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RESEARCH ARTICLE

R-S correspondence for the Hyper-octahedral group of type B_n — A different approach

M. Parvathi, B. Sivakumar[1] and A. Tamilselvi

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Abstract. In this paper we develop a Robinson Schensted algorithm for the hyperoctahedral group of type B_n on partitions of $(\frac{1}{2}r(r+1)+2n)$ whose $2-$core is δ_r, $r \geq 0$ where δ_r is the partition with parts $(r, r-1, \ldots, 0)$. We derive some combinatorial properties associated with this correspondence.

1. Introduction

The group algebra of the hyperoctahedral group $\widetilde{S}_n$ is a subalgebra of the signed Brauer algebra. In the process of constructing a cellular basis for the signed Brauer algebra [18,19], we constructed a R-S correspondence for the hyperoctahedral group. We distinguish between a positive image and a negative image of the signed permutation by a positive domino $\left(\begin{array}{|c|c|}\hline * & * \\ \hline\end{array}\right)$ and a negative domino $\left(\begin{array}{|c|}\hline * \\ \hline * \\ \hline\end{array}\right)$ respectively. The process of insertion described is independent of the core of the partition into which the dominoes are inserted. The positive dominoes are inserted along rows and negative dominoes inserted along columns. We give a different approach for the R-S correspondence by following closely the procedure of insertion and placement of the dominoes as that in the case of the symmetric group.

The irreducible modules of the hyper-octahedral group of type B_n $\widetilde{S}_n$ are indexed by partitions of $2n+1$ whose $2-$core is one [12]. There

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is a natural bijection between partitions of $2n+1$ whose $2-$core is one and partitions of $(\frac{1}{2}r(r+1)+2n)$ whose $2-$core is $\delta_r,\ r \geq 0$ where δ_r is the partition with parts $(r, r-1, \ldots, 0)$. In this paper we develop the Robinson-Schensted algorithm for the hyper-octahedral group which gives a correspondence between elements of the hyper-octahedral group and pairs of standard tableaux of shape λ, as λ varies over partitions of $(\frac{1}{2}r(r+1)+2n)$, whose $2-$core is $\delta_r,\ r \geq 0$. The insertion process is independent of the core. We study the growth and local rules for the R-S correspondence. We also study the Knuth relation for the hyper-octahedral group of type B_n as in the case of the symmetric group.

2. Preliminaries

2.1. R-S Correspondence for the Hyper-octahedral group of type B_n

In a 1982 paper Barbasch and Vogan [1] describe an insertion algorithm which identifies the hyperoctahedral permutations with domino tableau. They define this insertion using left-right insertion of a word and its negative, followed by a *jeu de taquin* that pairs up i and -i.

Subsequently Garfinkle [5] defined this insertion directly, both through a bumping algorithm and recursively in a manner similar to that used by Fomin [4]. Van Leeuwen [15] also describes this algorithm by translating Garfinkle's recursive definition into Fomin's language of shapes. He provides the first proof that the Garfinkle algorithm is the same as the Barbasch Vogan algorithm. He also defines insertion in the presence of a non empty 2-core.

Shimozono and White [14] described a semistandard generalization of Barbasch Vogan's domino insertion relating domino insertion to Haiman's mixed insertion.

The domino insertion given in this paper is such that it corresponds to the left cells of the modified Lusztig basis given in [6] . The algorithm is based on R-S correspondence for the symmetric group without splitting the diagram. We differ from the domino insertion given by Shimozono and White [14] in the sense that their construction involves splitting the tableau into two parts and successively attaching the dominos in a recursive manner to get the final shape while we use a process successive sliding of the dominos without splitting the tableau till the final shape is arrived at.

Definition 2.1.1. [9]. For an integer $n \geq 2$, the hyper-octahedral group of type B_n, denoted by $\widetilde{S}_n$ is defined to be subgroup of S_{2n} such that

$$\widetilde{S}_n = \{\theta \in S_{2n} | \theta(i) + \theta(-i) = 0, \forall i, 1 \leq i \leq n\}.$$

Definition 2.1.2. [3]. A sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ is called a partition of n, which is denoted by $\lambda \vdash n$, if

1. $\lambda_i \geq \lambda_{i+1}$, for every $i \geq 1$
2. $\sum\limits_{i=1}^{\infty} \lambda_i = n$

The λ_i are called the parts of λ.

Definition 2.1.3. [3]. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$. The Young diagram of λ is an array of n dots having l left justified rows with row i containing λ_i dots for $1 \leq i \leq l$.

Example:

$$[\lambda] := \begin{array}{llllll l} * & * & \cdot & \cdot & \cdot & * & \lambda_1 \text{ nodes} \\ * & * & \cdot & \cdot & * & & \lambda_2 \text{ nodes} \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ * & * & \cdots & * & & & \lambda_r \text{ nodes} \end{array}$$

Definition 2.1.4.[[8], §2]. Let α be a partition of n, denoted by $\alpha \vdash n$.

Then the (i,j)-hook of α, denoted by $H^{\alpha}_{i,j}$ which is defined to be a Γ-shaped subset of diagram α which consists of the (i,j)-node called the *corner* of the hook and all the nodes to the right of it in the same row together with all the nodes lower down and in the same column as the corner.

The number h_{ij} of nodes of H^{α}_{ij} i.e.,

$$h_{ij} = \alpha_i - j + \alpha'_j - i + 1$$

where α'_j = number of nodes in the jth column of α, is called the *length* of $H^{\alpha}_{i,j}$, where $\alpha = [\alpha_1, \cdots \alpha_k]$. A hook of length q is called a *q-hook.* Then $H[\alpha] = (h_{ij})$ is called the *hook graph* of α.

Definition 2.1.5.[[8], §2]. A hook of length two will be called a 2-hook.

Definition 2.1.6.[13]. We shall call the (i,j) node of λ, as r-node if and only if $j - i \equiv r(mod 2)$.

Definition 2.1.7.[13]. A node (i, j) is said to be a $(2, r)$ node if $h_{ij} = 2m$ and the residue of node (i, λ_i) in λ is r. i.e. $\lambda_i - i \equiv r(mod 2)$.

Theorem 2.1.8.[13]. If we delete all the elements in the hook graph $H[\lambda]$ not divisible by 2, then the remaining elements,

$$h_{ij} = h^r_{ij}(2), \ (r = 0, 1)$$

can be divided into disjoint sets whose $(2, r)$ nodes constitute the diagram $[\lambda]^r_2$, $(r = 0, 1)$ with hook graph (h^r_{ij}). The λ is written as (λ_1, λ_2) where the nodes in λ_1 correspond to $(2, 0)$ nodes and the nodes in λ_2 correspond to $(2, 1)$ nodes.

Definition 2.1.9.[[8], §2]. Let $[\lambda] \vdash n$. An (i, j)-node of $[\lambda]$ is said to be a rim node if there does not exist any $(i + 1, j + 1)$-node of $[\lambda]$.

Definition 2.1.10.[[8], §2]. A 2-hook comprising of rim nodes is called a rim 2-hook.

Definition 2.1.11.[[8], §2]. Diagrams $[\lambda]$ which do not contain any 2-hook are called 2-cores.

Definition 2.1.12.[8] Each 2×1 and 1×2 rectangular boxes consisting of two nodes is called as a domino.

Definition 2.1.13.[8] Let λ, μ be partitions of n. Then the dominance order $\trianglerighteq$ is defined as $\lambda \trianglerighteq \mu$ if $\sum\limits_{i=1}^{j} \lambda_i \geq \sum\limits_{i=1}^{j} \mu_i, \ \forall j$.

Definition 2.1.14.[8] Let (λ, μ) and (α, β) be bipartitions of n where $\lambda \vdash l, \ \mu \vdash m, \ l + m = n$ and $\alpha \vdash r, \ \beta \vdash s, \ r + s = n$. Then we write $(\lambda, \mu) \trianglerighteq (\alpha, \beta)$ if

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \alpha_i, \quad \forall j \ \text{ and } \ |\lambda| + \sum_{i=1}^{j} \mu_i \geq |\alpha| + \sum_{i=1}^{j} \beta_i, \quad \forall j$$

Definition 2.1.15. [[2],§3.7.] Let $S'_{\underset{\sim}{n}} = S_n \cup \{t_1, t_2, \ldots, t_n\}$. Then the extended left descent set of $w \in \widetilde{S}_n$ is defined by $\mathcal{L}'(w) := \{u \in S'_n | l(uw) < l(w)\} = \{u \in S'_n | uw < w\}$.

Let $x, y \in \widetilde{S}_n$ and $s \in \sum_n = S_n - \{t\}$, then we define

$$x \xrightarrow{s}_L y \overset{def}{\Longleftrightarrow} y = sx, \ l(y) > l(x) \text{ and } \mathcal{L}'(\text{x}) \nsubseteq \mathcal{L}'(\text{y}),$$

and we write $x \overset{s}{\longleftrightarrow}_L y$ if $x \overset{s}{\longrightarrow}_L y$ or $y \overset{s}{\longrightarrow}_L x$.

Finally, we write $x \longleftrightarrow_L y$ if there exists a sequence $x = x_1, x_2, \ldots, x_k = y$ and $s_i \in \sum_n$ such that $x_i \overset{s}{\longleftrightarrow}_L \; x_{i+1}$ for all i.

3. R-S correspondence for the Hyperoctahedral group of type B_n - a different approach

We define a R-S correspondence for the hyperoctahedral group of type B_n on partitions of $(\frac{1}{2}r(r+1)+2n)$ whose 2-core is δ_r(where δ_r is the partition $(r, r-1, r-2, ..., 2, 1, 0)$)

Definition 3.1.1. A domino in which both the nodes are filled with same number from the set $A = \{1, 2, \cdots n\}$ is defined as a tablet. i.e.,

$$\boxed{\text{x}}\boxed{\text{x}}\,, \quad \begin{array}{|c|}\hline \text{x} \\ \hline \text{x} \\ \hline \end{array}\,, \quad x \in A.$$

Notation 3.1.2. A partition λ of $(\frac{1}{2}r(r+1)+2n)$ whose $2-$core is δ_r, $r \geq 0$ where δ_r is the partition with parts $(r, r-1, \ldots, 0)$ is denoted by $\lambda \vdash_a (\frac{1}{2}r(r+1)+2n)$.

Definition 3.1.3. Given $\lambda \vdash_a (\frac{1}{2}r(r+1)+2n)$, by a Young tableau we mean the Young diagram of λ filled with tablets containing integers

$$A = \{1, 2, \cdots n\}.$$

Definition 3.1.4. Let $\lambda \vdash_a (\frac{1}{2}r(r+1)+2n)$. Let $\lambda = \lambda, \lambda^1, \ldots, \lambda^r = \delta_r$ be a sequence of partitions obtained by successive removal of rim 2-hooks. The reverse of such a sequence of partitions is called a path of λ.

Definition 3.1.5. A tableau is said to be standard if the entries are increasing along a path of λ.

Example $\lambda = \begin{array}{cccc} * & * & * & * \\ * & * & & \end{array}$

Then $\lambda^0 = \Phi$, $\lambda^1 = \; * \;\; *$, $\lambda^2 = \; * \;\; * \;\; * \;\; *$, $\lambda^3 = \begin{array}{cccc} * & * & * & * \\ * & * & & \end{array} = \lambda$.

A standard tableau of shape corresponding to the above path $(\lambda^0, \lambda^1, \lambda^2, \lambda^3,)$ is obtained as given below

$$\Phi, \quad 1 \;\; 1 \; , \quad 1 \;\; 1 \;\; 2 \;\; 2 \; , \quad \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 3 & 3 & & \end{array}$$

Another standard tableau of the same shape $\lambda = \begin{matrix} * & * & * & * \\ * & * & & \end{matrix}$ corresponding to the path

$$\lambda^0 = \Phi,\ \lambda^1 = \begin{matrix} * \\ * \end{matrix},\ \lambda^2 = \begin{matrix} * & * \\ * & * \end{matrix},\ \lambda^3 = \begin{matrix} * & * & * & * \\ * & * & & \end{matrix} = \lambda.$$

is obtained as follows

$$\lambda^0 = \Phi,\ \lambda^1 = \begin{matrix} 1 \\ 1 \end{matrix},\ \lambda^2 = \begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix},\ \lambda^3 = \begin{matrix} 1 & 2 & 3 & 3 \\ 1 & 2 & & \end{matrix} = \lambda.$$

3.1.6. Theorem.[R-S correspondence] The map $\pi \overset{R-S}{\longleftrightarrow} (P,Q)$ is a bijection between elements of $\widetilde{S}_n$ and pairs of standard tableaux of the same shape $\lambda \vdash_a (\frac{1}{2}r(r+1)+2n)$, where λ varies over partitions of $(\frac{1}{2}r(r+1)+2n)$ whose 2-core is δ_r, $r \geq 0$.

Proof. We construct the bijection denoted by $\pi \overset{R-S}{\longleftrightarrow} (P,Q)$ where $\pi \in \widetilde{S}_n$ and P, Q are $\lambda-$tableaux, $\lambda \vdash_a (\frac{1}{2}r(r+1)+2n)$ as follows. We first describe the map that, given a permutation, produces a pair of tableaux. "$\pi \overset{R-S}{\longrightarrow} (P,Q)$" Suppose that π is given in two-line notation as

$$\pi = \begin{pmatrix} 1 & 2 \dots & n \\ x_1 & x_2 \dots & x_n \end{pmatrix}$$

where $x_i \in \{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}$.

We construct a sequence of tableaux pairs

$$(P_0, Q_0),\ (P_1, Q_1),\ \dots, (P_n, Q_n) = (P, Q),$$

where P_0, Q_0 are the tableau of shape δ_r whose entries are 0's and $x_1, x_2, \dots, x_n$ are inserted into the P's as tablets and the corresponding $1, 2, \dots, n$ are placed in the Q's as tablets so that $shP_k = shQ_k$ for all k. The operations of insertion and placement will now be described.We define the insertion process inductively. Let P be a partial tableau whose 2-core is δ_r, i.e., an array with distinct entries which increase along a path of λ . (So a partial tableau with 2-core δ_r will be standard if its elements are precisely tablets of $\{1, 2, \dots, n\}$.)

Let x be an element not in P. To insert x into P, we proceed as follows:

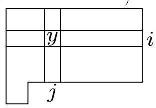
If x is positive then the horizontal tablet $\alpha_x = \boxed{x \mid x}$ is to be inserted into P along the cells (i,j) and $(i,j+1)$. The $(i,j+1)$th cell of α_x is called the head node of α_x and the (i,j)th cell of α_x is called the tail node of α_x.

If x is negative then $x := |x|$ and the vertical tablet $\beta_x = \begin{array}{|c|}\hline x \\ \hline x \\ \hline \end{array}$ is to be inserted into P along the cells (i,j) and $(i+1,j)$. The (i,j)th cell of β_x is called the head node of β_x and the $(i+1,j)$th cell of β_x is called the tail node of β_x.
If x is positive then,

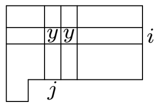
A Set Row $i := 1$, head node of $\alpha_x := x$ and tail node of $\alpha_x := x$.

B If head node of α_x is less than some element of Row i then,

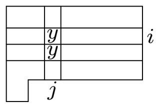
Let y be the smallest element of Row i greater than x which is in the cell (i,j).



(2 cases arise,
(BI) tablet containing y is horizontal i.e., α_y.

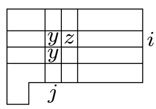


(BII) tablet containing y is vertical i.e., β_y.)

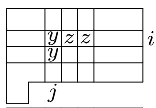


case BI If the tablet containing y is α_y. (i.e.head node of α_y is in the cell $(i,j+1)$ and the tail node of α_y is in the cell (i,j)) then, replace tablet α_y by tablet α_x.
Set tablet $\alpha_x :=$ tablet α_y, Row $i := i+1$ and go to **B**.

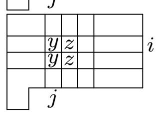
case BII If the tablet containing y is β_y. (i.e. head node of β_y is in the cell (i,j) and the tail node of β_y is in the cell $(i+1,j)$) then
Let z be the element in the cell $(i,j+1)$.



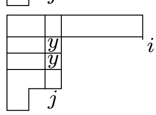
(3 cases arise,
(BIIa) tablet containing z is horizontal i.e., α_z.



(BIIb) tablet containing z is vertical i.e., β_z.

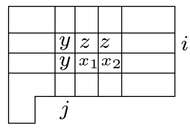


(BIIc) z is empty.



case BIIa If the tablet containing z is α_z (i.e., head node of α_z is in the cell $(i, j+2)$ and tail node of α_z is in the cell $(i, j+1)$) then replace head node of β_y and tail node of α_z by the tablet α_x.

Set $x_1 :=$ the element in the cell $(i+1, j+1)$ and
$x_2 :=$ the element in the cell $(i+1, j+2)$.



Place head node of β_y and tail node of α_z in the cells $(i+1, j+1)$ and $(i+1, j+2)$ respectively.

✠: (3 cases arise,
(BIIa1) x_1 is empty.
(BIIa2) $x_1 = x_2$.
(BIIa3) $x_1 \neq x_2$.)

case BIIa1 If x_1 is empty then stop.

case BIIa2 If $x_1 = x_2$ then
Set tablet $\alpha_x :=$ tablet α_{x_1}, Row $i := i + 2$ and go to **B**.

case BIIa3 If $x_1 \neq x_2$ then
(2 cases arise,
(BIIa3i) the element in the cell $(i+1, j+3) = x_2$, i.e., α_{x_2}
(BIIa3ii) the element in the cell $(i+2, j+2) = x_2$, i.e., β_{x_2})

case BIIa3i If the element in the cell $(i+1, j+3) = x_2$ then
Set $y_1 :=$ the element in the cell $(i+2, j+2)$ and
$y_2 :=$ the element in the cell $(i+2, j+3)$
replace the element in the cell $(i+2, j+2)$ and $(i+2, j+3)$ by
head node of β_{x_1} and tail node of α_{x_2} respectively.
Set $x_1 := y_1$, $x_2 := y_2$, Row $i = i+1$, Column $j = j+1$ and go to ✠.

case BIIa3ii If the element in the cell $(i+2, j+2)$ $= x_2$ then
Set tablet $\beta_x :=$ tablet β_{x_2},
replace the elements in the cell $(i+2, j+2)$ by head node of β_{x_1}.
Set Column $j := j + 3$ and go to B$'$. (B$'$ is the case as in B by replacing row by column, column by

row, positive tablet by negative tablet and negative tablet by positive tablet.)

case BIIb If the tablet containing z is β_z (i.e., head node of β_z is in the cell $(i, j+1)$ and tail node of β_z is in the cell $(i+1, j+1)$) then
replace head nodes of β_y and β_z by the tablet α_x and tail node of β_z by head node of β_y.
Set tablet $\beta_x :=$ tablet β_z, Column $j := j+3$ and go to $\mathbf{B}'$ (B$'$ is the case as in B by replacing row by column, column by row, positive tablet by negative tablet and negative tablet by positive tablet.)

case BIIc If z is empty then
replace head node of β_y by tail node of α_x, place head node ofα_x and head node of β_y in the cell $(i, j+1)$ and $(i+1, j+1)$ respectively and stop.

C Now head node of α_x is greater than every element of Row i so place the tablet α_x at the end of the Row i and stop.

If x is negative then, replace row by column, column by row, positive tablet by negative tablet and negative tablet by positive tablet in the positive case.

Placement of the tablet of an element in a tableau is even easier than insertion. Suppose that Q is a partial tableau of shape μ and if k is greater than every element of Q, then place the tablet of k in Q along the cells where the insertion in P terminates.

To build the sequence of tableaux pairs from the permutation

$$\pi = \begin{matrix} 1 & 2 & \cdots & n \\ x_1 & x_2 & \cdots & x_n \end{matrix}$$

start with a pair (P_0, Q_0) of tableaux of shape δ_r whose entries are 0's. Assuming that (P_{k-1}, Q_{k-1}) has been constructed, define (P_k, Q_k) by

$$\begin{aligned} P_k &= i_{x_k}(P_{k-1}), \\ Q_k &= \text{place tablet of } k \text{ into } Q_{k-1} \text{ where the insertion in P terminates.} \end{aligned}$$

Note that the definition of Q_k ensures that $\mathrm{sh}P_k = \mathrm{sh}Q_k$ for all k. The P tableau is called the insertion tableau and the Q tableau is called the recording tableau.

To show that we have a bijection, we construct an inverse correspondence.
$''P, Q) \overset{S-R}{\longrightarrow} \pi''$. We begin by defining $(P_n, Q_n) = (P, Q)$. We proceed

inductively, assuming that (P_k, Q_k) has been constructed, we will find x_k (the kth element of π) and (P_{k-1}, Q_{k-1}). To avoid double subscripting in what follows, we use $P_{i,j}$ to stand for the (i, j) entry of P_k.

Find the cells containing the tablet of k in Q_k.

2 cases arise,

† The cells containing tablet of k in Q_k are $(i, j-1)$ and (i, j)

‡ The cells containing tablet of k in Q_k are $(i-1, j)$ and (i, j)

case † If the cells containing tablet of k in Q_k are $(i, j-1)$ and (i, j), since this is the largest element whose tablet appears in Q_k, $P_{i,j-1}, P_{i,j}$ must have been the last element to be placed in the construction of P_k.

We can now use the following procedure to delete $P_{i,j-1}$, $P_{i,j}$ from P. For convenience, we assume the existence of an empty zeroth row above the first row of P_k and empty zeroth column to the left of the first column of P_k.

Set $x_1 := P_{i,j-1}$, $x_2 := P_{i,j}$ and erase $P_{i,j-1}, P_{i,j}$.

(2 cases arise,

(A) $x_1 = x_2$

(B) $x_1 \neq x_2$)

case A If $x_1 = x_2$ then

case AI Set head node of $\alpha_x := x_2$, tail node of $\alpha_x := x_1$ and Row $i := (i-1)^{\text{th}}$ row of P_k.

case AII If Row i is not the zeroth row of P_k then

Let y be the largest element of Row i smaller than x which is in the cell (i, l)

(2 cases arise,

(AIIa) the tablet containing y is α_y

(AIIb) the tablet containing y is β_y)

case AIIa If the tablet containing y is α_y then
replace tablet α_y by tablet α_x and
Set tablet $\alpha_x :=$ tablet α_y, Row $i := i - 1$ and goto **AII**

case AIIb If the tablet containing y is β_y then
Let z be the element in the cell $(i, l-1)$ and replace tail node of β_y and z by tablet α_x.
Set $x_1 := z$ and $x_2 :=$ tail node of β_y and go to **B**.

case AIII Now the tablet α_x has been removed from the first row, so

Set $\pi_k := x$.

case B If $x_1 \neq x_2$ then

(2 cases arise,

(B1) the tablet containing x_1 is β_{x_1}

(B2) the tablet containing x_1 is α_{x_1})

case BI If the tablet containing x_1 is β_{x_1} then
replace head node of β_{x_1} by tail node of β_{x_2}.
Set tablet $\beta_x :=$ tablet β_{x_1} and
Column $j := j - 2$ and go to **A′II**. (**A′II** is the case as in **AII** by replacing row by column, column by row, positive tablet by negative tablet and negative tablet by positive tablet.)

case B2 If the tablet containing x_1 is α_{x_1} then
replace the elements in the cell $(i-1, j-2)$ and $(i-1, j-1)$ by head node of α_{x_1} and tail node of β_{x_2} respectively.
Set $x_1 :=$ the element in the cell $(i-1, j-2)$
$x_2 :=$ the element in the cell $(i-1, j-1)$ and
if $x_1 = x_2$ then Row $i := i - 1$ and go to **A**
else Column $j = j - 1$ go to **B**.

Case ‡ follows as in case † by replacing row by column, column by row, positive tablet by negative tablet and negative tablet by positive tablet.

It is easy to see that P_{j-1} is P_j after the deletion process just described is complete and Q_{j-1} is Q_j with the tablet of j erased. Continuing in this way, we eventually recover all the elements of π in reverse order.

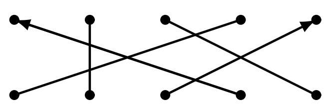
This completes the proof of the theorem.

Now we consider an example of the complete algorithm. Boldface numbers are used for the elements of the lower line of π and hence also for the elements of the P_k.

Example:

Let

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \mathbf{-4} & \mathbf{2} & \mathbf{5} & \mathbf{1} & \mathbf{-3} \end{pmatrix}.$$



Then the tableau whose 2-core is empty is constructed by the algorithm are

$$P_k = \phi,\quad \begin{array}{l}4,\\4\end{array}\quad \begin{array}{ll}2&2\\4&4\end{array},\quad \begin{array}{llll}2&2&5&5\\4&4\end{array},\quad \begin{array}{llll}1&1&5&5\\2&2\\4&4\end{array},\quad \begin{array}{llll}1&1&5&5\\2&2\\3&4\\3&4\end{array} = P$$

$$Q_k = \phi\ ,\quad \begin{array}{l}1\\1\end{array},\quad \begin{array}{ll}1&2\\1&2\end{array},\quad \begin{array}{llll}1&2&3&3\\1&2\end{array},\quad \begin{array}{llll}1&2&3&3\\1&2\\4&4\end{array},\quad \begin{array}{llll}1&2&3&3\\1&2\\4&4\\5&5\end{array} = Q$$

So the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \mathbf{-4} & \mathbf{2} & \mathbf{5} & \mathbf{1} & \mathbf{-3} \end{pmatrix}$ gives the pair of tableaux

$$\left(\begin{array}{llll}1&1&5&5\\2&2\\3&4\\3&4\end{array}\ ,\quad \begin{array}{llll}1&2&3&3\\1&2\\4&4\\5&5\end{array} \right)$$

For the same permutation, the tableau whose 2-core is $(4,3,2,1,0)$ is constructed by the algorithm are

$$P_k = \begin{array}{llll}0&0&0&0\\0&0&0\\0&0\\0\end{array},\quad \begin{array}{llll}0&0&0&0\\0&0&0\\0&0\\0\\4\\4\end{array},\quad \begin{array}{llllll}0&0&0&0&2&2\\0&0&0\\0&0\\0\\4\\4\end{array},\quad \begin{array}{llllllll}0&0&0&0&2&2&5&5\\0&0&0\\0&0\\0\\4\\4\end{array},$$

$$\begin{array}{llllllll}0&0&0&0&1&1&5&5\\0&0&0&2&2\\0&0\\0\\4\\4\end{array},\quad \begin{array}{llllllll}0&0&0&0&1&1&5&5\\0&0&0&2&2\\0&0\\0&4\\3&4\\3\end{array} = P$$

$$Q_k = \begin{array}{llll}0&0&0&0\\0&0&0\\0&0\\0\end{array},\quad \begin{array}{llll}0&0&0&0\\0&0&0\\0&0\\0\\1\\1\end{array},\quad \begin{array}{llllll}0&0&0&0&2&2\\0&0&0\\0&0\\0\\1\\1\end{array},\quad \begin{array}{llllllll}0&0&0&0&2&2&3&3\\0&0&0\\0&0\\0\\1\\1\end{array},$$

$$\begin{array}{llllllll}0&0&0&0&2&2&3&3\\0&0&0&4&4\\0&0\\0\\1\\1\end{array},\quad \begin{array}{llllllll}0&0&0&0&2&2&3&3\\0&0&0&4&4\\0&0\\0&5\\1&5\\1\end{array} = Q$$

So the pair of tableaux associated with the signed permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \mathbf{-4} & \mathbf{2} & \mathbf{5} & \mathbf{1} & \mathbf{-3} \end{pmatrix}$$

is given by

$$\left(\begin{array}{cccccccccccccccccc}
0 & 0 & 0 & 0 & 1 & 1 & 5 & 5 & \quad & 0 & 0 & 0 & 0 & 2 & 2 & 3 & 3 \\
0 & 0 & 0 & 2 & 2 & & & & & 0 & 0 & 0 & 4 & 4 & & & \\
0 & 0 & & & & & & & & 0 & 0 & & & & & & \\
0 & 4 & & & & & & & & 0 & 5 & & & & & & \\
3 & 4 & & & & & & & & 1 & 5 & & & & & & \\
3 & & & & & & & & & 1 & & & & & & &
\end{array}\right)$$

4. Combinatorial properties of the R-S Correspondence

Proposition 4.1.1. Let $\pi \in \widetilde{S}_n$ such that $\pi = x_1 x_2 \ldots x_n$. Then $P(\pi^-) = P^t$ and $Q(\pi^-) = Q^t$ where $\pi^- = (-x_1)(-x_2)\ldots(-x_n)$ and P^t is the conjugate of the P-tableau.

Proof. The insertion of positive elements is done along the rows and the insertion of negative elements along the columns. The process of insertion of positive and negative dominoes gets interchanged for π^- as compared to that of π. Hence $P(\pi^-) = P^t$ and $Q(\pi^-) = Q^t$.

4.2. Growth and Local rules

We define the growth and local rules for the RS algorithm developed in previous section following that for the symmetric group.

Definition 4.2.1. We define a partial order $\leq$ on the set of partitions of $(\frac{1}{2}r(r+1)+2n)$ whose $2-$core is $\delta_r,\ r \geq n-1$ as, $\lambda \leq \mu$ if λ is obtained from μ by successive removal of rim 2-hooks.

Definition 4.2.2. Let λ, μ be partitions of $(\frac{1}{2}r(r+1)+2n)$ whose $2-$core is $\delta_r,\ r \geq n-1$. The smallest partition containing both λ and μ is said to be the join of λ and μ, denoted by $\lambda \cup \mu$.

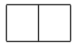
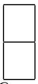
Definition 4.2.3. Let λ, μ be partitions of $(\frac{1}{2}r(r+1)+2n)$ whose $2-$core is $\delta_r,\ r \geq n-1$. The largest partition which is contained in both λ and μ is said to be the meet of λ and μ, denoted by $\lambda \cap \mu$.

The RS-insertion algorithm is defined in a fashion very similar to that for the symmetric group. The growth and local rules also follows closely as that for the symmetric group.

Let Y be the lattice of partitions of $(\frac{1}{2}r(r+1)+2n)$ whose $2-$core is $\delta_r,\ r \geq n-1$.

Let C_n be the n-chain. i.e. $1-2-3-\cdots-n$ lattice in set $\{1, 2, \ldots, n\}$.

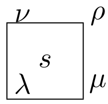
We will represent the elements of $C_n \times C_n$ as vertices $(i, j), 0 \leq i, j \leq n$ and covering relations as lines. The squares thus formed will be cordinated with square (i, j) being the one whose north east vertex is (i, j).

Let $\pi = x_1 x_2 \ldots x_n$ be the second line notation of the permutation $\pi \in \widetilde{S}_n$. The element x_i is represented by a domino  if $x_i > 0$ and by the domino  if $x_i < 0$ in the square $(i, |x_i|)$.

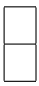
We now define the growth $g_\pi : C_n \times C_n \to Y$ that will depend on π. We start by letting

$$g_\pi(i, j) = \emptyset \text{ if } \mathrm{i} = 0, \mathrm{j} = 0.$$

Consider the square with coordinates (i, j) labeled as in diagram



Note :

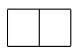
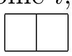
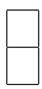
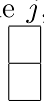
- s can be empty
- s can be 
- s can be 

Suppose by induction on $i+j$ that we have defined $g_\pi(i-1, j-1), g_\pi(i, j-1)$ and $g_\pi(i-1, j)$ which we denote by λ, μ and ν respectively, we then define $g_\pi(i, j)$, denoted by ρ using the following local rules.

4.2.4. Local Rules

LR1 If $\mu \neq \nu$ then $\rho = \mu \cup \nu$.

LR2 If $\lambda < \mu = \nu$. Two cases arise.

1. If μ is obtained fromλ by adding a positive domino  to λ_i for some i, then, ρ is obtained from μ by adding a positive domino  to μ_{i+1}.
2. If μ is obtained fromλ by adding a negative domino  to λ'_j for some j, then, ρ is obtained from μ by adding a negative domino  to μ'_{j+1}.

LR3 If $\lambda = \mu = \nu$, then let

$$\rho = \begin{cases} \lambda & , \text{ if the box does not contain a domino} \\ \lambda \text{ with } \square\square \text{ added to first row } \lambda_1 , & \text{if } \square\square \text{ appears in box} \\ \lambda \text{ with } \begin{smallmatrix}\square\\\square\end{smallmatrix} \text{ added to first column } \lambda'_1, & \text{if } \begin{smallmatrix}\square\\\square\end{smallmatrix} \text{ appears in box} \end{cases}$$

Lemma 4.2.5. Let $\pi \in \widetilde{S}_n$ then $P(\pi^{-1}) = Q(\pi)$.
Proof. This is a consequence of the fact that growth diagram and local rules are symmetric.

4.3. Relation between the sets of bipartitions of n and partitions of $2n$ whose 2-core is δ_r

In this section, we give a bijection between the set of bipartitions of n and that of single partitions of $2n$ whose 2-core is $\delta_r,\ r \geq n-1$, which is dominance order preserving.

Definition 4.3.1. Let ρ be a partition of $(\frac{1}{2}r(r+1)+2n)$ whose 2−core is $\delta_r,\ r \geq n-1$. We define a map

$$\eta : \rho \mapsto (\lambda, \mu), \quad \lambda \vdash l,\ \mu \vdash m,\ l+m=n$$

such that if r is even

$$\begin{aligned} \lambda_i &= \frac{1}{2}(\rho_i - (n-i)) \\ \mu_i &= \sum_{\substack{j \\ \mu'_j \geq i}} 1 \text{ where } \mu'_{\mathrm{j}} = \frac{1}{2}(\rho'_{\mathrm{j}} - (\mathrm{n}-\mathrm{j})) \end{aligned}$$

if r is odd

$$\begin{aligned} \lambda_i &= \frac{1}{2}(\rho_i - (n-i)+1) \\ \mu_i &= \sum_{\substack{j \\ \mu'_j \geq i}} 1 \text{ where } \mu'_{\mathrm{j}} = \frac{1}{2}(\rho'_{\mathrm{j}} - (\mathrm{n}-\mathrm{j})+1) \end{aligned}$$

Lemma 4.3.2. Let ρ, ν be partitions of $(\frac{1}{2}r(r+1)+2n)$ whose 2−core is $\delta_r,\ r \geq n-1$ ($r = n-1$, if n is odd and $r = n$, if n is even) such that $\rho = (\lambda, \mu)$ and $\nu = (\alpha, \beta)$. Then $\rho \trianglerighteq \nu \Leftrightarrow (\lambda, \mu) \trianglerighteq (\alpha, \beta)$.

Proof. When r is even (r-odd follows similarly), let $\rho \trianglerighteq \nu$. i.e. $\sum_{i=1}^{j} \rho_i \geq \sum_{i=1}^{j} \nu_i, \ \forall j$.

$$\begin{aligned}
\lambda_i &= \frac{1}{2}(\rho_i - (n-i)) \\
\sum_{i=1}^{j} \lambda_i &= \frac{1}{2}\sum_{i=1}^{j}(\rho_i - (n-i)) \geq \frac{1}{2}\sum_{i=1}^{j}(\nu_i - (n-i)) = \sum_{i=1}^{j} \alpha_i
\end{aligned}$$

To show $|\lambda| + \sum_{i=1}^{j} \mu_i \geq |\alpha| + \sum_{i=1}^{j} \beta_i, \quad \forall j$.

Since $\rho \trianglerighteq \nu \Leftrightarrow \nu' \trianglerighteq \rho'$, we have $\sum_{i=1}^{j} \rho_i \geq \sum_{i=1}^{j} \nu_i \Leftrightarrow \sum_{i=1}^{j} \rho'_i \leq \sum_{i=1}^{j} \nu'_i$.

By an argument similar to the one in the first part we have,

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{j}(\rho'_i - (n-i)) &\leq \frac{1}{2}\sum_{i=1}^{j}(\nu'_i - (n-i)), \ \forall j \\
\sum_{i=1}^{j} \mu'_i &\leq \sum_{i=1}^{j} \beta'_i, \ \forall j \\
\sum_{i=1}^{j} \mu_i &\geq \sum_{i=1}^{j} \beta_i, \ \forall j \ \text{ since } \beta' \trianglerighteq \mu' \Leftrightarrow \mu \trianglerighteq \beta \\
|\alpha| + \sum_{i=1}^{j} \mu_i &\geq |\alpha| + \sum_{i=1}^{j} \beta_i, \ \forall j \\
|\lambda| + \sum_{i=1}^{j} \mu_i &\geq |\alpha| + \sum_{i=1}^{j} \beta_i, \ \forall j, \ \text{since } |\lambda| \geq |\alpha|
\end{aligned}$$

Since the map η defined above is a bijection, the other part follows. Hence the proof.

Lemma 4.3.3. Let ρ be a partition of $(\frac{1}{2}r(r+1)+2n)$ whose 2-core is δ_r. Then ρ can be associated to a pair of partitions as in Definition 2.1.8., but when associated to a pair of partitions through the map η we have, every domino in row i of ρ corresponds to a node of λ_i and every domino in column j of ρ corresponds to a node of μ'_j.

Proof. The proof follows by observing that every ρ_i is of the form $j+2k$ for some $0 \leq k \leq n$, where the head node of a positive domino is in the cell (i,j) and hence $\rho_i - i \equiv j - i \pmod{2}$. Similarly for a negative domino with tail node in the cell (i, j), the lemma follows from the fact that every ρ'_j is of the form $i+2k$ for some $0 \leq k \leq n$ and hence $\rho'_j - j \equiv j - i \pmod{2}$.

Remark 4.3.4. In [2], Bonaffé and Iancu constructed a generalised R-S correspondence for the set of bipartitions of n. Any x_i is inserted along the rows of partitions depending on being greater than or less than zero. i.e. the negative and positive in the permutation are inserted separately, which gives the tableaux $((P^{(+)}, P^{(-)}), (Q^{(+)}, Q^{(-)}))$ which follows the insertion procedure for the symmetric group.

We define a procedure of insertion of positive domino of $x_i > 0, \forall i$ along the rows using dominoes and negative domino $x_j < 0, \forall j$ along the columns using dominoes, which gives the pair of tableaux (P, Q). By lemma 4.3.3, we may split $P = (P^{(0)}, P^{(1)}), Q = (Q^{(0)}, Q^{(1)})$ where for $i = 0, 1$

$$\begin{aligned} P^{(i)} &= \text{tableau formed by the tablets whose head nodes are in} \\ &\quad i^{\text{th}}\text{–residue of } P \\ Q^{(i)} &= \text{tableau formed by the tablets whose head nodes are in} \\ &\quad i^{\text{th}}\text{–residue of } Q \end{aligned}$$

If δ_r is such that r is even then since positive values are inserted along the rows in both $P^{(0)}, P^{(+)}$, and Q is recording tableau we have

$$P^{(0)} = P^{(+)}, \quad Q^{(0)} = Q^{(+)}$$

and since negative values are inserted along the rows in $P^{(-)}$ and along the columns in $P^{(1)}$, and Q is recording tableau we have

$$P^{(1)} = (P^{(-)})^{\text{t}}, \quad Q^{(1)} = (Q^{(-)})^{\text{t}}.$$

If δ_r is such that r is odd then since positive values are inserted along the rows in both $P^{(1)}, P^{(+)}$, and Q is recording tableau we have

$$P^{(1)} = P^{(+)}, \quad Q^{(1)} = Q^{(+)}$$

and since negative values are inserted along the rows in $P^{(-)}$ and along the columns in $P^{(0)}$, and Q is recording tableau we have

$$P^{(0)} = (P^{(-)})^{\text{t}}, \quad Q^{(0)} = (Q^{(-)})^{\text{t}}.$$

The dominance ordering for the indexing set of the cellular basis happens naturally in our insertion process, whereas for the Bonnafé and Iancu [2] insertion process, one has to take the following:

$$(\lambda_1, \lambda_2) \trianglerighteq (\mu_1, \mu_2^{\text{t}})$$

Proposition 4.3.5. Let $w \in \widetilde{S}_n$. The R-S correspondence gives a pair of standard tableaux $w \longleftrightarrow (P(w), Q(w))$. For any $x, y \in \widetilde{S}_n$, $P(x) = P(y)$ if and only if $x \longleftrightarrow_L y$, where $P(x)$, $P(y)$ are the tableaux of shape $\lambda \vdash_a 2n + |\delta_r|,\ r \geq n-1$.
Proof. As in [[2] Prop.3.8.], we define admissible transformations in $\widetilde{S}_n$. Let $x \in \widetilde{S}_n$ be represented as

$$x = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & n \\ \varepsilon_1 p_1 & \varepsilon_2 p_2 & \dots & \varepsilon_i p_i & \varepsilon_{i+1} p_{i+1} & \dots & \varepsilon_n p_n \end{pmatrix}$$

$\varepsilon_j \in \{1, -1\},\ p_j \in \{1, 2, \dots, n\},\ p_i \neq p_j$ for $i \neq j$. The proof of the proposition is similar to the proof given in [2]. Since we give insertion in a single tableau, the proof closely follows that of the symmetric group.

Interchanging $\varepsilon_i p_i$ and $\varepsilon_{i+1} p_{i+1}$ is an admissible transformation if

(a) $2 \leq i \leq n-1$, $\varepsilon_{i-1} = \varepsilon_i = \varepsilon_{i+1}$ and p_{i-1} lies between p_i and p_{i+1}, denoted by $x \overset{a}{\sim} s_{i+1}x$

(b) $1 \leq i \leq n-2$, $\varepsilon_i = \varepsilon_{i+1} = \varepsilon_{i+2}$ and p_{i+2} lies between p_i and p_{i+1}, denoted by $x \overset{b}{\sim} s_{i+1}x$

(c) $\varepsilon_i = -\varepsilon_{i+1}$, denoted by $x \overset{c}{\sim} s_{i+1}x$.

The group theoretical description of these admissible transformation is given by $xs_i < x$ if and only if $x(i+1) < x(i)$ ($\varepsilon_i = \varepsilon_{i+1}$ or $\varepsilon_i = -\varepsilon_{i+1}$). The following cases arise

1. Each of (a) and (b) have two cases

   (a^+) $\varepsilon_{i-1} = \varepsilon_i = \varepsilon_{i+1} = 1$
   (a^-) $\varepsilon_{i-1} = \varepsilon_i = \varepsilon_{i+1} = -1$

   Similarly,

   (b^+) $\varepsilon_i = \varepsilon_{i+1} = \varepsilon_{i+2} = 1$
   (b^-) $\varepsilon_i = \varepsilon_{i+1} = \varepsilon_{i+2} = -1$

In the case of the symmetric group

(a) type transformation corresponds to Knuth relation of first kind and

(b) type transformation corresponds to Knuth relation of second kind.

Proof in case (a^+) We prove as in the case of symmetric group.

$$\begin{aligned} x &= p_1 \cdots p_{i-1} p_i p_{i+1} \cdots p_n \\ y &= p_1 \cdots p_{i-1} p_{i+1} p_i \cdots p_n. \end{aligned}$$

Since all elements inserted before p_{i-2} are same, it suffices to prove that for any partial tableau inserting p_{i-1}, p_i, p_{i+1} and p_{i-1}, p_{i+1}, p_i in the respective order yield the same tableau. (We are in the positive case and all dominoes inserted are along the first row)

If we denote by $I_{p_i}(P)$ the tableau rewriting from inserting p_i in P, we have to prove

$$I_{p_{i-1}} I_{p_i} I_{p_{i+1}}(P) = I_{p_{i-1}} I_{p_{i+1}} I_{p_i}(P) \qquad (1)$$

The proof is same as in the case of symmetric group, we give it here for the sake of completion. We prove this claim by induction on the number of rows in P. For $P = \delta_r$, both sides of (1) yields the same tableau.

$$\begin{array}{cccccccccc}
\times & \times & \cdot & \cdot & \cdot & \times & p_i & p_i & p_{i-1} & p_{i-1} \\
\times & \times & \cdot & \cdot & \cdot & p_{i+1} & p_{i+1} & & & \\
\times & \cdot & \cdot & \cdot & \times & & & & & \\
\times & \times & \times & & & & & & & \\
\times & \times & & & & & & & & \\
\times & & & & & & & & &
\end{array}$$

Now assume that P has r rows. Suppose the tablet $\boxed{p_{i-1}}\boxed{p_{i-1}}$ enters in the first row along k^{th} and $(k+1)^{\text{th}}$ column by replacing $\boxed{p'_{i-1}}\boxed{p'_{i-1}}$, we examine both sides of (1). Assume $\boxed{p_i}\boxed{p_i}$ is inserted next. Since $p_i < p_{i-1}$, $\boxed{p_i}\boxed{p_i}$ replaces some $\boxed{p'_i}\boxed{p'_i}$ from columns $j, j+1$ with $j < k$. Also $p'_i < p'_{i-1}$. This follows from Lemma 4.3.3.

Similarly $p_{i+1} > p_i$ that $\boxed{p_{i+1}}\boxed{p_{i+1}}$ replaces some element $\boxed{p'_{i+1}}\boxed{p'_{i+1}}$ from column $l, l+1$ with $l > k$ and $p'_{i+1} > p'_i$. Considering the right hand side of (1), we get that $\boxed{p_{i+1}}\boxed{p_{i+1}}$ and $\boxed{p_i}\boxed{p_i}$ if inserted in this respective order, replace same elements $\boxed{p'_{i+1}}\boxed{p'_{i+1}}$ and $\boxed{p'_i}\boxed{p'_i}$ from same column $l, l+1$ and $j, j+1$.

Therefore the first rows of two tableaux obtained are same. Moreover the rest of tableau is obtained by inserting p'_{i-1}, p'_i, p'_{i+1} and p'_{i-1}, p'_{i+1}, p'_i in a tableau of a strictly smaller number of rows in this respective order. Since the same order $p'_i < p'_{i-1} < p'_{i+1}$ still holds we appeal to induction to asset that the rest of the tableau are also the same.

Proof in case (a^-) is obtained by replacing rows by columns in proof of (a^+). The argument is the same since positive dominoes insertion along the rows is replaced by negative dominoes insertions along columns.

Since (b) part corresponds to the Knuth relation of second kind, a similar argument like that of (a) works. This completes the first half of the proof.

Since by lemma 4.1.3, the dominoes in the 0-residue and 1-residue classes split we can get the word associated to a tableau as in the case of the symmetric group. Given a tableau P, we can associate a pair of elements $(w_p, w_p^\dagger)$ called the row word pair of P.

$$w_p = w_p^0 w_p^1 \quad \text{and} \quad w_p^\dagger = w_p^1 w_p^0$$

where w_p^0 is the row word associated to the 0-residue and w_p^1 is the row word associated to the 1-residue.

We will show that $\pi \overset{a^+}{\sim} \pi_P$. (Note that in the case of a^-, $\pi \overset{a^-}{\sim} \pi_P^\dagger$.) Since Knuth relations are transitive the converse of the theorem will follow. We induct on n. The base case is trivial for $n = 1$, $\pi = \pi_P$. Now, assume that x is the last element of π, i.e, π is written in one line notation as $\pi = \ldots x$ if $x > 0$.

On $\pi = \pi' x$, where π' is a sequence of $n-1$ elements. Therefore, by induction we have $\pi' \overset{a^+}{\sim} \pi'_P$ where $P' = P(\pi')$. Thus it suffices to prove that $\pi_{P'} x \overset{a^+}{\sim} \pi_P$.

Let $R_1, \ldots, R_l, C_1, \ldots, C_m$ be the rows and columns of P'. Assume that $R_1 = p_1 \cdots p_k$. If the domino x enters P' in the column $j, j+1$, then $p_1 < \ldots < p_{j-1} < x < p_j < \ldots < p_k$.

Therefore, we have the following Knuth operation

$$\begin{aligned}
\pi'_P x &= C_l \cdots C_1 R_l \cdots R_2 p_1 \cdots p_k x \\
&\overset{a^+}{\sim} C_l \cdots C_1 R_l \cdots R_2 p_1 \cdots p_{k-1} x p_k \\
&\vdots \\
&\overset{a^+}{\sim} C_l \cdots C_1 R_l \cdots R_2 p_1 \cdots p_{j-1} p_j x p_{j+1} \cdots p_k \\
&\overset{b^+}{\sim} C_l \cdots C_1 R_l \cdots R_2 p_1 \cdots p_j p_{j-1} x p_{j+1} \cdots p_k \\
&\vdots \\
&\overset{b^+}{\sim} C_l \cdots C_1 R_l \cdots R_2 p_j p_1 \cdots p_{j-1} x p_{j+1} \cdots p_k
\end{aligned}$$

Therefore the Knuth relations generate exactly the first row of $P(\pi)$. Also the element replaced by x from the first row comes at the end of R_2. The above sequence of operations can be repeated for each row to get the same tableau. The other case is done by replacing row by column. Since the (c) type of transformations do not change the relative ordering of elements within the residues the P-tableau remain the same.

Remark 4.3.6. The partitions of $2n$ whose 2-core is δ_r gives rise to an alternative description of the indexing set of the cellular basis in the sense of Graham and Lehrer [7] in the asymptotic case in type B_n the Kazhdan-Lusztig basis $\{C_w | w \in \mathcal{H}_n\}$ given in [6].

Note 4.3.7. The signed Brauer algebra was introduced in 1997 by Parvathi and Kamaraj [10]. These algebras contain Brauer algebras and the group algebra of the hyperoctahedral group of type B_n, as subalgebras. The study of the signed Brauer algebras has assumed importance since the realization of these algebras as homomorphic image of centralizer of certain closed subgroup $O_t(n)$ of orthogonal group [11]. We are on the process of constructing a cellular basis for the signed Brauer algebras using the cellular basis of the hyperoctahedral group of type B_n.

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References

[1] D.Barbasch, D.Vogan *Primitive ideals and orbital integrals on complex classical groups*, Math.Ann., 259 (1982), 153-199.

[2] C.Bonnafé and L.Iancu *Left cells in type B_n with unequal parameters*, preprint, 2003; available at arXiv: math.RT/0302037.

[3] Bruce E.Sagan *The symmetric group*, second edition, Springer 1991.

[4] S.Fomin *Schensted algorithms for dual graded graphs*, J.Alg.Combin. 4 (1995), 5-45.

[5] D.Garfinkle *On the classification of Primitive ideals for complex classical Lie algebras II*, Compositio Math., 81 (1992), 307-336.

[6] M.Geck *Relative Kazhdan-Lusztig cells*, preprint(2005), available at http://www.arXiv.org/math.RT/0504216.

[7] Graham and Lehrer *Cellular algebra*, Invent. Math., 123 (1996), 1-34.

[8] G.James and A.Kerber *The Representation theory of the symmetric group*, Encyclopedia of mathematics and its applications, Addison Wesley publishing company 1981.

[9] C.Musili *Representations of finite groups*, Hindustan book agency.

[10] M.Parvathi and M.Kamaraj *Signed Brauer Algebras*, Comm. in Algebra, 26(3) (1998), 839-855.

[11] M.Parvathi and C.Selvaraj *Signed Brauer Algebras as centralizer Algebras*, Comm. in Algebra 27(12) (1999), 5985-5998.

[12] M.Parvathi and C.E.Shanthi *Representation of Hecke algebra of type B_n*, Comm. in Algebra 27(11) (1999), 5203-5234.

[13] G.De.B.Robinson *Representation theory of symmetric group*, Edinburg University press, 1961.

[14] M.Shimozono And D.E.White *Color to spin ribbon Schensted algorithms*, Discrete Math., 246(2002), 295-316.

[15] M.A.A.Van Leeuwen *The Robinson Schensted and Shutzenberger algorithms an elementary approach*, Electronic J. Combin., 3 (1996).

Contact information

M. Parvathi,
B. Sivakumar,
A. Tamilselvi

Ramanujan Institute for Advanced study in mathematics, University of Madras, Chennai-600 005, India
E-Mail: sparvathi.riasm@gmail.com, b.sivakumar@hotmail.com, tamilselvi.riasm@gmail.com

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