parapermanents analogous to the one for determinants and permanents of a square matrix holds.

Theorem 3. *[18] (Decomposition of paradeterminant and parapermanent by elements of an inscribed rectangular table)* Let A be triangular matrix (1.1) and $T(i)$ be some inscribed rectangular table. Then the following equalities hold:

$$ddet(A) = \sum_{s=1}^{i} \sum_{r=i}^{n} \{a_{rs}\} \cdot D_{rs}, \tag{2.1}$$

$$pper(A) = \sum_{s=1}^{i} \sum_{r=i}^{n} \{a_{rs}\} \cdot P_{rs}, \tag{2.2}$$

where D_{rs} and P_{rs} are the algebraic complements to the factorial product of the key element a_{rs}, which belongs to the inscribed rectangular table $T(i)$.

Corollary 1. For $i = 1$ by formulae (2.1)–(2.2) and 3, obtain the decomposition of paradeterminant and parapermanent by elements of their first column

$$ddet(A) = \sum_{r=1}^{n} (-1)^{r+1} \cdot \{a_{r1}\} \cdot ddet(R_{n,r+1}),$$

$$pper(A) = \sum_{r=1}^{n} \{a_{r1}\} \cdot pper(R_{n,r+1}).$$

For $i = n$ obtain the respective decompositions by elements of last row

$$ddet(A) = \sum_{s=1}^{n} (-1)^{n+s} \cdot \{a_{ns}\} \cdot ddet(R_{s-1,1}),$$

$$pper(A) = \sum_{s=1}^{n} \{a_{ns}\} \cdot pper(R_{s-1,1}).$$

It is known that the question of Polia "Does there exist a way to assign + and − sign to each element of a square matrix of order n, $n \geq 3$, in such a way that its permanent would be equal the determinant?" doesn't have a positive answer. Moreover, Markus and Minc proved [9] that for $n \geq 3$ there is no linear transformation T on the set of all matrices of order n such that $per(T(A)) = det(A)$.

But this is not the case for paradeterminants and parapermanents.

Proposition.[18] *If A is a triangular matrix (1.1), then the following equality holds*

$$ddet((-1)^{\delta_{ij}+1} \cdot a_{ij})_{1 \leq j \leq i \leq n} = pper(a_{ij})_{1 \leq j \leq i \leq n}. \tag{2.3}$$

Remark 4. Using multiplication of vectors by paradeterminants (see (1.6), p. 112), equality (2.3) can be written as

$$((-1)^{n-1}, (-1)^{n-2}, \cdots, (-1)^{0}) \cdot \left\langle \begin{matrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right\rangle =$$

$$= \left[\begin{matrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right].$$

The following theorem provides a convenient way of computing paradeterminants and parapermanents.

Theorem 4. *(Lishchinsky I.I.)* The following equalities hold:

$$\left\langle \begin{matrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right\rangle = (-1)\cdot \left\langle \begin{matrix} (a_{21}-a_{11})\cdot a_{22} & & & \\ (a_{31}-a_{11})\cdot a_{32} & a_{33} & & \\ \vdots & \vdots & \ddots & \\ (a_{n1}-a_{11})\cdot a_{n2} & a_{n3} & \cdots & a_{nn} \end{matrix} \right\rangle,$$

$$\left[\begin{matrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right]_n = \left[\begin{matrix} (a_{21}+a_{11})\cdot a_{22} & & & \\ (a_{31}+a_{11})\cdot a_{32} & a_{33} & & \\ \vdots & \vdots & \ddots & \\ (a_{n1}+a_{11})\cdot a_{n2} & a_{n3} & \cdots & a_{nn} \end{matrix} \right]_{n-1}.$$

Remark 5. To find the value of a paradeterminant (parapermanent) it is enough to perform $\frac{n\cdot(n-1)}{2}$ multiplications and the same number of additions. Since a triangular matrix contains $\frac{n\cdot(n+1)}{2}$ entries and all of them affect the value, the proposed algorithm cannot be essentially improved.

Note that we listed only a few established paradeterminant and parapermanent properties. Other properties can be found in the paper [18].

3. Applications of parapermanents in the investigation of linear recurrent sequences of k-th order

Parapermanents of triangular matrices are convenient tools for investigation of linear recurrent sequences. They can be used to solve linear recurrent equations of k-th order, and to establish some important functional relations between the terms of sequences generated by recurrent equations. Also they are used to define an important class of so called *normal* initial conditions in linear recurrent equations.

Theorem 5. *[18]* Consider a linear equation of k-th order

$$u_n = a_1\cdot u_{n-1} + \ldots + a_k\cdot u_{n-k},\ \ a_1 \neq 0,\ \ 1 \le k \le n,\ \ n = k+1, k+2, \ldots, \tag{3.1}$$

with initial conditions

$$u_i = a_i^{(0)},\ \ a_i^{(0)} \in R,\ \ i = 1, \ldots, k. \tag{3.2}$$

Then the following equality holds

$$u_n = \left[\begin{array}{cccccccc} a_1x_1x_2 & & & & & & & \\ \frac{a_2}{a_1}x_1 & a_1x_3 & & & & & & \\ \cdots & \cdots & \ddots & & & & & \\ \frac{a_{k-1}}{a_{k-2}}x_1 & \frac{a_{k-2}}{a_{k-3}} & \cdots & a_1x_k & & & & \\ \frac{a_k}{a_{k-1}}x_1 & \frac{a_{k-1}}{a_{k-2}} & \cdots & \frac{a_2}{a_1} & a_1 & & & \\ 0 & \frac{a_k}{a_{k-1}} & \cdots & \frac{a_3}{a_2} & \frac{a_2}{a_1} & a_1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \\ 0 & 0 & \cdots & 0 & \frac{a_k}{a_{k-1}} & \cdots & \frac{a_2}{a_1} & a_1 \end{array}\right]_{n-1}, \tag{3.3}$$

$$u_1 = x_1,\ n = k+1, k+2, \cdots,$$

where the corrections x_i are defined by the equalities

$$x_1 = a_1^{(0)},\ x_i = \frac{a_i^{(0)}}{a_1a_{i-1}^{(0)} + a_2a_{i-2}^{(0)} + \cdots + a_{i-2}a_2^{(0)} + a_{i-1}a_1^{(0)}},\ i = 2, \ldots, k,$$

The following is also true

$$\frac{a_1^{(0)} + a_2^{(0)} \cdot (1 - \frac{1}{x_2}) \cdot x^1 + \ldots + a_k^{(0)} \cdot (1 - \frac{1}{x_k}) \cdot x^{k-1}}{1 - a_1x - \ldots - a_kx^k} = \sum_{i=1}^{\infty} u_ix^{i-1}. \tag{3.4}$$

Remark 6. For $k = 1$ equality (3.3) is of the form

$$u_n = \left[\begin{array}{cccc} a_1x_1 & & & \\ 0 & a_1 & & \\ \cdots & \cdots & \ddots & \\ 0 & 0 & \cdots & a_1 \end{array}\right]_{n-1}.$$

Remark 7. If some coefficient a_i, $i = 2, \ldots, k-1$ in the recurrent equation (3.1) equals zero, then the zeros cancel out in the evaluation of paradeterminants or parapermanents and the indefiniteness disappears.

Let us denote

$$\overline{u}_i = a_1a_{i-1}^{(0)} + a_2a_{i-2}^{(0)} + \ldots + a_{i-1}a_i^{(0)},\ \ i = 2, \ldots, k.$$

The differences $u_i - \overline{u}_i$ will be called *defects*. The equality (3.4) can then be written as

$$\frac{u_1 + \sum_{i=2}^{k}(u_i - \overline{u}_i) \cdot x^{i-1}}{1 - a_1x - \ldots - a_kx^k} = \sum_{i=1}^{\infty} u_ix^{i-1} \tag{3.5}$$

Definition 10. *Let us call initial conditions*

$$u_1 = 1,\ u_i = \begin{bmatrix} a_1 & & & \\ \frac{a_2}{a_1} & a_1 & & \\ \cdots & \cdots & \ddots & \\ \frac{a_{i-1}}{a_{i-2}} & \frac{a_{i-2}}{a_{i-3}} & \cdots & a_1 \end{bmatrix}_{i-1},\ i = 2, \ldots, k \tag{3.6}$$

of the recurrent equation (3.1) normal initial conditions.

Remark 8. If initial conditions in the equation (3.1) are normal, then all corrections $x_i,\ i = 1, \ldots, k$, in the equality (3.3) are equal to one, and all defects in the equality (3.5) are equal to zero. Then equality (3.5) may be written as

$$\frac{1}{1 - a_1x - \ldots - a_kx^k} = \sum_{i=1}^{\infty} u_i x^{i-1}.$$

As noted before, it is easy to establish general functional relations between terms of sequences generated by linear recurrent equations of k-th order by using parapermanents, which are relatively hard to prove even in the case $k = 2$.

Theorem 6. If sequences $\{u_n^*\}_{n=1}^{\infty},\ \{u_n\}_{n=1}^{\infty}$ satisfy the recurrent equation (3.1) of k-th order with initial conditions (3.2) and (3.6) respectively, and $k < r$, then

$$u_{r+s}^* = \sum_{i=1}^{k} a_i \left(\sum_{j=r-i+1}^{r} u_j^* u_{r+s-i-j+1} \right). \tag{3.7}$$

Corollary 2. If a sequence $\{u_n\}_{n=1}^{\infty}$ satisfies the recurrent equation (3.1) with normal initial conditions (3.6), then

$$u_{r+s} = \sum_{i=1}^{k} a_i \left(\sum_{j=r-i+1}^{r} u_j u_{r+s-i-j+1} \right). \tag{3.8}$$

To establish relation (3.8) it is enough to decompose the parapermanent which is the solution of equation (3.1) with normal initial conditions, by elements of some inscribed rectangular table.

The next two theorems illustrate application of parapermanents in investigating number-theoretic properties of sequences generated by linear recurrent equation of second order.

We essentially use relations (3.7), (3.8) for $k = 2$.

Theorem 7. Let the sequence $\{u_k\}_{k=1}^{\infty}$ satisfy the recurrent equation of second order

$$u_{n+2} = a_1 u_{n+1} + a_2 u_n \tag{3.9}$$

with integer non-zero coefficients and normal initial conditions

$$u_1 = 1,\ u_2 = a_1.$$

Then

1) the following equalities hold:

$$u_{r+s} = u_{r+1}u_s + a_2 u_r u_{s-1},\ r = 1, 2, \dots;\ s = 2, 3, \dots,$$

$$u_{sr} \equiv 0 (mod\, u_r),\ s, r = 1, 2, \dots;$$

2) if the coefficients in (3.9) are relatively prime, i.e. $(a_1, a_2) = 1$, then

$$(u_s, u_r) = u_{(s,r)}.$$

Corollary 3. If the sequence $\{u_k\}_{k=1}^{\infty}$ satisfies the conditions of theorem 7 and $u_k \neq 1,\ 2 \leq k$, then u_s is a prime if and only if s is a prime.

Corollary 4. Let the sequence $\{u_k\}_{k=1}^{\infty}$ satisfy conditions of theorem 7 and p be a prime. Then u_p is relatively prime with all previous terms of this sequence.

Corollary 5. Let the sequence $\{u_k\}_{k=1}^{\infty}$ satisfy the conditions of theorem 7. If $a_2 = b^2$, where b is an integer, then each term $u_{2m+1},\ 1 < m$ of this sequence can be represented as a sum of squares of two natural numbers

$$u_{2m+1} = u_{m+1}^2 + (bu_m)^2.$$

Corollary 6. For each natural number $m > 0$ the following holds:

$$\sum_{i=0}^{m} (-1)^i \binom{2m-i}{i} a^{2(m-i)} b^{2i} =$$

$$= \left(\sum_{i=0}^{[m/2]} (-1)^i \binom{m-i}{i} a^{m-2i} b^{2i} - \right.$$

$$\left. - \sum_{i=0}^{[(m-1)/2]} (-1)^i \binom{m-i-1}{i} a^{m-2i-1} b^{2i+1} \right) \times$$

$$\times \left(\sum_{i=0}^{[m/2]} (-1)^i \binom{m-i}{i} a^{m-2i} b^{2i} + \right.$$

$$\left. + \sum_{i=0}^{[(m-1)/2]} (-1)^i \binom{m-i-1}{i} a^{m-2i-1} b^{2i+1} \right)$$

Example 2. For $m = 7$ and $m = 11$ identity (2) can be written as:

$$a^6 - 5a^4b^2 + 6a^2b^4 - b^6 = (a^3 - 2ab^2 - a^2b + b^3) \cdot (a^3 - 2ab^2 + a^2b - b^3),$$

$$a^{10} - 9a^8b^2 + 28a^6b^4 - 35a^4b^6 + 15a^2b^8 - b^{10} =$$
$$= (a^5 - a^4b - 4a^3b^2 + 3a^2b^3 + 3ab^4 - b^5)\cdot$$
$$\cdot(a^5 + a^4b - 4a^3b^2 - 3a^2b^3 + 3ab^4 + b^5).$$

Remark 9. Since the recurrent equation $u_{n+2} = u_{n+1} + u_n$ generates the Fibonacci sequence, theorem 7 can be considered as generalization of some relations between Fibonacci numbers (see [7], p. 325-327).

Theorem 8. Let the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{u_n^*\}_{n=1}^{\infty}$ satisfy a recurrent equation of second order

$$u_{n+2} = a_1u_{n+1} + a_2u_n$$

with initial conditions

$$u_1 = 1,\ u_2 = a_1;\ u_1^* = k,\ u_2^* = a_1,$$

respectively. Then:

(1)for each n, $n \geq 3$, the following holds

$$u_n^* = u_n + (k-1)a_2u_{n-2};$$

(2) If

$$k = a_2 = s^2 + 1,$$

and $0 < a_1$, then for each n, $n \geq 3$, the number u_{2n-1}^* is a sum of three squares:

$$u_{2n-1}^* = (u_n)^2 + ((s^2+1)\cdot u_{n-1})^2 + ((s^3+s)\cdot u_{n-2})^2; \qquad (3.10)$$

(3) If

$$k = s^2 + 1,\ a_2 = b^2,$$

then for each $n,\ 2 \leq n$ the number u_{2n+1}^* is a sum of four squares:

$$u_{2n+1}^* = u_{n+1}^2 + (bu_n)^2 + (sbu_n)^2 + (sb^2u_{n-1})^2. \qquad (3.11)$$

Example 3. Let $a_1 = 4,\ s = 2$ in theorem 8 . Then $u_1 = a_2 = k = 5$,

$$u_n^* = \frac{1}{2}\cdot(3\cdot 5^{n-1} + 7\cdot(-1)^{n-1}),\ u_n = \frac{1}{6}\cdot(5^n + (-1)^{n-1})$$

and equality (3.10) can be written as

$$\frac{1}{2}\cdot(3\cdot 5^{2n-2}+7) == \left(\frac{1}{6}\cdot(5^n+(-1)^{n-1})\right)^2+$$
$$\left(\frac{5}{6}\cdot(5^{n-1}+(-1)^{n-2})\right)^2+\left(\frac{5}{3}\cdot(5^{n-2}+(-1)^{n-3})\right)^2.$$

In particular, for $n=13$ the last equality provides a decomposition of the prime number 89406967163085941 into sum of three squares:

$$89406967163085941 = 203450521^2+203450520^2+81380210^2.$$

Example 4. If in theorem 8, (3) we put $a_1=3$, $a_2=b=2$, $s=1$, then $u_n^*=4^{n-1}+(-1)^{n-1}$, $u_n=\frac{1}{5}\cdot(4^n+(-1)^{n-1})$ and equality (3.11) transforms to

$$2^{4m}+1=\left(\frac{4^{m+1}+(-1)^m}{5}\right)^2+\left(\frac{2^{2m+1}+(-1)^{m-1}\cdot 2}{5}\right)^2+$$
$$+\left(\frac{2^{2m+1}+(-1)^{m-1}\cdot 2}{5}\right)^2+\left(\frac{2^{2m}+(-1)^{m-2}\cdot 4}{5}\right)^2.$$

Since Fermat numbers $F_n=2^{2^n}+1$ can be written as $2^{4m}+1$ for all $n\geq 2$, they can be decomposed into sum of four squares of positive integers for all $n\geq 3$.

4. Applications of parapermanents to the study of continued fractions

Suppose we are given some continued periodic fraction

$$\delta = a_0+\frac{b_1}{a_1+}\frac{b_2}{a_2+\cdots+}\frac{b_n}{a_n+\cdots}=a_0+\mathop{K}_{i=1}^{\infty}\left(\frac{b_i}{a_i}\right),$$

where

$$a_{sk+m}=a_m>0;\ b_{sk+m}=b_m>0;\ m=1,\ldots,k;\ s=0,1,\ldots;\ k\geq 2$$

and its n-th approaching fraction

$$\delta_n == \frac{P_n}{Q_n}=a_0+\frac{b_1}{a_1+}\frac{b_2}{a_2+\cdots+}\frac{b_n}{a_n}=a_0+\mathop{K}_{i=1}^{n}\left(\frac{b_i}{a_i}\right),$$

where

$$P_n=a_nP_{n-1}+b_nP_{n-2},\ \ P_{-2}=0, P_{-1}=1,$$

$$Q_n = a_n Q_{n-1} + b_n Q_{n-2},\ \ Q_{-2} = 1, P_{-1} = 0,\ b_0 = 1.$$

It is easy to see that

$$P_n = \begin{bmatrix} a_0 & & & & \\ \frac{b_1}{a_1} & a_1 & & & \\ 0 & \frac{b_2}{a_2} & a_2 & & \\ \cdots & \cdots & \cdots & \ddots & \\ 0 & \ldots & 0 & \frac{b_n}{a_n} & a_n \end{bmatrix},\ n = 0, 1, \ldots;$$

$$Q_n = \begin{bmatrix} a_1 & & & & \\ \frac{b_2}{a_2} & a_2 & & & \\ 0 & \frac{b_3}{a_3} & a_3 & & \\ \cdots & \cdots & \cdots & \ddots & \\ 0 & \ldots & 0 & \frac{b_n}{a_n} & a_n \end{bmatrix},\ n = 1, 2, \ldots$$

Theorem 9. *[23]* The sequence

$$\delta_r = a_0 + b_1 \cdot \frac{B_{r-1}}{A_r},$$

where

$$A_r = b_1 \cdot \alpha \cdot B_{r-2} + \beta \cdot A_{r-1},\ \ B_{r-1} = b_1 \cdot \gamma \cdot B_{r-2} + \lambda \cdot A_{r-1},$$

$$\alpha = \begin{bmatrix} a_1 & & & & \\ \frac{b_2}{a_2} & a_2 & & & \\ 0 & \frac{b_3}{a_3} & a_3 & & \\ \cdots & \cdots & \cdots & \ddots & \\ 0 & \ldots & 0 & \frac{b_{k-1}}{a_{k-1}} & a_{k-1} \end{bmatrix},\ \beta = A_1 = \begin{bmatrix} a_1 & & & & \\ \frac{b_2}{a_2} & a_2 & & & \\ 0 & \frac{b_3}{a_3} & a_3 & & \\ \cdots & \cdots & \cdots & \ddots & \\ 0 & \ldots & 0 & \frac{b_k}{a_k} & a_k \end{bmatrix},$$

$$\gamma = \begin{bmatrix} a_2 & & & & \\ \frac{b_3}{a_3} & a_3 & & & \\ 0 & \frac{b_4}{a_4} & a_4 & & \\ \cdots & \cdots & \cdots & \ddots & \\ 0 & \ldots & 0 & \frac{b_{k-1}}{a_{k-1}} & a_{k-1} \end{bmatrix},\ \lambda = B_0 = \begin{bmatrix} a_2 & & & & \\ \frac{b_3}{a_3} & a_3 & & & \\ 0 & \frac{b_4}{a_4} & a_4 & & \\ \cdots & \cdots & \cdots & \ddots & \\ 0 & \ldots & 0 & \frac{b_k}{a_k} & a_k \end{bmatrix},$$

such that the following inequality is satisfied

$$\omega = \frac{b_1}{\beta^2} \cdot |\beta\gamma - \alpha\lambda| < 1,$$

converges to δ, and moreover there is an error bound of the form

$$|\delta - \delta_r| < \sigma \cdot \frac{\omega^r}{1 - \omega},$$

where

$$\sigma = \frac{b_1\lambda\beta}{b_1\lambda\alpha + \beta^2}.$$

Remark 10. *When $k = 2$ we assume $\gamma = 1$.*

Analogous theorem on combined continued fractions is proved in [23].

5. Paradeterminants and partition polynomials

The notion of *partition polynomials* introduced by Bell [8] has wide range of applications in discrete mathematics. They appear in differentiation of composite functions [10], [12], in number theory [11], algebra, etc. In this section identities between some important partition polynomials and paradeterminants of triangular matrices are established.

Consider a triangular matrix of the form

$$A = \begin{pmatrix} k_{11} \cdot x_1 & & & \\ k_{21} \cdot \frac{x_2}{x_1} & k_{22} \cdot x_1 & & \\ \cdots & \cdots & \cdots & \\ k_{n1} \cdot \frac{x_n}{x_{n-1}} & k_{n2} \cdot \frac{x_{n-1}}{x_{n-2}} & \cdots & k_{nn} \cdot x_1 \end{pmatrix}_n = \left(k_{ij} \cdot \frac{x_{i-j+1}}{x_{i-j}} \right)_{1 \le j \le i \le n}, \tag{5.1}$$

$x_0 = 1$, where k_{ij} is some rational function of i and j.

Let M be a fixed multiset with *primary specification of Sachkov* [17] of the form $\left[1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n}\right]$.

If the exponents of the primary specification of a multiset M satisfy the equation $\lambda_1 + 2\lambda_2 + \ldots + n\lambda_n = n$ then such a multiset is called an unordered partition of the positive integer n and is denoted by $\pi(n)$.

The sum of all elements of the primary specification of a partition $\pi(n)$ is denoted by $\lambda(\pi)$, i.e.

$$\lambda(\pi) = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$

Note that $\lambda(\pi)$ has certain combinatorial sense, namely it is the number of components of the partition $\pi(n)$.

Let $\Pi(n)$ be the set of all multisets $\pi(n)$, and $\Pi_k(n)$ be the set of all multisets $\pi(n)$ such that the exponents of their primary specifications satisfy the equality $\lambda(\pi) = k$.

Definition 11. Partition polynomials are polynomials of the form

$$P(x_1, \ldots, x_n) = \sum_{k=1}^{n} y_k \cdot \sum_{\pi(n) \in \Pi_k(n)} c(n; \lambda_1, \ldots \lambda_n) \cdot x_1^{\lambda_1} \cdot \ldots \cdot x_n^{\lambda_n}, \tag{5.2}$$

where y_k, $c(n; \lambda_1, \ldots, \lambda_n)$ are some rational numbers.

Remark 11. If all y_k in the equality (5.2) are equal to 1, then it can be transformed to the following form

$$\begin{aligned} P(x_1,\ldots,x_n) &= \sum_{k=1}^{n} \sum_{\pi(n)\in\Pi_k(n)} c(n;\lambda_1,\ldots,\lambda_n)\cdot x_1^{\lambda_1}\cdot\ldots\cdot x_n^{\lambda_n} = \\ &= \sum_{\pi(n)\in\Pi(n)} c(n;\lambda_1,\ldots\lambda_n)\cdot x_1^{\lambda_1}\cdot\ldots\cdot x_n^{\lambda_n}. \end{aligned} \tag{5.3}$$

Polynomials of the form (5.3) are called *primary partition polynomials.*

Theorem 10. Paradeterminants and parapermanents of triangular matrices of the form (5.1) are primary partition polynomials, i.e. the equalities

$$pper(A) = \sum_{\pi(n)\in\Pi(n)} c(n;\lambda_1,\ldots,\lambda_n)\cdot x_1^{\lambda_1}\cdot\ldots\cdot x_n^{\lambda_n}, \tag{5.4}$$

$$ddet(A) = \sum_{\pi(n)\in\Pi(n)} (-1)^{n-\lambda(\pi)}\cdot c(n;\lambda_1,\ldots,\lambda_n)\cdot x_1^{\lambda_1}\cdot\ldots\cdot x_n^{\lambda_n}. \tag{5.5}$$

are satisfied.

If the equalities (5.4)-(5.5) are satisfied, then the partition polynomials

$$\sum_{k=1}^{n} y_k\cdot\left(\sum_{\pi(n)\in\Pi_k(n)} c(n;\lambda_1,\ldots,\lambda_n)\cdot x_1^{\lambda_1}\cdot\ldots\cdot x_n^{\lambda_n}\right), \tag{5.6}$$

$$\sum_{k=1}^{n} y_k\cdot\left(\sum_{\pi(n)\in\Pi_k(n)} (-1)^{n-k}\cdot c(n;\lambda_1,\ldots,\lambda_n)\cdot x_1^{\lambda_1}\cdot\ldots\cdot x_n^{\lambda_n}\right),$$

according to the definition (1.6) of a product of a vector and a paradeterminant and the definition (1.7) of a product of a vector and a parapermanent, may be written in the form of products:

$$(y_1,\ldots,y_n)\cdot\left\langle\begin{array}{cccc} \tau_{11}\cdot x_1 & & & \\ \tau_{21}\cdot\frac{x_2}{x_1} & \tau_{22}\cdot x_1 & & \\ \vdots & \cdots & \ddots & \\ \tau_{n1}\cdot\frac{x_n}{x_{n-1}} & \tau_{n2}\cdot\frac{x_{n-1}}{x_{n-2}} & \cdots & \tau_{nn}\cdot x_1 \end{array}\right\rangle$$

$$(y_1,\ldots,y_n)\cdot\left[\begin{array}{cccc} \tau_{11}\cdot x_1 & & & \\ \tau_{21}\cdot\frac{x_2}{x_1} & \tau_{22}\cdot x_1 & & \\ \cdots & \cdots & \cdots & \\ \tau_{n1}\cdot\frac{x_n}{x_{n-1}} & \tau_{n2}\cdot\frac{x_{n-1}}{x_{n-2}} & \cdots & \tau_{nn}\cdot x_1 \end{array}\right]$$

Definition 12. A triangular matrix of the form

$$B_n(a_1, a_2, \ldots, a_n) =$$

$$= \begin{pmatrix} a_1 & & & & & \\ \frac{1}{1}\cdot\frac{a_2}{a_1} & a_1 & & & & \\ \frac{1}{2}\cdot\frac{a_3}{a_2} & \frac{2}{1}\cdot\frac{a_2}{a_1} & a_1 & & & \\ \vdots & \ldots & \ldots & \ddots & & \\ \frac{1}{n-2}\cdot\frac{a_{n-1}}{a_{n-2}} & \frac{2}{n-3}\cdot\frac{a_{n-2}}{a_{n-3}} & \frac{3}{n-4}\cdot\frac{a_{n-3}}{a_{n-4}} & \ldots & a_1 & \\ \frac{1}{n-1}\cdot\frac{a_n}{a_{n-1}} & \frac{2}{n-2}\cdot\frac{a_{n-1}}{a_{n-2}} & \frac{3}{n-3}\cdot\frac{a_{n-2}}{a_{n-3}} & \ldots & \frac{n-1}{1}\cdot\frac{a_2}{a_1} & a_1 \end{pmatrix}_n =$$

$$= \left(\frac{j}{i-j+j\cdot\delta_{ij}} \cdot \frac{a_{i-j+1}}{a_{i-j}} \right)$$

is called a triangular Bell matrix.

Proposition 4. The right part of the Faa di Bruno formula

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{m=1}^{n} \frac{d^m}{dg^m} f(g(x)) \sum_{\pi(n)\in\Pi_m(n)} \frac{n!}{\lambda_1!\cdot\ldots\cdot\lambda_n!\cdot(1!)^{\lambda_1}\cdot\ldots\cdot(n!)^{\lambda_n}} \times$$

$$\times (g'(x))^{\lambda_1}\cdot\ldots\cdot(g^{(n)}(x))^{\lambda_n}$$

is a partition polynomial and the formula can be expressed in the form

$$\frac{d^n}{dx^n} f(g(x)) = (f_g^{'}, f_g^{''}, \ldots, f_g^{(n)}) \cdot pper(B(g_x^{'}, g_x^{''}, \ldots, g_x^{(n)}).$$

Proposition 5. *For a triangular matrix of the form*

$$Z\left(x_1, x_2, \ldots, x_n\right) = \begin{pmatrix} x_1 & & & \\ \frac{x_2}{x_1} & x_1 & & \\ \cdots & \cdots & \cdots & \\ \frac{x_n}{x_{n-1}} & \frac{x_{n-1}}{x_{n-2}} & \cdots \ \frac{x_2}{x_1} & x_1 \end{pmatrix}_n = \left(\frac{x_{i-j+1}}{x_{i-j}}\right)_{1\le j\le i\le n} \tag{5.7}$$

the next identities hold:

$$pper\left(Z(x_1, \ldots, x_n)\right) = \sum_{\pi(n)\in\Pi(n)} \frac{(\lambda_1+\ldots+\lambda_n)!}{\lambda_1!\cdot\ldots\cdot\lambda_n!} \cdot x_1^{\lambda_1}\cdot\ldots\cdot x_n^{\lambda_n}, \tag{5.8}$$

$$ddet\left(Z(x_1, \ldots, x_n)\right) = \tag{5.9}$$

$$= \sum_{\pi(n)\in\Pi(n)} (-1)^{n-(\lambda_1+\ldots+\lambda_n)} \cdot \frac{(\lambda_1+\ldots+\lambda_n)!}{\lambda_1!\cdot\ldots\cdot\lambda_n!} \cdot x_1^{\lambda_1}\cdot\ldots\cdot x_n^{\lambda_n}.$$

By replacing x_i, $i = 1, \ldots, n$,, in identities (5.8), (5.9) by $\frac{y_i}{i}$ and $\frac{y_i}{i!}$, respectively, we obtain new important identities:

$$pper\left(Z\left(\frac{y_1}{1}, \frac{y_2}{2}, \ldots, \frac{y_n}{n}\right)\right) = \left[\frac{i-j+\delta_{ij}}{i-j+1} \cdot \frac{y_{i-j+1}}{y_{i-j}}\right]_{1\le j\le i\le n} =$$

$$= \sum_{\Pi(n)} \frac{(\lambda_1 + \ldots + \lambda_n)!}{1^{\lambda_1}\lambda_1! \cdot \ldots \cdot n^{\lambda_n}\lambda_n!} \cdot y_1^{\lambda_1} \cdot \ldots \cdot y_n^{\lambda_n},$$

$$pper\left(Z\left(\frac{y_1}{1!}, \frac{y_2}{2!}, \ldots, \frac{y_n}{n!}\right)\right) = \left[\frac{1}{i-j+1} \cdot \frac{y_{i-j+1}}{y_{i-j}}\right]_{1\le j\le i\le n} =$$

$$= \sum_{\Pi(n)} \frac{(\lambda_1 + \ldots + \lambda_n)!}{(1!)^{\lambda_1}\lambda_1! \cdot \ldots \cdot (n!)^{\lambda_n}\lambda_n!} \cdot y_1^{\lambda_1} \cdot \ldots \cdot y_n^{\lambda_n},$$

$$ddet\left(Z\left(\frac{y_1}{1}, \frac{y_2}{2}, \ldots, \frac{y_n}{n}\right)\right) = \left\langle\frac{i-j+\delta_{ij}}{i-j+1} \cdot \frac{y_{i-j+1}}{y_{i-j}}\right\rangle_{1\le j\le i\le n} =$$

$$= \sum_{\Pi(n)} (-1)^{n-(\lambda_1+\ldots+\lambda_n)} \cdot \frac{(\lambda_1 + \ldots + \lambda_n)!}{1^{\lambda_1}\lambda_1! \cdot \ldots \cdot n^{\lambda_n}\lambda_n!} \cdot y_1^{\lambda_1} \cdot \ldots \cdot y_n^{\lambda_n},$$

$$ddet\left(Z\left(\frac{y_1}{1!}, \frac{y_2}{2!}, \ldots, \frac{y_n}{n!}\right)\right) = \left\langle\frac{1}{i-j+1} \cdot \frac{y_{i-j+1}}{y_{i-j}}\right\rangle_{1\le j\le i\le n} =$$

$$= \sum_{\Pi(n)} (-1)^{n-(\lambda_1+\ldots+\lambda_n)} \cdot \frac{(\lambda_1 + \ldots + \lambda_n)!}{(1!)^{\lambda_1}\lambda_1! \cdot \ldots \cdot (n!)^{\lambda_n}\lambda_n!} \cdot y_1^{\lambda_1} \cdot \ldots \cdot y_n^{\lambda_n}.$$

Consider one more example of a primitive partition polynomial.

Proposition 6. If

$$C = \left(\frac{j}{i-j+1} \cdot \frac{x_{i-j+1}}{x_{i-j}}\right)_{1\le j\le i\le n},$$

then the following identities hold:

$$pper(C) = \sum_{\pi(n)\in\Pi(n)} \frac{n! \cdot (\lambda(\pi))!}{\lambda_1! \cdot \ldots \cdot \lambda_n! \cdot (1!)^{\lambda_1} \cdot \ldots \cdot (n!)^{\lambda_n}} \cdot x_1^{\lambda_1} \cdot \ldots \cdot x_n^{\lambda_n},$$

$$ddet(C) =$$

$$= \sum_{\pi(n)\in\Pi(n)} (-1)^{n-\lambda(\pi)} \cdot \frac{n! \cdot (\lambda(\pi))!}{\lambda_1! \cdot \ldots \cdot \lambda_n! \cdot (1!)^{\lambda_1} \cdot \ldots \cdot (n!)^{\lambda_n}} \cdot x_1^{\lambda_1} \cdot \ldots \cdot x_n^{\lambda_n}.$$

The following propositions also hold.

Proposition 7.

$$a)\ \left[\frac{j}{i-j+j\cdot\delta_{ij}}\right]_{1\le j\le i\le n}=$$

$$=\sum_{\pi(n)\in\Pi(n)}\frac{n!}{\lambda_1!\cdot\ldots\cdot\lambda_n!\cdot(1!)^{\lambda_1}\cdot\ldots\cdot(n!)^{\lambda_n}}=$$

$$=\sum_{k=1}^{n}\sum_{\pi(n)\in\Pi_k(n)}\frac{n!}{\lambda_1!\cdot\ldots\cdot\lambda_n!\cdot(1!)^{\lambda_1}\cdot\ldots\cdot(n!)^{\lambda_n}}=\sum_{k=1}^{n}S(n,k)\ ;$$

$$b)\ \left\langle\frac{j}{i-j+j\cdot\delta_{ij}}\right\rangle_{1\le j\le i\le n}=$$

$$=\sum_{\pi(n)\in\Pi(n)}(-1)^{n-\lambda(\pi)}\cdot\frac{n!}{\lambda_1!\cdot\ldots\cdot\lambda_n!\cdot(1!)^{\lambda_1}\cdot\ldots\cdot(n!)^{\lambda_n}}=$$

$$=\sum_{k=1}^{n}\sum_{\pi(n)\in\Pi_k(n)}(-1)^{n-k}\cdot\frac{n!}{\lambda_1!\cdot\ldots\cdot\lambda_n!\cdot(1!)^{\lambda_1}\cdot\ldots\cdot(n!)^{\lambda_n}}=$$

$$=\sum_{k=1}^{n}(-1)^{n-k}S(n,k)\ ,$$

where $S(n,k)$ denote Stirling numbers of the second kind.

Proposition 8.

$$a)\ [j-(j-1)\cdot\delta_{ij}]_{1\le j\le i\le n}=\sum_{\pi(n)\in\Pi(n)}\frac{n!}{\lambda_1!\cdot\ldots\cdot\lambda_{n!}\cdot1^{\lambda_1}\cdot\ldots\cdot n^{\lambda_n}}=$$

$$=\sum_{k=1}^{n}(-1)^{n-k}\cdot s(n,k)=n!,$$

$$b)\ \langle(j-(j-1)\cdot\delta_{ij})\rangle_{1\le j\le i\le n}=$$

$$=\sum_{\pi(n)\in\Pi(n)}(-1)^{n-(\lambda_1+\ldots+\lambda_n)}\frac{n!}{\lambda_1!\ldots\cdot\lambda_n!\cdot1^{\lambda_1}\ldots\cdot n^{\lambda_n}}=$$

$$=\sum_{k=1}^{n}s(n,k)=\begin{cases}1\ , & n=1,\\ 0\ , & n>1.\end{cases}$$

where $s(n,k)$ denote Stirling numbers of the first kind.

Some results of this section were announced by the author in [22] and an extended version of the content is accepted for publication.

6. Paradeterminants and formal operations with formal power series

One of the central methods of combinatorial analysis is the method of generating functions [4], which uses formal operations with power series. However operations such as inversion, composition of series and some other operations with formal series, as it is well known, present some difficulties. In this section, using techniques of paradeterminants and parapermanents of triangular matrices, we construct some recurrent algorithms for formal operations with series.

Some results of this section were announced by the author in [24] and an extended version of the content is accepted to print.

We consider formal power series with nonzero constant term.

Theorem 11. Let $A(x)=\sum_{i=0}^{\infty}a_ix^i$, $a_0=1$, be some formal power series. Then the following equalities hold:

$$(A(x))^n = 1+\sum_{k=1}^{\infty}(-1)^k\left\langle \frac{(i-j+1)n-(j-1)}{(i-j)n-j}\cdot\frac{a_{i-j+1}}{a_{i-j}}\right\rangle_{1\le j\le i\le k}\cdot x^k,$$

$$(A(x))^{-n} = 1+\sum_{k=1}^{\infty}(-1)^k\left\langle \frac{(i-j+1)n+(j-1)}{(i-j)n+j}\cdot\frac{a_{i-j+1}}{a_{i-j}}\right\rangle_{1\le j\le i\le k}\cdot x^k,$$

$$(A(x))^{\frac{1}{n}} = 1+\sum_{k=1}^{\infty}(-1)^k\left\langle \frac{-(i-j+1)+(j-1)n}{-(i-j)+jn}\cdot\frac{a_{i-j+1}}{a_{i-j}}\right\rangle_{1\le j\le i\le k}\cdot x^k,$$

$$(A(x))^{-\frac{1}{n}} = 1+\sum_{k=1}^{\infty}(-1)^k\left\langle \frac{(i-j+1)+(j-1)n}{(i-j)+jn}\cdot\frac{a_{i-j+1}}{a_{i-j}}\right\rangle_{1\le j\le i\le k}\cdot x^k.$$

Theorem 12. Let $A(x)=1+\sum_{i=1}^{\infty}a_ix^i$, $B(x)=1+\sum_{i=1}^{\infty}b_ix^i$, $C(x)=1+\sum_{i=1}^{\infty}c_ix^i$ be some formal power series such that

$$C(x)=\frac{A(x)}{B(x)}.$$

Then

$$c_i=\sum_{j=0}^{i-1}(a_{i-j}-b_{i-j})\cdot\left\langle\frac{b_{s-r+1}}{b_{s-r}}\right\rangle_{1\le r\le s\le j},\quad i=1,2,\ldots.$$

We assume that $\left\langle\frac{b_{s-r+1}}{b_{s-r}}\right\rangle_{1\le r\le s\le 0}=1$.

Corollary 7. If $A(x)$ is the formal power series from Theorem 12 then

$$\frac{1}{A(x)} = 1 - \langle a_1 \rangle \cdot x^1 + \left\langle \begin{matrix} a_1 & \\ \frac{a_2}{a_1} & a_1 \end{matrix} \right\rangle \cdot x^2 - \ldots +$$

$$+(-1)^i \left\langle \begin{matrix} a_1 & & & \\ \frac{a_2}{a_1} & a_1 & & \\ \vdots & \cdots & \ddots & \\ \frac{a_i}{a_{i-1}} & \frac{a_{i-1}}{a_{i-2}} & \cdots & a_1 \end{matrix} \right\rangle \cdot x^i + \ldots$$

Theorem 13. Let $f(x)$ and $g(x)$ be two infinitely differentiable functions such that

$$g_x^i(0) = a_i,\ f_x^i(a_0) = b_i,\ i = 0, 1, 2, \ldots.$$

Then

$$f(g(x)) = b_0 + \frac{1}{1!} pper(B_1(a_1) \cdot (b_1) \cdot x + \frac{1}{2!} pper(B_2(a_1, a_2)) \cdot (b_1, b_2) \cdot x^2 + \ldots,$$

where $B_n(a_1, a_2, \ldots, a_n)$ is a triangular Bell matrix.

Corollary 8. If the function $f(x)$ is infinitely differentiable and $g(x)$ can be presented as a series

$$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots,$$

such that equalities $f(a_0) = b_0,\ f_x^{(i)}(a_0) = b_i,\ i = 1, 2, \ldots,$ hold, then

$$f(g(x)) = b_0 + pper(Z_1(a_1) \cdot \left(\frac{b_1}{1!} \right) \cdot x + pper(Z_2(a_1, a_2)) \cdot \left(\frac{b_1}{1!}, \frac{b_2}{2!} \right) \cdot x^2 + \ldots,$$

where $Z_n(a_1, a_2, \ldots, a_n)$ is a triangular matrix of the form (5.7*)*.

We consider now formal operations with formal power series with zero constant term.

Theorem 14. *(Theorem on composition of series)* If formal power series $c(x) = \sum_{i=1}^{\infty} c_i x^i$ is a composition of formal power series $b(x) = \sum_{i=1}^{\infty} b_i x^i$ and $a(x) = \sum_{i=1}^{\infty} a_i x^i$, i.e. $c(x) = b(a(x)) = \sum_{i=1}^{\infty} b_i a^i(x)$, then

$$c_i = (b_1, b_2, \ldots, b_i) \cdot \begin{bmatrix} a_1 & & & \\ \frac{a_2}{a_1} & a_1 & & \\ \vdots & \vdots & \ddots & \\ \frac{a_i}{ai-1} & \frac{a_{i-1}}{a_{i-2}} & \cdots & a_1 \end{bmatrix}_i.$$

Theorem 15. *(Theorem on inversion of a series)* Let $a(x) = \sum_{i=1}^{\infty} a_i x^i$ and $b(x) = \sum_{i=1}^{\infty} b_i x^i$ be some formal series such that

$$b(a(x)) = 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \dots .$$

Then the following equalities hold

$$b_i = \frac{(-1)^{i-1}}{a_1^i} \cdot \left(\frac{(i+1)^{\overline{0}}}{1!}, \frac{(i+1)^{\overline{1}}}{2!}, \dots, \frac{(i+1)^{\overline{i-2}}}{(i-1)!} \right) \times \tag{6.1}$$

$$\times \begin{bmatrix} \frac{a_2}{a_1} & & & \\ \frac{a_3}{a_2} & \frac{a_2}{a_1} & & \\ \vdots & \vdots & \ddots & \\ \frac{a_i}{a_{i-1}} & \frac{a_{i-1}}{a_{i-2}} & \cdots & \frac{a_2}{a_1} \end{bmatrix}, \; i = 1, 2, \dots .$$

Theorem 16. Let $a(x) = \sum_{i=1}^{\infty} a_i x^i$ и $c(x) = \sum_{i=1}^{\infty} c_i x^i$ be some formal series and $\omega(x) = \sum_{i=1}^{\infty} \omega_i \cdot x^i$. Then the equalities

$$\omega(a(x)) = c(x) \qquad \text{and} \qquad a(\omega(x)) = c(x)$$

imply the equalities

$$\omega_n = pper(Z(b_1, \dots, b_n)) \cdot (c_1, \dots, c_n)$$

and

$$\omega_n = pper(Z(c_1, \dots, c_n)) \cdot (b_1, \dots, b_n),$$

where $b_i, \; i = 1, 2, \dots, n$, are defined by equalities (6.1).

7. Application of paradeterminants to the solution of problems on paths in skew diagrams and Ferrer graphs

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be some disordered partition of a number n. A partition $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ is called a subpartition of λ (denoted $\mu \preceq \lambda$), if the inequalities $\mu_i \leq \lambda_i, \; i = 1, \dots, r$ hold. To any pair of such partitions one can associate some skew diagram ([14], 12–15).

$$diagr(\lambda, \mu) = \begin{pmatrix} \lambda_1, \dots, \lambda_{n-1}, \lambda_n \\ \mu_1, \dots, \mu_{n-1}, 0 \end{pmatrix} \tag{7.1}$$

Distance between two points on the diagram is the shortest distance between these points. A path in a skew diagram between the lowest right point A and the upper left point B is a shortest path between these points. Furthermore, we will move only in two directions: up and left.

Theorem 17. *[21]* The number of the shortest paths between the lowest right point and the upper left point on the skew diagram (7.1) is equal to

$$ddet\left(\left(\frac{1-\delta_{ij}}{i-j+1}+\delta_{ij}\right)\cdot\frac{(\lambda_i-\mu_j+j-i+1)^{\overline{i-j+1}}}{(\lambda_i-\mu_{j+1}+j-i+2)^{\overline{i-j}}}\right)_{1\le j\le i\le n} \tag{7.2}$$

where δ_{ij} is the Kronecker symbol.

Remark 12. If in the paradeterminant (7.2) we have the inequality $\lambda_i-\mu_j+j-i+1\le 0$, then the corresponding element of this paradeterminant is assumed to be zero. Moreover, the value of the expression $(\lambda_i-\mu_{j+1}+j-i+2)^{\overline{i-j}}$ for $i=j=n$ is assumed to be 1.

Corollary 9. Suppose the diagram (7.1) has the form

$$diagr(\lambda,0)=\begin{pmatrix}\lambda_1,\dots,\lambda_{n-1},\lambda_n\\ 0,\quad\dots,\quad 0,\quad 0\end{pmatrix},$$

i.e. it is a Ferrer graph. Then the number of shortest paths in the Ferrer graph, or the number of standard tableaux of this graph can be found by the formula

$$(-1)^{\frac{n(n-1)}{2}}\cdot\frac{N!}{\prod_{i=1}^{n}(\lambda_i+n-i)!}\cdot det((\lambda_i-i)^{j-1})_{i,j=1,\dots,n}, \tag{7.3}$$

where N is the weight of the Ferrer graph.

Remark 13. Thus the Frame-Robinson-Thrall hook rule [15] giving the number of standard Young tableaux on Ferrer deagrams can be represented using formula (7.3) and Vandermonde determinant.

The reader can find an extended version of this material in [21].

8. A connection between determinants and paradeterminants

The existing analogy between the properties of determinants and paradeterminants can be explained mostly by the close connections between them. In some sense determinants can be reduced to paradeterminants. Replacement of determinants by paradeterminants could essentially simplify calculations in many cases since the latter can be calculated faster.

Consider a matrix

$$B = \begin{pmatrix} b_{11} & 1 & 0 & \dots & 0 & 0 \\ b_{21} & b_{22} & 1 & \dots & 0 & 0 \\ b_{31} & b_{32} & b_{33} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & 1 \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{n,n-1} & b_{nn} \end{pmatrix}, \qquad (8.1)$$

which we will call lower quasi-triangular.

Theorem 18. *(Lischinski I.I.)* For any triangular matrix (1.1)

$$\left\langle \begin{matrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right\rangle = \begin{vmatrix} b_{11} & 1 & 0 & \dots & 0 & 0 \\ b_{21} & b_{22} & 1 & \dots & 0 & 0 \\ b_{33} & b_{32} & b_{33} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & 1 \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{n,n-1} & b_{nn} \end{vmatrix},$$

$$\begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} b_{11} & 1 & 0 & \dots & 0 & 0 \\ b_{21} & b_{22} & 1 & \dots & 0 & 0 \\ b_{33} & b_{32} & b_{33} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & 1 \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{n,n-1} & b_{nn} \end{bmatrix},$$

where

$$b_{ij} = \{a_{ij}\} = \prod_{k=j}^{i} a_{ik}, \ 1 \le j \le i \le n. \qquad (8.2)$$

Corollary 10. For any lower triangular matrix (8.1)

$$\begin{vmatrix} b_{11} & 1 & 0 & \dots & 0 & 0 \\ b_{21} & b_{22} & 1 & \dots & 0 & 0 \\ b_{31} & b_{32} & b_{33} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & 1 \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{n,n-1} & b_{nn} \end{vmatrix} =$$

$$= \left\langle \begin{matrix} b_{11} & & & & \\ \frac{b_{21}}{b_{22}} & b_{22} & & & \\ \frac{b_{31}}{b_{32}} & \frac{b_{32}}{b_{33}} & b_{33} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{b_{n1}}{b_{n2}} & \frac{b_{n2}}{b_{n3}} & \frac{b_{n3}}{b_{n4}} & \cdots & b_{nn} \end{matrix} \right\rangle.$$

Observe that the elements b_{ij}, $1 \leq j \leq i \leq n$ in equality (??) can take values from a numeric field, as follows from Remark 7.

Theorem 19. Let A be a square matrix of order n

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix},$$

such that the minors of A satisfy the inequalities:

$$A\begin{pmatrix} 1 \\ 2 \end{pmatrix} \neq 0,\ A\begin{pmatrix} 1\,2 \\ 2\,3 \end{pmatrix} \neq 0,\ \ldots, A\begin{pmatrix} 1\,2\,\ldots\,n-2 & n-1 \\ 2\,3\,\ldots\,n-1 & n \end{pmatrix} \neq 0.$$

Then the following identity holds

$$det(A) = a_{12}a_{23}^{(1)}a_{34}^{(2)} \cdot \ldots \cdot a_{n-1,n}^{(n-2)} \cdot ddet\left\langle \frac{a_{ij}^{(j-2)}}{a_{i,j+1}^{(j-1)}} \right\rangle_{1 \leq j \leq i \leq n},$$

where

$$a_{i,p}^{(p-2)} = \frac{A\binom{1\,2\,\ldots\,p-2\ i}{2\,3\,\ldots p-1\ p}}{A\binom{1\,2\,\ldots\,p-2}{2\,3\ldots\,p-1}},\ p = 3, 4, \ldots, n,\ i = p-1, p, \ldots, n,$$

$$a_{ip}^{(p-2)} = a_{ip}\ , p = 1, 2,\ a_{n,n+1}^{n-1} = 1.$$

Proposition 9. For any matrix (19) and $n = 3, 4, \ldots$, the following identity holds

$$det(A) \cdot A\binom{1\,2\cdots n-2}{2\,3\cdots n-1} = A\binom{1\,2\cdots n-1}{1\,2\cdots n-1}A\binom{1\,2\cdots n-2\,n}{2\,3\cdots n-1\,n} -$$
$$-A\binom{1\,2\cdots n-2\,n-1}{2\,3\cdots n-1\ \ n}A\binom{1\,2\cdots n-2\ \ n}{1\,2\cdots n-2\,n-1}.$$

Thus, by virtue of the proposition cited above, the determinant of the matrix (19), for any $n = 3, 4, \ldots$, can be expressed through four minors of order $n-1$ and one minor of order $(n-2)$.

9. Principles of calculus for triangular matrices

We define the basic operations on triangular matrices: addition of triangular matrices, multiplication of a triangular matrix by a number and multiplication of triangular matrices.

Let $A = (a_{ij})_{1 \leq j \leq i \leq n}$, $B = (b_{ij})_{1 \leq j \leq i \leq n}$ and $C = (c_{ij})_{1 \leq j \leq i \leq n}$ be some triangular matrices of order n.

Definition 13. Sum of two triangular matrices A and B is the matrix C, whose elements are equal to the sum of the corresponding elements in A and B, i.e. $c_{ij} = a_{ij} + b_{ij},\ 1 \le j \le i \le n$.

It follows directly from the definition that the addition operation is commutative and associative.

Definition 14. Product of a triangular matrix A by a number α from some numeric field is the matrix C with elements $c_{ij} = \alpha \cdot a_{ij},\ 1 \le j \le i \le n$.

Obviously:

$$\alpha(A + B) = \alpha A + \alpha B,\ \ (\alpha + \beta)A = \alpha A + \beta A,\ \ (\alpha\beta)A = \ \alpha(\beta A).$$

To define the product of two triangular matrices we give some preliminary definitions:

Definition 15. An element $\xi_1 \in \Xi(n)$ is not related to an element $\xi_2 \in \Xi(n)$ if $[\xi_1] \bigcap [\xi_2] = \{n\}$, where $[\cdot]$ denotes the basis of the corresponding multiset (see definition 6). Otherwise these elements are called related.

The set of all elements of $\Xi(n)$ not related to an element ξ is denoted by $\Xi_\xi(n)$.

Proposition 10. There are 3^{n-1} pairs (ξ_1, ξ_2) in the Cartesian product $\Xi(n) \times \Xi(n)$ whose components are not related.

Two summands $a_{\xi_1(1),1} a_{\xi_1(2),2} \cdot \ldots \cdot a_{\xi_1(n),n}$ and $b_{\xi_2(1),1} b_{\xi_2(2),2} \cdot \ldots \cdot b_{\xi_2(n),n}$ in the paradeterminant (parapermanent) of the triangular matrices A and B are not related, if the elements $\xi_1 = \{\xi_1(1), \xi_1(2), \ldots, \xi_1(n)\},\ \xi_2 = \{\xi_2(1), \xi_2(2), \ldots, \xi_2(n)\}$, which belong to $\Xi(n)$, are not related.

Definition 16. Incomplete product of the paradeterminants of matrices A and B is the sum of products of all pairwise not related components of these paradeterminants taken with appropriate signs, i.e.

$$ddet(A) \circ ddet(B) =$$

$$= \sum_{(\xi_1,\xi_2) \in \Xi(n) \times \Xi(n)} (-1)^{\varepsilon(\xi_1) + \varepsilon(\xi_2)} \cdot k(\xi_1, \xi_2) \cdot a_{\xi_1(1),1} \cdot \ldots \cdot a_{\xi_1(n),n} \cdot b_{\xi_2(1),1} \cdot \ldots \cdot b_{\xi_2(n),n},$$

where $\varepsilon(\xi_1)$ and $\varepsilon(\xi_2)$ are the numbers of different elements in the multiset ξ_1 and ξ_2, respectively, and $k(\xi_1, \xi_2)$ is defined by the equality

$$k(\xi_1, \xi_2) = \begin{cases} 1, & \text{if } [\xi_1] \cap [\xi_2] = \{n\}, \\ 0, & \text{if } [\xi_1] \cap [\xi_2] \ne \{n\}. \end{cases}$$

Example 5. *The incomplete product of paradeterminants of matrices A and B of order 3 is given by*

$$\begin{aligned} & ddet(A)\circ ddet(B) = \\ = \; & a_{11}a_{22}a_{33}b_{31}b_{32}b_{33} + a_{21}a_{22}a_{33}b_{11}b_{32}b_{33} - a_{21}a_{22}a_{33}b_{31}b_{32}b_{33} + \\ + \; & a_{11}a_{32}a_{33}b_{21}b_{22}b_{33} - a_{11}a_{32}a_{33}b_{31}b_{32}b_{33} + a_{31}a_{32}a_{33}b_{11}b_{22}b_{33} - \\ - \; & a_{31}a_{32}a_{33}b_{21}b_{22}b_{33} - a_{31}a_{32}a_{33}b_{11}b_{32}b_{33} + a_{31}a_{32}a_{33}b_{31}b_{32}b_{33} \end{aligned}$$

Incomplete product of parapermanents is defined in the same way, except that the sing $(-1)^{\varepsilon(\xi)+\varepsilon(\bar{\xi})}$. is disregarded.

Definition 17. The paradeterminant (parapermanent) product of two triangular matrices A and B of order n is the matrix $C = A\cdot B$ of the same order with elements:

$$c_{ij}(A,B) = (-1)^{\delta_{ij}+1}\cdot \frac{d_{ij}(A,B)}{d_{i,j+1}(A,B)}$$

$$\left(c_{ij}(A,B) = \frac{p_{ij}(A,B)}{p_{i,j+1}(A,B)}\right),$$

where δ_{ij} is the Kronecker symbol, and $d_{ij}(A,B), p_{ij}(A,B)$ — incomplete product of paradeterminants (parapermanents) of the corners R_{ij} of the triangular matrices A and B, i.e.

$$d_{ij}(A,B) = ddet(R_{ij}(A))\circ ddet(R_{ij}(B))$$

$$(p_{ij}(A,B) = pper(R_{ij}(A))\circ pper(R_{ij}(B))),$$

where $1\le j\le i\le n$.

Proposition 11. The following equalities hold: $AB = BA$, $(AB)C = A(BC)$, $A(B+\ C) = AB + AC$, $ddet(AB) = ddet(A)\cdot ddet(B)$,

$$pper(AB) = pper(A)\cdot pper(B).$$

Identity matrix is the matrix of the form

$$E = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

References

[1] Babai L., Frankl P. *Linear algebra methods in combinatories with Applications to Geometry and Computer Science* p. 234.

[2] Protasov I.V., Khromulyak O.M. *Methods of Linear Algebra in Combinatorics.* Kyiv, 1997, 40 p (in Ukrainian).

[3] Stechkin B.S. *Tuples and their usage in combinatorial schemes (on one combinatorial formalization)* // Sb. "Comb. and asympt. analysis", Kras. GU, 1977., p. 44-54 (in Russian).

[4] Doubilet P., Rota G.-C., Stanley R. *On the foundations of combinatorial theory. VI. The idea of generating function. Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory, pp. 267–318. Univ. California Press, Berkeley, Calif., 1972.

[5] Stanley R. *Enumerative combinatorics*: Translation from English — Moscow: Mir, 1990. — 440 p.

[6] Andrews G. *The theory of partitions.* Encyclopedia of Mathematics and its Applications, Vol. 2. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. xiv+255 pp.

[7] Graham R.L., Knuth D.E., Patashnik O. *Concrete mathematics.* A foundation for computer science. Second edition. Addison-Wesley Publishing Company, Reading, MA, 1994. xiv+657 pp.

[8] Bell E.T. *Partition polynomials.* — Ann. Math. 29, 1927, p. 38 – 46.

[9] M.Marcus, H.Minc. *On the relation between the determinant and the permanent,* Illinois J. Math. 5 (1961), 376-381.

[10] Riordan J. *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.

[11] Fine N.J. *Sums over partitions.*— Report of the Institute in the Theory of Numbers, Boulder, 1959, p. 86-94.

[12] Riordan J. *Kombinatornye tozhdestva.* (Russian) [Combinatorial identities] Translated from the English by A. E. Zhukov. "Nauka", Moscow, 1982. 256 pp.

[13] Aĭgner M. *Kombinatornaya teoriya.* (Russian) [Combinatorial theory] Translated from the English by V. V. Ermakov and V. N. Lyamin. Translation edited and with a preface by G. P. Gavrilov. "Mir", Moscow, 1982. 558 pp.

[14] Macdonald I.G. *Simmetricheskie funktsii i mnogochleny Kholla.* (Russian) [Symmetric functions and Hall polynomials] Translated and with a preface by A.V. Zelevinskiĭ. "Mir", Moscow, 1985. 224 pp.

[15] Frame J.S., Robinson G. de B., Thrall R.M. *The hook graphs of Sn.* // Canad. J. Math. 6, 1954, p. 316-324.

[16] Tarakanov V.E. *Kombinatornye zadachi i* $(0, 1)$*-matritsy.* (Russian) [Combinatorial problems and $(0, 1)$-matrices] Problemy Nauki i Tekhnicheskogo Progressa. [Problems of Science and Technological Progress] "Nauka", Moscow, 1985. 192 pp.

[17] Sachkov V.N. *Vvedenie v kombinatornye metody diskretnoĭmatematiki.* (Russian) [Introduction to combinatorial methods of discrete mathematics] "Nauka", Moscow, 1982. 384 pp.

[18] Zatorsky R.A. *On paradeterminants and parapermanents of triangular matrices.* (Ukrainian) Mat. Stud. 17 (2002), no. 1, 3–17.

[19] Zatorsky R.A. *Paradeterminants and parapermanents of triangular matrices.* (Ukrainian) Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki 2002, no. 8, 21–28.

[20] ZatorskiĭR.A. *Parapermanents and linear recurrent relations.* In the Proceedings of Internation Algebraic conference in memory of D.Grave, Kyiv University, Kyiv, Ukraine, June, 2002., p. 138.

[21] Zatorsky R.A. *Determinants of triangular matrices and trajectories on Ferrers diagrams.* (Russian) Mat. Zametki 72 (2002), no. 6, 834–852; translation in Math. Notes 72 (2002), no. 5-6, 768–783.

[22] Zatorsky R.A. *On partition polynomials and their paradeterminants and parapermanents.* In the Proceedings of X Kravchuk International Conference, Kyiv: NTUU (KPI). - 2004. - p. 383.

[23] Zatorsky R.A. *Continued fractions, K-polynomials and parapermanents.* (Ukrainian) Mat. Metodi Fiz.-Mekh. Polya 44 (2001), no. 4, 12–21.

[24] R.Zatorsky. *Paradeterminants and formal operations on series* // 4th International Algebraic Conference in Ukraine, Lviv, 2003., 242.

[25] Ganyushkin O.G.,Zatorsky R.A., Lishcinski I.I. *On paradeterminants and parapermanents of triangular matrices* (Ukrainian) Bull. Kyiv Univ. Ser. Phys. and Math. (2005) no. 1, 40–45.

[26] Dyakiv N.M., Zatorsky R.A.*On $F-$paradeterminants and $F-$parapermanents of triangular matrices* (Ukrainian) Mat. Metodi Fiz.-Mekh. Polya (2005), no. 1, 21–24.

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