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RESEARCH ARTICLE

Natural dualities for varieties generated by a set of subalgebras of a semi-primal algebra

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Abstract. The main contribution of this paper is the construction of a strong duality for the varieties generated by a set of subalgebras of a semi-primal algebra. We also obtain an axiomatization of the objects of the dual category and develop some algebraic consequences (description of the dual of the finite structures and algebras, construction of finitely generated free algebras,...). Eventually, we illustrate this work for the finitely generated varieties of MV-algebras.

1. Introduction

The most famous example of equivalence between algebras and topological spaces appeared in 1936 in a pioneering paper [18] of Stone in which he developed a duality between the category of Boolean algebras and the category of Boolean spaces. This duality gives a way to translate the algebraic properties of Boolean algebras in the language of topology and vice versa.

His ideas were later adapted to a wide range of algebras, especially to classes of algebras arising from logic. Let us quote, for example, Pontryagin's duality for Abelian groups (see [14] and [15]), Priestley's dualities for distributive bounded lattices (see [16] and [17]) or Heyting algebras etc. In all these situations, the results were obtained for a quasi-variety of algebras on the one hand and a category of topological structures on the other hand.

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In the early eighties DAVEY and WERNER initiated a systematic approach of the problem of construction of dualities for classes of algebras (see [9] for one of the first papers and [4] for a monograph on the subject). This was the starting point of the theory of natural dualities. One of the main features of this theory is to provide sufficient conditions for the existence of a duality for a finitely generated quasi-variety of algebras, and a canonical construction of the duality. The objects of the dual category are topological spaces with a structure, and the theory suggests which structure one has to add to these spaces to obtain a duality. When a class of algebras is dualisable, one can translate algebraic problems into equivalent topological problems that are sometimes more easily solved in this language. The most interesting cases arise when the duality is *strong*. Indeed, it means that the representation theorem can be extended to a categorical equivalence between the categories of algebras and topological structures. Universal problems can then be solved by duality. This theory has been widely developed and applied since its birth (see [7], [10], [5], [8], [13] for example or [4] and the bibliography therein).

One of the most beautiful applications of the theory of natural dualities is the case of a variety generated by a semi-primal algebra. Indeed, in this particular situation, the dual category admits a very nice axiomatization (see theorem 3.14 in [4]). Moreover, some interesting varieties fall in this scope. For instance, any MV-chain $Ł_n$ ($n \in \mathbb{N}$) is a semi-primal algebra and generates a dualisable variety (which is the variety of algebras of the $n+1$-valued ŁUKASIEWICZ logic). As noticed by R. CIGNOLI in [3], this result can be seen as a consequence of Theorem 6.5 in [11]. The paper [13] deals with some of the applications of this duality such as the description of the finitely (or countably) generated free algebras, or the finite projective members in $\mathbb{HSP}(Ł_n)$.

The first motivation of the present paper is to generalize the strong dualities for the varieties $\mathbb{HSP}(Ł_n)$ to *strong* dualities for varieties $\mathbb{HSP}(Ł_{n_1}, \ldots, Ł_{n_r})$ generated by a finite number of finite MV-chains. These latter varieties deserve a special interest for they are exactly, according to KOMORI's classification work [12], the finitely generated subvarieties of the variety of MV-algebras.

Now, remark that if n is the least common multiple of $\{n_1, \ldots, n_r\}$, then for $1 \leq i \leq r$, $Ł_{n_i}$ is a subalgebra of $Ł_n$ and the variety $\mathbb{HSP}(Ł_{n_1}, \ldots, Ł_{n_r}) = \mathbb{ISP}(Ł_{n_1}, \ldots, Ł_{n_r})$ is a subvariety of $\mathbb{HSP}(Ł_n) = \mathbb{ISP}(Ł_n)$.

Therefore, with a duality for $\mathbb{HSP}(Ł_n)$, any member of $\mathbb{HSP}(Ł_{n_1}, \ldots, Ł_{n_r})$ can be represented as a concrete algebra of morphisms and a duality for $\mathbb{HSP}(Ł_{n_1}, \ldots, Ł_{n_r})$ might seem to be useless. But a strong duality is much more than a representation theorem : it is the expression of a (dual) equivalence between two categories. So, in order to translate properties that involve the category $\mathbb{HSP}(Ł_{n_1}, \ldots, Ł_{n_r})$ and no longer a single object (such as the universal problems), it is the duality for $\mathbb{HSP}(Ł_n)$ that would be useless.

The method we develop to construct a strong duality for $\mathbb{HSP}(Ł_{n_1}, \ldots, Ł_{n_r})$ relies on the following observation: for any algebra A of $\mathbb{HSP}(Ł_{n_1}, \ldots, Ł_{n_r})$, if n is the least common multiple of $\{n_1, \ldots, n_r\}$, then for any homomorphism u from A to $Ł_n$, there exists a i in $\{1, \ldots, r\}$ such that u is a homomorphism form A to $Ł_{n_i}$. It means that, roughly speaking, the information contained in the union for $1 \leq i \leq r$ of the sets of the homomorphisms from A to $Ł_{n_i}$ is the same as the information contained in the set of the homomorphisms from A to $Ł_n$. Thus, by transporting the structure of the dual of any algebra A under the duality for $\mathbb{HSP}(Ł_n)$ to the disjoint union for $1 \leq i \leq r$ of the sets of the homomorphisms from A to $Ł_{n_i}$ (which is the base set for the construction of the dual of A in the multi-sorted approach), there is a hope to obtain a duality. The interesting point is that, by proceeding carefully (by identifying what should be identified; this is obviously made precise in the paper), it is possible to obtain a *strong* duality.

Now, it appears clearly that this line of argument does not depend on the nature of the algebras $Ł_n$. The semi-primality of the algebras is not even necessary. So, one can restate the problem in these more general settings. Suppose that Π is a set of subalgebras of a finite algebra D and that a duality for $\mathbb{ISP}(D)$ already exists. Can one hope to construct a duality for $\mathbb{ISP}(\Pi)$ by transporting the structure of the duality for $\mathbb{ISP}(D)$?

In the very first part of the paper, we study some conditions under which one can construct a duality for $\mathbb{ISP}(\Pi)$ by this argument. Then, the results are particularized to varieties generated by a set of subalgebras of a semi-primal algebra. In this case, the obtained duality reveals to be a strong one. In Theorem 2.5, we even obtain a nice axiomatization of the objects of the dual category (this axiomatization is the counterpart of Theorem 3.14 in [4]).

Then, we study some algebraic consequences of the duality. For instance, when we deal with the strong dualities for the varieties generated by subalgebras of a semi-primal algebra, we are able to describe the dual of finite structures and finite algebras, to construct the finitely generated free algebras, and to characterize finite projective algebras in $\mathbb{HSP}(\Pi)$.

The final part of the paper is dedicated to a concrete illustration of our developments. We indeed use our results to construct a strong duality for any of the finitely generated subvarieties of the variety of MV-algebras and give some of its applications such as the description of finitely generated free algebras in these varieties.

2. Construction of a duality

2.1. Notations

Let us first set the general notations and assumptions that are in use throughout the paper. We use the standard notations of the theory of natural dualities and category theory. Hence, we denote the algebras by underlined Roman capital letters and the topological structures by "undertilded" Roman capital letters.

We start with a finite dualisable algebra $\underline{D}$ on the language $\mathcal{L}$. We denote by $\mathcal{D}$ the category with algebras of $\mathbb{ISP}(\underline{D})$ for objects and homomorphisms for arrows.

We set a structure $\underset{\sim}{D} = \langle D; G_0, H_0, R_0, \tau\rangle$ that generates a duality for $\mathcal{D}$ where τ is the discrete topology and G_0 (resp. H_0, R_0) is a set of operations (resp. partial operations, relations). We denote by $\mathcal{E}$ the dual category: objects of $\mathcal{E}$ are the topological structures of $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{D})$ and arrows of $\mathcal{E}$ are the continuous maps that preserve the structure defined by G_0, H_0 and R_0. The canonical functors of this duality are denoted by $\mathsf{H} : \mathcal{D} \to \mathcal{E}$ and $\mathsf{K} : \mathcal{E} \to \mathcal{D}$.

For the first part of the paper, we do not assume that $\underline{D}$ is a semi-primal algebra.

Our goal is to construct a (strong) duality for the quasi-variety $\mathcal{A} = \mathbb{ISP}(\underline{\Pi})$ where $\underline{\Pi} = \{\underline{P}_1, \ldots, \underline{P}_r\}$ is a set of subalgebras of $\underline{D}$. To this aim, our approach in natural duality theory is the multi-sorted one. It means that the objects of the dual category are the members of $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi})$ for a suitable $\underline{\Pi}$-indexed structure $\underset{\sim}{\Pi}$ and that the dual of an algebra $\underline{A}$ of $\mathcal{A}$ is a $\underline{\Pi}$-indexed topological structure defined on $\dot\cup_{i\in\{1,\ldots,r\}}\mathcal{A}(\underline{A}, \underline{P}_i)$ (where $\dot\cup$ stands for the disjoint union).

We are going to restrict ourselves to some particular sets of subalgebras $\underline{\Pi}$. Indeed, in this paper, we only consider the sets $\underline{\Pi}$ such that

$$\forall\ \underline{A} \in \mathcal{A},\ \mathcal{D}(\underline{A}, \underline{D}) = \bigcup_{i\in\{1,\ldots,r\}} \mathcal{A}(\underline{A}, \underline{P}_i), \tag{2.1}$$

(note that $\mathcal{D}(\underline{A}, \underline{P}_i) = \mathcal{A}(\underline{A}, \underline{P}_i)$ for every $\underline{A}$ in $\mathcal{A}$ and $\underline{P}_i$ in $\underline{\Pi}$). This is a rather strong restriction. It means that, disregarding the structure that could be defined on $\dot\cup_{i\in\{1,\ldots,r\}}\mathcal{A}(\underline{A}, \underline{P}_i)$, this set contains the same information as $\mathsf{H}(\underline{A})$. But, as we shall see, condition (2.1) is satisfied for any set $\underline{\Pi}$ of proper subalgebras of a semi-primal algebra $\underline{D}$. Note that the converse property always holds: if $\underline{A}$ is an algebra of $\mathcal{D}$ such that $\cup_{i\in\{1,\ldots,r\}}\mathcal{A}(\underline{A}, \underline{P}_i) = \mathcal{D}(\underline{A}, \underline{D})$, then $\underline{A}$ is an algebra of $\mathcal{A}$.

Now, in order to obtain a duality for $\mathcal{A}$, it suffices to define a structure $\underset{\sim}{\Pi}$ on Π in such a way that the algebra of morphisms from $\dot\cup_{i\in\{1,\ldots,r\}}\mathcal{A}(\underline{A},\underline{P}_i)$ (seen as a structure of $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi})$) to $\underset{\sim}{\Pi}$ is isomorphic to the algebra of morphisms from $\mathsf{H}(\underline{A})$ to $\underset{\sim}{D}$ (which is the bidual of $\underline{A}$ under the duality between $\mathcal{D}$ and $\mathcal{E}$). The idea is to define $\underset{\sim}{\Pi}$ by transporting in Π the structure of $\underset{\sim}{D}$ and to identify the elements of $\dot\cup_{i\in\{1,\ldots,r\}}P_i$ which are equal in D. This should ensure that any morphism from the $\underline{\Pi}$-indexed structure $\dot\cup_{i\in\{1,\ldots,r\}}\mathcal{A}(\underline{A},\underline{P}_i)$ to $\underset{\sim}{\Pi}$ can be seen as an $\mathcal{E}$-morphism and conversely. More precisely, we define $\underset{\sim}{\Pi}$ as the $\underline{\Pi}$-indexed structure

$$\underset{\sim}{\Pi} = \langle \dot{\bigcup_{i\in\{1,\ldots,r\}}} P_i; \{f_{ij} \mid i,j \in \{1,\ldots,r\}\}, \bigcup_{s\in R_0^+} \{s^{-1}_{i_1,\ldots,i_{k_s}} \mid 1 \le i_1 \le \cdots \le i_{k_s} \le r\}; \tau\rangle, \quad (2.2)$$

where

- τ is the discrete topology;
- for every i and j in $\{1,\ldots,r\}$, the map f_{ij} is a partial operation defined on the subset $P_i \cap P_j$ of the j^{th} summand P_j of $\dot\cup_{k\in\{1,\ldots,r\}}P_k$, which is valued in the subset $P_i \cap P_j$ of the i^{th} summand P_i of $\dot\cup_{k\in\{1,\ldots,r\}}P_k$ and which identifies the element that are equal in $\cup_{k\in\{1,\ldots,r\}}P_k$ (the latter union is not taken disjoint):
$$f_{ij} : P_i \cap P_j \subset P_j \to P_i : q \mapsto q;$$
- R_0^+ denotes the set whose elements are the relations of R_0 and the graphs of the (partial) operations of G_0 and H_0;
- for every relation s in R_0^+ with arity k_s and every $i_1,\ldots,i_{k_s} \in \{1,\ldots,r\}$, the relation $s^{-1}_{i_1,\ldots,i_{k_s}}$ is defined by
$$s^{-1}_{i_1,\ldots,i_{k_s}} = \{(q_1,\ldots,q_{k_s}) \in P_{i_1} \times \cdots \times P_{i_{k_s}} \mid (q_1,\ldots,q_{k_s}) \in s\}.$$
We denote by $R_0^{+,-1}$ the set of the relations $s^{-1}_{i_1,\ldots,i_{k_s}}$ with $s \in R_0^+$ and $1 \le i_1 \le \cdots \le i_{k_s} \le r$.

We denote by $\mathcal{X}$ the category of the topological structures of $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi})$ with the obvious morphisms (the continuous maps that preserve the structure). If $\underline{A}$ is an algebra of $\mathcal{A}$, we denote by $\mathsf{D}(\underline{A})$ the $\underline{\Pi}$-indexed structure induced by $\underset{\sim}{\Pi}^A$ on $\dot\cup_{i\in\{1,\ldots,r\}}\mathcal{A}(\underline{A},\underline{P}_i)$. If $\underset{\sim}{X}$ is an element of $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi})$, we denote by $\mathsf{E}(\underset{\sim}{X})$ the $\mathcal{A}$-algebra of the $\mathcal{X}$-morphisms from $\underset{\sim}{X}$ to $\underset{\sim}{\Pi}$.

Note that, strictly speaking, since we compute the structure $\underset{\sim}{\Pi}$ on $\dot\bigcup_{i\in\{1,\ldots,r\}} P_i$, we should have replaced P_i by $P_i\times\{i\}$ for every i in $\{1,\ldots,r\}$ in order to ensure that the P_i are disjoint. By doing so, the definition of f_{ij} would have become $f_{ij} : \{(q,j) \mid q \in P_i \cap P_j\} \to P_i \times \{i\} : (q,j) \mapsto (q,i)$. Since the idea behind the construction of $\underset{\sim}{\Pi}$ is clear, we have decided not to bother with such heavier notations.

2.2. The duality theorem

With the notations and assumptions introduced in the previous section in mind, we can prove the following theorem.

Theorem 2.1. *If $\underline{\Pi} = \{\underline{P}_1,\ldots,\underline{P}_r\}$ is a set of subalgebras of a finite algebra $\underline{D}$ that satisfies condition (2.1) and if $\underset{\sim}{\Pi}$ is the $\underline{\Pi}$-indexed structure defined by (2.2), then the structure $\underset{\sim}{\Pi}$ generates a duality between $\mathcal{A} = \mathbb{ISP}(\underline{\Pi})$ and $\mathcal{X} = \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi})$.*

Proof. Assume that $\underline{A}$ is an algebra of $\mathcal{A}$. Since $\underset{\sim}{\Pi}$ is a $\underline{\Pi}$-indexed structure that is algebraic over $\underline{\Pi}$, it suffices to prove that the evaluation map $e^{\mathcal{A}}_{\underline{A}} : \underline{A} \hookrightarrow \mathsf{ED}(\underline{A})$ is onto. In fact, one can prove that $\psi : \mathsf{ED}(\underline{A}) \to \mathsf{KH}(\underline{A})$ defined by $\psi(\alpha)(u) = \alpha(u)$ for every $\alpha \in \mathsf{ED}(\underline{A})$ and every $u \in \mathsf{H}(\underline{A})$ is an isomorphism such that $\psi \circ e^{\mathcal{A}}_{\underline{A}} = e^{\mathcal{D}}_{\underline{A}}$ (where $e^{\mathcal{D}}_{\underline{A}}$ is the evaluation map $e^{\mathcal{D}}_{\underline{A}} : \underline{A} \to \mathsf{KH}(\underline{A})$). The details are left the reader. □

Note that the structure $\underset{\sim}{\Pi}$ could also have been defined with the help of the *Piggyback duality theorem* (see [6]).

Naturally, we denote by $\mathsf{D} : \mathcal{A} \to \mathcal{X}$ and $\mathsf{E} : \mathcal{X} \to \mathcal{A}$ the functors defined by the duality of Theorem 2.1.

Theorem 2.1 provides a tool to obtain a representation of any algebra of $\mathcal{A}$ as a concrete algebra of morphisms. But we could already obtain such a representation with the duality between $\mathcal{D}$ and $\mathcal{E}$. So, the advantage of this result must be found elsewhere.

In fact, the best way to derive benefit from Theorem 2.1 is to consider it as a step toward the construction of a strong duality for $\mathcal{A}$ (and so a dual equivalence for the category $\mathcal{A}$). Indeed, the category $\mathcal{X}$ that we have defined can be seen as the category of structures of $\mathcal{E}$ that have nothing to share with algebras of $\mathcal{D} \setminus \mathcal{A}$. It means exactly that the dual $\mathsf{E}(\underset{\sim}{X})$ of any structure $\underset{\sim}{X}$ of $\mathcal{X}$ is an algebra of $\mathcal{A}$ while the dual $\mathsf{K}(\underset{\sim}{Y})$ of a structure $\underset{\sim}{Y}$ of $\mathcal{E}$ is not necessarily an algebra of $\mathcal{A}$.

Of course, we would be very lucky if the duality of Theorem 2.1 were a strong duality (even when the duality between $\mathcal{D}$ and $\mathcal{E}$ is strong). But in the next section, we prove that it can happen.

2.3. When $\underline{D}$ is a semi-primal algebra

Expression of the duality

When $\underline{D}$ is a semi-primal algebra, it is possible to simplify the expression of the dual $\mathsf{D}(\underline{A})$ of an algebra $\underline{A}$ of $\mathcal{A}$ (which proves in this case to be a variety), similarly as in the uni-sorted case (see [4] for example). Moreover, this duality turns out to be a strong duality. Finally, we can obtain an axiomatization of the dual class of $\mathcal{A}$.

For the sake of convenience, we suppose that the language of the algebras contains at least two constants (as a consequence, there is no one-element subalgebra of $\underline{D}$). The duality that we consider for $\mathcal{D}$ is the strong duality of Theorem 3.14 of Chapter 3 in [4].

Recall that we denote by $\mathcal{P}_n$ and $\mathcal{B}_n$ the set of the n-ary (partial) operations and relations respectively which are algebraic on $\underline{\Pi}$. Let us also recall that if $\underline{F}$ is a finite algebra, we define $\mathrm{irr}(\underline{F})$ as the least n such that the zero congruence on $\underline{F}$ is a meet of n meet-irreducible non zero congruences. The irreducibility index $\mathrm{Irr}(\underline{P})$ of a finite algebra $\underline{P}$ is the maximum of the $\mathrm{irr}(\underline{F})$ where $\underline{F}$ is a subalgebra of $\underline{P}$. The irreducibility index $\mathrm{Irr}(\underline{\Pi})$ of the set $\underline{\Pi} = \{\underline{P}_1, \ldots, \underline{P}_r\}$ is $\mathrm{Irr}(\underline{\Pi}) = \max_{1 \leq i \leq r} \mathrm{Irr}(\underline{P}_i)$.

Proposition 2.2. *If $\underline{\Pi} = \{\underline{P}_1, \ldots, \underline{P}_r\}$ is a set of subalgebras of a semi-primal algebra $\underline{D}$ and if $\underset{\sim}{\Pi}$ denotes the structure*

$$\underset{\sim}{\Pi} =< \dot{\bigcup_{i \in \{1,\ldots,r\}}} P_i; \{f_{ij} \mid i,j \in \{1,\ldots,r\}\}, \bigcup_{i=1}^{r} \mathbb{S}(\underline{P}_i); \tau >,$$

where

- *for every i in $\{1,\ldots,r\}$, $\mathbb{S}(\underline{P}_i)$ denotes the set of subalgebras of $\underline{P}_i$ (viewed as unary relations);*
- *for every i and j in $\{1,\ldots,r\}$, the partial operation $f_{ij} : P_i \cap P_j \subseteq P_j \to P_i$ is defined by $f_{ij}(q) = q$ (see (2.2) for details);*
- *τ is the discrete topology;*

then $\underset{\sim}{\Pi}$ generates a strong natural duality on $\mathcal{A}$.

Proof. It is an application of the *Multi Sorted N.U. Strong Duality Theorem* (see Theorem 1.2 of Chapter 7 in [4]). Indeed, the algebras $\underline{P}_i$ ($i \in \{1,\ldots,r\}$) have a common Pixley term, and so a common ternary near-unanimity term. Furthermore, since $\underset{\sim}{\Pi}$ generates a natural duality on $\mathcal{A}$ (it is the content of Theorem 2.1 in this more restricted context of semi-primal algebras), this structure entails all the finitary algebraic relations on $\underset{\sim}{\Pi}$ and in particular the relations of $\mathcal{B}_2$. Finally, since the algebras of $\underline{\Pi}$ are hereditarily simple, we have $\mathrm{Irr}(\underline{\Pi}) = 1$. It then follows that $\mathcal{P}_1 = \bigcup_{1 \leq i,j \leq r} \{f \in \mathcal{A}(\underline{B}, \underline{P}_j) \mid \underline{B} \in \mathbb{S}(\underline{P}_i)\}$. So, if i and j are in $\{1, \dots, r\}$, if $\underline{B}$ is an element of $\mathbb{S}(\underline{P}_i)$ and if f is a homomorphism of $\mathcal{A}(\underline{B}, \underline{P}_j)$ then we can deduce that $\underline{B}$ is an algebra of $\mathbb{S}(\underline{P}_i \cap \underline{P}_j)$ and that $f = f_{ji}|_{\underline{B}}$. We can thus conclude that $\mathcal{B}_2 \cup \mathcal{P}_1$ is strongly entailed by $\{f_{ij} \mid 1 \leq i, j \leq r\} \cup \bigcup_{i=1}^{r} \mathbb{S}(\underline{P}_i)$. □

As in the previous section, we denote by $\mathsf{D} : \mathcal{A} \to \mathcal{X}$ and $\mathsf{E} : \mathcal{X} \to \mathcal{A}$ the canonical functors defined by this strong duality (where $\mathcal{X} = \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi})$).

Axiomatization of the dual class

Our next task is to use this strong duality to obtain an axiomatization of the class $\mathsf{D}(\mathcal{A}) = \mathcal{X}$. First, we study the properties of the members of $\mathcal{X}$.

Proposition 2.3. *For any algebra $\underline{A}$ of $\mathcal{A}$, any element i of $\{1, \dots, r\}$ and any subalgebras $\underline{F}$ and $\underline{F}'$ of $\underline{P}_i$,*

1. *the interpretation $\underline{F}^{\mathsf{D}(\underline{A})_i}$ of $\underline{F}$ on the i^{th} summand of $\mathsf{D}(\underline{A})$ is a closed subspace of $\mathcal{A}(\underline{A}, \underline{P}_i)$;*
2. *$\underline{P}_i^{\mathsf{D}(\underline{A})_i} = \mathcal{A}(\underline{A}, \underline{P}_i)$;*
3. *$(\underline{F} \cap \underline{F}')^{\mathsf{D}(\underline{A})_i} = \underline{F}^{\mathsf{D}(\underline{A})_i} \cap \underline{F}'^{\mathsf{D}(\underline{A})_i}$;*

Proof. The proof is straightforward. □

The next proposition gives the properties of the partial operations f_{ij}. Once more, these are direct consequences of the definitions.

Proposition 2.4. *If $\underline{A}$ is a member of $\mathcal{A}$, if i, j and k are elements of $\{1, \dots, r\}$ and if $\underline{F}$ is a subalgebra of $\underline{A}$, then*

1. *the composition $f_{kj}^{\mathsf{D}(\underline{A})} \circ f_{ji}^{\mathsf{D}(\underline{A})}$ is equal to $f_{ki}^{\mathsf{D}(\underline{A})}$ on $(\underline{P}_i \cap \underline{P}_j \cap \underline{P}_k)^{\mathsf{D}(\underline{A})_i}$ and $f_{ii}^{\mathsf{D}(\underline{A})} = \mathrm{id}|_{\mathcal{A}(\underline{A}, \underline{P}_i)}$;*
2. *the partial operation $f_{ji}^{\mathsf{D}(\underline{A})}$ is a homeomorphism from $(\underline{P}_i \cap \underline{P}_j)^{\mathsf{D}(\underline{A})_i}$ to $(\underline{P}_i \cap \underline{P}_j)^{\mathsf{D}(\underline{A})_j}$;*
3. *if x is an element of $\underline{F}^{\mathsf{D}(\underline{A})_i}$, then $f_{ji}^{\mathsf{D}(\underline{A})}(x)$ is an element of $\underline{F}^{\mathsf{D}(\underline{A})_j}$.*

□

In fact, these properties are exactly the ones required on a $\underset{\sim}{\Pi}$-indexed structure of the same type as $\underset{\sim}{\Pi}$ to be a member of $\mathcal{X}$. Thus, the following proposition is the counterpart to the well-known axiomatization of the dual class of a variety generated by a single semi-primal algebra (see Proposition 3.14 in Chapter 3 of [4]).

Theorem 2.5. *If $\underset{\sim}{\Pi}$ denotes the structure defined in Proposition 2.2, then a topological structure $\underset{\sim}{X}$ is a member of $\mathcal{X} = \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi})$ if and only if*

$$\underset{\sim}{X} =< \dot{\bigcup_{1\leq i\leq r}} X_i; \{f_{ij}^{\underset{\sim}{X}} \mid 1 \leq i,j \leq r\}, \bigcup_{i=1}^{r}\{r_{\underline{F}}^{\underset{\sim}{X_i}} \mid \underline{F} \in \mathbb{S}(\underline{P}_i)\}; \tau >,$$

where for any i and j in $\{1,\ldots,r\}$ and any subalgebras $\underline{F}$ and $\underline{F}'$ of $\underline{P}_i$,

1. *τ is the disjoint union of Boolean topologies τ_i on each X_i;*
2. *the set $r_{\underline{F}}^{\underset{\sim}{X_i}}$ is a closed subspace of X_i and*
   - $r_{\underline{P}_i}^{\underset{\sim}{X_i}} = X_i$
   - $r_{\underline{F}}^{\underset{\sim}{X_i}} \cap r_{\underline{F}'}^{\underset{\sim}{X_i}} = r_{\underline{F}\cap\underline{F}'}^{\underset{\sim}{X_i}}$;
3. *the partial operation $f_{ji}^{\underset{\sim}{X}}$ is a homeomorphism from $r_{\underline{P}_i\cap\underline{P}_j}^{\underset{\sim}{X_i}}$ to $r_{\underline{P}_i\cap\underline{P}_j}^{\underset{\sim}{X_j}}$ such that*
   - $f_{kj}^{\underset{\sim}{X}} \circ f_{ji}^{\underset{\sim}{X}} = f_{ki}^{\underset{\sim}{X}}$ *for every $k \in \{1,\ldots,r\}$*
   - $f_{ii}^{\underset{\sim}{X}} = \mathrm{id}|_{X_i}$
   - *if $x \in r_{\underline{F}\cap\underline{P}_j}^{\underset{\sim}{X_i}}$ then $f_{ji}^{\underset{\sim}{X}}(x) \in r_{\underline{F}}^{\underset{\sim}{X_j}}$.*

Proof. The necessity of the condition is the content of Propositions 2.3 and 2.4.

Now, let us consider the equivalence Θ whose elements are the $(x_i,x_j) \in X_i \times X_j$ (where $i,j \in \{1,\ldots,r\}$) such that $x_i \in r_{\underline{P}_i\cap\underline{P}_j}^{\underset{\sim}{X_i}}$ and $x_j \in r_{\underline{P}_j\cap\underline{P}_i}^{\underset{\sim}{X_j}}$ with $f_{ji}^{\underset{\sim}{X}}(x_i) = x_j$. We are about to define a subset $r_{\underline{F}}^{X/\Theta}$ of X/Θ for every subalgebra $\underline{F}$ of $\underline{D}$ in such a way that

$$\langle X/\Theta, \{r_{\underline{F}}^{X/\Theta} \mid \underline{F} \in \mathbb{S}(\underline{D})\}; \tau\rangle,$$

(where τ is the quotient topology) is a member $\mathsf{H}(\underline{A})$ of $\mathcal{E}$ (recall that the duality between $\mathcal{D}$ and $\mathcal{E}$ is strong since $\underline{D}$ is a semi-primal algebra) with $\mathsf{D}(\underline{A}) \cong \underset{\sim}{X}$.

For every $\underline{F}$ in $\mathbb{S}(\underline{D})$ we define $r_{\underline{F}}^{X/\Theta}$ as the set that collects the classes x^Θ such that x is in $\cup_{i\in\{1,\dots r\}} r_{\underline{F}}^{\underset{\sim}{X}_i}$.

With this definition, the only non trivial part consists in proving that X/Θ is a Boolean space. Let us consider two points $x, y \in X$ and suppose that they are not equivalent. We will show that there exists a saturated clopen set Ω of X such that $x \in \Omega$ and $y \notin \Omega$. We may suppose that $x \in X_1$, and say $y \in X_j$.
For any $k \in \{1, \dots, r\}$ such that $y \in r_{\underline{P}_j\cap\underline{P}_k}^{\underset{\sim}{X}_j}$, we set $y_k = f_{kj}^{\underset{\sim}{X}}(y)$.

Now, we will build by induction clopen sets Ω_k of X_k ($k \in \{1, \dots, r\}$) such that

1. $x \in \Omega_1$
2. $\forall k \in \{1, \dots, r\}, y_k \notin \Omega_k$
3. $\forall k, l \in \{1, \dots, r\}, f_{kl}^{\underset{\sim}{X}}(\Omega_l \cap r_{\underline{P}_k\cap\underline{P}_l}^{\underset{\sim}{X}_l}) = \Omega_k \cap r_{\underline{P}_k\cap\underline{P}_l}^{\underset{\sim}{X}_k}$.

Then the set $\Omega = \cup_{1\leq k\leq r}\Omega_k$ will allow us to conclude the proof.

The existence of Ω_1 is obvious since X_1 is a Boolean space and $x \neq y_1$. Now, let us suppose that we have constructed clopen sets $\Omega_1, \dots, \Omega_{i-1}$ ($i \leq r$) fulfilling conditions (1), (2) and (3) and let us show how to define Ω_i. For any $k < i$ we set

$$\omega_{i,k} = f_{ik}^{\underset{\sim}{X}}(\Omega_k \cap r_{\underline{P}_k\cap\underline{P}_i}^{\underset{\sim}{X}_k})$$

and $\omega = \cup_{1\leq k<i}\,\omega_{i,k}$. In view of condition (2), the set ω does not contain y_i. It is also a clopen subset of $\cup_{1\leq k<i}\, r_{\underline{P}_k\cap\underline{P}_i}^{\underset{\sim}{X}_i}$. Indeed, on the one hand, for $k < i$, $\omega_{i,k}$ is a closed subset of $r_{\underline{P}_k\cap\underline{P}_i}^{\underset{\sim}{X}_i}$ and therefore of X_i. On the other hand, since

$$f_{ki}^{\underset{\sim}{X}}(\omega_{i,l} \cap r_{\underline{P}_k\cap\underline{P}_i}^{\underset{\sim}{X}_i}) \subset f_{ki}^{\underset{\sim}{X}} \circ f_{il}^{\underset{\sim}{X}}(\Omega_l \cap r_{\underline{P}_i\cap\underline{P}_k\cap\underline{P}_l}^{\underset{\sim}{X}_l}) \subset \Omega_k \cap r_{\underline{P}_i\cap\underline{P}_k}^{\underset{\sim}{X}_k}$$

holds for any $k, l < i$, we have

$$\omega_{i,l} \cap r_{\underline{P}_k\cap\underline{P}_i}^{\underset{\sim}{X}_i} \subset \omega_{i,k},$$

and then

$$(\cup_{k<i} r_{\underline{P}_k\cap\underline{P}_i}^{\underset{\sim}{X}_i}) \setminus (\cup_{k<i}\omega_{i,k}) = \cup_{k<i}(r_{\underline{P}_k\cap\underline{P}_i}^{\underset{\sim}{X}_i} \setminus \omega_{i,k}).$$

This set is open since $\omega_{i,k}$ is closed in $r_{\underline{P}_k\cap\underline{P}_i}^{\underset{\sim}{X}_i}$.

Then, there exists a clopen set Ω_i of X_i such that

$$\Omega_i \cap (\bigcup_{k<i} r_{\underline{P}_k\cap\underline{P}_i}^{\underset{\sim}{X}_i}) = \omega,$$

so that for any $k < i$,

$$\Omega_i \cap r^{\underset{\sim}{X}_i}_{\underline{P}_k \cap \underline{P}_i} = \omega \cap r^{\underset{\sim}{X}_i}_{\underline{P}_k \cap \underline{P}_i} = \omega_{i,k} = f_{ik}(\Omega_k \cap r^{\underset{\sim}{X}_k}_{\underline{P}_k \cap \underline{P}_i}).$$

Eventually, we can require in addition that $y_i \notin \Omega_i$ since $y_i \notin \omega$. □

3. Algebraic aspects of the duality

In this section, we gather algebraic consequences of the duality between $\mathcal{A}$ and $\mathcal{X}$.

3.1. General Results

We do not assume initially the semi-primality of $\underline{D}$. Our first result shows that if the duality between $\mathcal{D}$ and $\mathcal{E}$ is strong, then there is a natural bijection between onto $\mathcal{A}$-morphisms and $\mathcal{X}$-embeddings.

Proposition 3.1. *Suppose that the duality between $\mathcal{D}$ and $\mathcal{E}$ is strong and consider two algebras $\underline{A}$ and $\underline{B}$ of $\mathcal{A}$. If $\psi : \mathsf{D}(\underline{B}) \to \mathsf{D}(\underline{A})$ is an embedding, then $\mathsf{E}(\psi) : \mathsf{ED}(\underline{A}) \to \mathsf{ED}(\underline{B})$ is an onto $\mathcal{A}$-morphism. Conversely, if $u : \underline{A} \to \underline{B}$ is an onto $\mathcal{A}$-morphism then $\mathsf{D}(u) : \mathsf{D}(\underline{B}) \to \mathsf{D}(\underline{A})$ is an $\mathcal{X}$-embedding.*

Proof. For the first part of the assertion, it suffices to prove that if $\psi : \mathsf{D}(\underline{B}) \to \mathsf{D}(\underline{A})$ is an embedding, then the map $\psi' : \mathsf{H}(\underline{B}) \to \mathsf{H}(\underline{A}) : x \mapsto \psi(x)$ is a well defined $\mathcal{E}$-embedding and satisfies $\mathsf{E}(\psi) = \mathsf{K}(\psi')$ (up to the canonical isomorphism $f : \mathsf{ED}(\underline{A}) \to \mathsf{KH}(\underline{A})$ defined by $f(\alpha) : x \mapsto \alpha(x)$). Indeed, the map $\mathsf{K}(\psi')$ is an onto homomorphism according to Lemma 2.6 of Chapter 3 in [4].

Conversely, its suffices to note that the map $\mathsf{D}(u) : \mathsf{D}(\underline{B}) \to \mathsf{D}(\underline{A})$ is in fact defined by $(\mathsf{D}(u))(x) = (\mathsf{H}(u))(x)$. The result follows from the fact that $\mathsf{H}(u)$ is an $\mathcal{E}$-embedding following the previously cited Lemma. □

The connection between $\mathcal{A}$-embeddings and onto $\mathcal{X}$-morphisms is not so perfect, since the dual of an $\mathcal{X}$-morphism is an $\mathcal{A}$-embedding but not conversely. Moreover, this result requires more restrictive hypotheses.

Proposition 3.2. *Suppose that the duality between $\mathcal{D}$ and $\mathcal{E}$ is strong and that $\underline{D}$ is injective in $\mathcal{D}$. Consider two algebras $\underline{A}$ and $\underline{B}$ of $\mathcal{A}$. If $\psi : \mathsf{D}(\underline{B}) \to \mathsf{D}(\underline{A})$ is an onto $\mathcal{X}$-morphism, then $\mathsf{E}(\psi) : \mathsf{ED}(\underline{A}) \to \mathsf{ED}(\underline{B})$ is an $\mathcal{A}$-embedding.*

Proof. Follow the idea of the proof of Proposition 3.1 and apply Lemma 2.8 of Chapter 3 in [4]. □

Example 4.2 shows that the converse of the previous proposition does not hold.

3.2. Results involving the semi-primality of $\underline{D}$

Much more consequences can be derived from the duality between $\mathcal{A}$ and $\mathcal{X}$ when $\underline{D}$ is a semi-primal algebra.

Our first result is a description of finite dual spaces. To obtain it, we need to introduce some new unary relations.

Definition 3.3. For each member $\underline{B}$ of $\mathcal{D}$ and each subalgebra $\underline{F}$ of $\underline{D}$, the unary relation $s_{\underline{F}}^{\mathsf{H}(\underline{B})}$ is defined by

$$s_{\underline{F}}^{\mathsf{H}(\underline{B})} = \underline{F}^{\mathsf{H}(\underline{B})} \setminus \bigcup_{\underline{F}' \in \mathbb{S}(\underline{F}) \setminus \{\underline{F}\}} \underline{F}'^{\mathsf{H}(\underline{B})}.$$

Similarly, for each algebra $\underline{A}$ of $\mathcal{A}$, each i in $\{1, \dots, r\}$ and each subalgebra $\underline{F}$ of $\underline{P}_i$, we define the unary relation $s_{\underline{F}}^{\mathsf{D}(\underline{A})_i}$ on the i^{th} summand of $\mathsf{D}(\underline{A})$ by

$$s_{\underline{F}}^{\mathsf{D}(\underline{A})_i} = \underline{F}^{\mathsf{D}(\underline{A})_i} \setminus \bigcup_{\underline{F}' \in \mathbb{S}(\underline{F}) \setminus \{\underline{F}\}} \underline{F}'^{\mathsf{D}(\underline{A})_i}.$$

If $\underline{F}$ is a subalgebra of $\underline{D}$ and if $\underline{B}$ is a member of $\mathcal{D}$, then

$$r_{\underline{F}}^{\mathsf{H}(\underline{B})} = \dot{\bigcup_{\underline{F}' \in \mathbb{S}(\underline{F})}} s_{\underline{F}'}^{\mathsf{H}(\underline{B})},$$

and a similar result holds for the relations $r_{\underline{F}}^{\mathsf{D}(\underline{A})_i}$ (where $\underline{A} \in \mathcal{A}$, $i \in \{1, \dots, r\}$ and $\underline{F} \in \mathbb{S}(\underline{P}_i)$).

With these new relations, we can give a new expression of Proposition 2.2 in [13].

Proposition 3.4. *If* $\underline{B} \cong \prod_{\underline{F} \in \mathbb{S}(\underline{D})} \underline{F}^{f_{\underline{B}}(\underline{F})}$ *(where the* $f_{\underline{B}}(\underline{F})$ *are non-negative integers) then* $\mathsf{H}(\underline{B})$ *is a discrete topological space such that* $|s_{\underline{F}}^{\mathsf{H}(\underline{B})}| = f_{\underline{B}}(\underline{F})$ *for every* $\underline{F}$ *in* $\mathbb{S}(\underline{D})$*. Conversely, if* $\underset{\sim}{Y}$ *is a finite (discrete) member of* $\mathcal{E}$*, then* $\mathsf{E}(\underset{\sim}{Y}) \cong \prod_{\underline{F} \in \mathbb{S}(\underline{D})} \underline{F}^{|s_{\underline{F}}^{\underset{\sim}{Y}}|}$. □

With the help of this proposition we can describe the dual of the finite members of $\mathcal{A}$.

Proposition 3.5. *If* $\underline{A} = \prod_{\underline{F} \in \bigcup_{1 \le i \le r} \mathbb{S}(\underline{P}_i)} \underline{F}^{f_{\underline{A}}(\underline{F})}$ *(where the* $f_{\underline{A}}(\underline{F})$ *are non negative integers) is a finite member of* $\mathcal{A}$*, then* $\mathsf{D}(\underline{A})$ *is a finite member of* $\mathcal{X}$ *such that for every* $i \in \{1, \dots, r\}$ *and* $\underline{F} \in \mathbb{S}(\underline{P}_i)$

$$|\mathsf{D}(\underline{A})_i| = \textstyle\sum_{\underline{F} \in \mathbb{S}(\underline{P}_i)} f_{\underline{A}}(\underline{F}) \quad \text{and} \quad |s_{\underline{F}}^{\mathsf{D}(\underline{A})_i}| = f_{\underline{A}}(\underline{F}).$$

Conversely, if $\underset{\sim}{X}$ is a finite (thus discrete) member of $\mathcal{X}$, then

$$\mathsf{E}(\underset{\sim}{X}) = \prod_{\underline{F} \in \bigcup_{i \in \{1,\ldots,r\}} \mathbb{S}(\underline{P}_i)} \underline{F}^{|s_{\underline{F}}^{\underset{\sim}{X}_i}|}.$$

Proof. By the previous proposition, we know that $\mathsf{H}(\underline{A})$ is a discrete space such that

$$|s_{\underline{F}}^{\mathsf{H}(\underline{A})}| = \begin{cases} f_{\underline{A}}(\underline{F}) \text{ if } \underline{F} \in \bigcup_{1 \leq i \leq r} \mathbb{S}(\underline{P}_i) \\ 0 \text{ if } \underline{F} \in \mathbb{S}(D) \setminus \bigcup_{1 \leq i \leq r} \mathbb{S}(\underline{P}_i). \end{cases}$$

It follows that

$$|\mathsf{D}(\underline{A})_i| = |\underline{P}_i^{\mathsf{H}(\underline{A})}| = \sum_{\underline{F} \in \mathbb{S}(\underline{P}_i)} |s_{\underline{F}}^{\mathsf{H}(\underline{A})}| = \sum_{\underline{F} \in \mathbb{S}(\underline{P}_i)} f_{\underline{A}}(\underline{F}).$$

We get the first part of the proposition if we note that $|s_{\underline{F}}^{\mathsf{D}(\underline{A})_i}| = |s_{\underline{F}}^{\mathsf{H}(\underline{A})}|$ for every $i \in \{1, \ldots, r\}$ and $\underline{F} \in \mathbb{S}(\underline{P}_i)$.

Conversely, if $\underset{\sim}{X}$ is a finite member of $\mathcal{X}$ and if X/Θ is the $\mathcal{E}$-structure defined in the proof of Theorem 2.5, we see that

$$\mathsf{K}(X/\Theta) \cong \prod_{\underline{F} \in \bigcup_{1 \leq i \leq r} \mathbb{S}(\underline{P}_i)} \underline{F}^{|s_{\underline{F}}^{X/\Theta}|},$$

since $|s_{\underline{F}}^{X/\Theta}| = 0$ if $\underline{F} \in \mathbb{S}(\underline{D}) \setminus \bigcup_{1 \leq i \leq r} \mathbb{S}(\underline{P}_i)$. Then,

$$\mathsf{E}(\underset{\sim}{X}) \cong \mathsf{K}(X/\Theta) \cong \prod_{\underline{F} \in \bigcup_{1 \leq i \leq r} \mathbb{S}(\underline{P}_i)} \underline{F}^{|s_{\underline{F}}^{X/\Theta}|} \cong \prod_{\underline{F} \in \bigcup_{1 \leq i \leq r} \mathbb{S}(\underline{P}_i)} \underline{F}^{|s_{\underline{F}}^{X_i}|},$$

and the conclusion follows. □

Our next purpose is to construct the finitely generated free algebras over $\mathcal{A}$. We first recall the definition of the classical MÖBIUS*' function* of lattice theory.

Definition 3.6. If O is a finite partially ordered set, the MÖBIUS*' function* associated to O is the function μ_O defined on $O \times O$ by

$$\mu_O(x, y) = \begin{cases} 0 & \text{if } x \not\leq y \\ 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu_O(x, z) & \text{if } x < y. \end{cases}$$

We denote by $\mathbf{Sub}(\underline{D})$ the lattice of the subalgebras of $\underline{D}$. Let us recall that we denote by $\mathcal{F}_{\mathcal{A}}(k)$ the free algebra with k generators over the class $\mathcal{A}$.

Proposition 3.7. *For every positive integer k, we have*

$$\mathcal{F}_{\mathcal{A}}(k) \cong \prod_{\underline{F} \in \bigcup_{1 \leq i \leq r} \mathbb{S}(\underline{P}_i)} \underline{F}^{f_{\mathcal{A}}(k, \underline{F})},$$

where

$$f_{\mathcal{A}}(k, \underline{F}) = \sum_{\underline{F}' \in \mathbb{S}(\underline{F})} \mu_{\mathbf{Sub}(\underline{D})}(\underline{F}', \underline{F}).|F'|^k.$$

Proof. Since, by Proposition 3.5,

$$\mathcal{F}_{\mathcal{A}}(k) \cong \mathsf{E}(\underset{\sim}{\Pi}^k) \cong \prod_{\underline{F} \in \bigcup_{1 \leq i \leq r} \mathbb{S}(\underline{P}_i)} \underline{F}^{|s_{\underline{F}}^{P_i^k}|},$$

we just have to prove that $|s_{\underline{F}}^{P_i^k}|$ is equal to the proposed $f_{\mathcal{A}}(k, \underline{F})$ for every subalgebra $\underline{F}$ of $\underline{P}_i$. An application of the MÖBIUS' inversion formula to

$$|\underline{F}|^k = |\underline{F}^{P_i^k}| = \sum_{\underline{F}' \in \mathbb{S}(\underline{F})} |s_{\underline{F}'}^{P_i^k}|$$

allows us to give the desired value for each of the $f_{\mathcal{A}}(k, \underline{F})$. □

Our last application is a characterization of finite projective algebras in $\mathcal{A}$. In the following proposition, we denote by $\underline{F}_0$ the algebra $\bigcap_{i \in \{1,\ldots,r\}} \underline{P}_i$ (this intersection is not empty since we have assumed that the language $\mathcal{L}$ of the algebras contains at least two constants).

Proposition 3.8. *If $\underline{A}$ is a member of $\mathcal{A}$, the following conditions are equivalent:*

1. *$\underline{A}$ is a finite projective member of $\mathcal{A}$;*
2. *there is an element i of $\{1, \ldots, r\}$ such that $r_{\underline{F}_0}^{\mathsf{D}(\underline{A})_i}$ is not empty;*
3. *there is a finite member $\underline{B}$ of $\mathcal{A}$ such that $\underline{A}$ is isomorphic to $\underline{F}_0 \times \underline{B}$.*

Proof. Since the duality between $\mathcal{A}$ and $\mathcal{X}$ is strong, a finite algebra $\underline{A}$ of $\mathcal{A}$ is projective in $\mathcal{A}$ if and only if $\mathsf{D}(\underline{A})$ is a finite injective member of $\mathcal{X}$.

Suppose that t_1 belongs to $r_{\underline{F}_0}^{\mathsf{D}(\underline{A})_1}$ and denote by t_i the element $f_{i1}(t_1)$ of $\mathsf{D}(\underline{A})_i$ for every i in $\{2, \ldots, r\}$. If $\psi : \underset{\sim}{Y} \to \underset{\sim}{Z}$ is an $\mathcal{X}$-embedding between two members $\underset{\sim}{Y}$ and $\underset{\sim}{Z}$ of $\mathcal{X}$ and if $\phi : \underset{\sim}{Y} \to \mathsf{D}(\underline{A})$ is an $\mathcal{X}$-morphism, we know that there is a continuous application $\theta : Z \to \mathsf{D}(\underline{A})$ respecting the $\underline{\Pi}$-indexed structure such that $\theta \circ \psi = \phi$ (this is a consequence of the injectivity of finite Boolean spaces in the class of Boolean spaces). As ϕ is an $\mathcal{X}$-morphism, we thus know that $\theta|_{\psi(Y)}$ is also an $\mathcal{X}$-morphism.

Now, for each i in $\{1,\dots,r\}$ and each x in $\mathsf{D}(\underline{A})_i$, denote by $\underline{F}_x$ the unique subalgebra of $\underline{P}_i$ such that x belongs to $s_{\underline{F}_x}^{\mathsf{D}(\underline{A})_i}$. Let us also define the closed subspace r_x of $\underset{\sim}{Z}_i$ by

$$r_x = \theta^{-1}(x) \cap \\ \cap \{ (\bigcup_{\substack{\underline{F}\in\mathbb{S}(\underline{P}_i)\\ \underline{F}\not\geq\underline{F}_x}} r_{\underline{F}}^{\widetilde{Z_i}}) \cup (\bigcup_{\substack{j\in\{1,\dots,r\}\\ i\neq j}} \bigcup_{\substack{z\in r^{\mathsf{D}(\underline{A})_j}_{\underline{P}_i\cap\underline{P}_j}\\ z\neq f_{ji}(x)}} \{ y \in r_{\underline{P}_i\cap\underline{P}_j}^{\widetilde{Z_i}} \mid \theta(f_{ji}(y)) = z \}) \}.$$

Denote by r_i the subspace $\bigcup_{x\in\mathsf{D}(\underline{A})_i} r_x$ and by r the subspace $\bigcup_{i\in\{1,\dots,r\}} r_i$. These are closed in Z_i and Z respectively. Since $\theta\mid_{\psi(Y)}$ is an $\mathcal{X}$-morphism, the intersection $\psi(Y)\cap r$ is empty. So, for every i in $\{1,\dots,r\}$, we can find an open subset ω_i of Z_i which contains r_i but no element of $\psi(Y)_i$. Thus, the map

$$\lambda : Z \to \mathsf{D}(\underline{A}) : x \in Z_i \mapsto \begin{cases} t_i & \text{if } x \in \omega_i \\ \theta(x) & \text{if } x \notin \omega_i \end{cases}$$

is an $\mathcal{X}$-morphism such that $\psi\circ\theta = \phi$.

Conversely, let us consider $\underset{\sim}{Y} = \mathsf{D}(\underline{P}_1 \times \cdots \times \underline{P}_r)$ and $\underset{\sim}{Z} = \mathsf{D}(\underline{F}_0 \times \underline{P}_1 \times \cdots \times \underline{P}_r)$. According to Proposition 3.5, the sets Y_i and Z_i are discrete spaces with $Y_i = s_{\underline{P}_i} = \{y_i\}$ and $Z_i = s_{\underline{F}_0} \cup s_{\underline{P}_i} = \{f_i\}\cup\{z_i\}$ for all i in $\{1,\dots,r\}$. If $\mathsf{D}(\underline{A})$ is a finite injective member of $\mathcal{X}$, if ψ denotes the unique $\mathcal{X}$-morphism from Y to Z and if $\phi : Y \to \mathsf{D}(\underline{A})$ is an $\mathcal{X}$-morphism, there exists an $\mathcal{X}$-morphism $\theta : Z \to \mathsf{D}(\underline{A})$ such that $\theta\circ\psi = \phi$. It follows that $\theta(f_i)$ belongs to $\underline{F}_0^{\mathsf{D}(\underline{A})_i}$.

The equivalence between (2) and (3) is a consequence of Proposition 3.5 □

4. Natural dualities on finitely generated varieties of MV-algebras

In this section we illustrate the previous developments by giving an application to the finitely generated varieties of MV-algebras. In this way, we finish the work begun in [13].

MV-algebras were introduced in 1958 by Chang (see [1] and [2]) as a many-valued counterpart of Boolean algebras. Their study in a logical and algebraic aspect led to numerous interesting results, as for instance an algebraic proof of the completeness theorem of Łukasiewicz's infinite-valued propositional calculus (see [2]).

An MV-algebra can be viewed as an algebra $\underline{A} = \langle A; \oplus, \odot, \neg, 0, 1\rangle$ of type (2,2,1,0,0) such that $\langle A; \oplus, 0\rangle$ is an Abelian monoid and that satisfies the following identities: $\neg\neg x = x$, $x \oplus 1 = 1$, $\neg 0 = 1$, $x \odot y = \neg(\neg x \oplus \neg y)$, $(x \odot \neg y) \oplus y = (y \odot \neg x) \oplus x$.

One of the most simple (and most important) example of MV-algebra is the real interval $[0, 1]$ endowed with the operations $x \oplus y = \min(x+y, 1)$, $x \odot y = \max(x + y - 1, 0)$ and $\neg x = 1 - x$.

KOMORI's classification of the subvarieties of the variety $\mathcal{M}$ of MV-algebras (see [12]) underlines the importance of the subalgebras $\underline{Ł}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ (where n is a positive integer) of $[0, 1]$. Indeed, the finitely generated subvarieties of $\mathcal{M}$ are exactly the ones generated by a finite number of $Ł_n$. For example, the variety of MV-algebras that satisfies the additional equation $(m + 1).x = m.x$ for an integer $m \geq 1$ is exactly the variety generated by $\underline{Ł}_1, \ldots, \underline{Ł}_m$.

We thus now apply our results to the finitely generated varieties of MV-algebras.

From now on, we denote by r an element of $\mathbb{N} \setminus \{1\}$ and by $n_1, \ldots, n_r$ some positive integers such that n_i does not divide n_j for every $i \neq j$ in $\{1, \ldots, r\}$. The set $\underline{\Pi}$ is the set of the algebras $\underline{Ł}_{n_1}, \ldots, \underline{Ł}_{n_r}$ and m is the lowest common multiple to $n_1, \ldots, n_r$. The class $\mathcal{A}$ is the variety $\mathbb{ISP}(\underline{\Pi})$ and $\mathcal{D}$ is the variety $\mathbb{ISP}(\underline{Ł}_m)$ (on which a natural duality was developed in [13]). We also make use of the previously defined notations concerning functors and duality.

The duality is obtained with the help of the results of section 2.3. In the sequel of the paper, we denote by $\mathrm{div}(n)$ the set of the positive divisors of the integer n.

Proposition 4.1. *Let us define the structure*

$$\underset{\sim}{\Pi} = < \dot{\bigcup_{1 \leq i \leq r}} Ł_{n_i}; \{f_{ji} \mid 1 \leq i, j \leq r\}, \bigcup_{i=1}^{r} \{Ł_k \mid k \in \mathrm{div}(n_i)\}; \tau >,$$

where τ is the discrete topology and where for every i, j in $\{1, \ldots, r\}$,

$$f_{ji} : Ł_{\gcd(n_i, n_j)} \subset Ł_{n_i} \to Ł_{n_j} : q \mapsto q.$$

Then $\underset{\sim}{\Pi}$ generates a strong duality on $\mathcal{A}$. □

The results of Proposition 3.1 and 3.2 apply here. We have announced a counterexample for the converse of Proposition 3.2. We are now able to produce it.

Example 4.2. Let us set $\underline{\Pi} = \{\underline{Ł}_6, \underline{Ł}_{10}\}$ and $\underline{D} = \underline{Ł}_{30}$. Consider the two algebras $\underline{A} = \underline{Ł}_2 \times \underline{Ł}_2$ and $\underline{B} = \underline{Ł}_6 \times \underline{Ł}_{10}$ of $\mathcal{A} = \mathbb{ISP}(\underline{\Pi})$. With the help of Proposition 3.5, we obtain on the one hand that the dual spaces $\mathsf{H}(\underline{A})$ and $\mathsf{H}(\underline{B})$ are two discrete spaces with two elements. On the other hand, each of the two summands of $\mathsf{D}(\underline{A})$ is made of two elements, but each summand of $\mathsf{D}(\underline{B})$ contains a single element. If we consider the dual $\mathsf{H}(p)$ of the canonical $\mathcal{A}$-embedding of $\underline{A}$ into $\underline{B}$, we obtain an onto $\mathcal{E}$-morphism, since $\underline{\text{Ł}}_{30}$ is injective in $\mathcal{D} = \mathbb{ISP}(\underline{\text{Ł}}_{30})$. On the other hand, it is clear that $\mathsf{D}(p) : \mathsf{D}(\underline{B}) \to \mathsf{D}(\underline{A})$ can not be onto.

Using Proposition 3.7 and the MÖBIUS function μ defined on $\mathbb{N}$ by

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square free} \\ (-1)^{|P(n)|} & \text{if } n \text{ is square free,} \end{cases}$$

where $P(n)$ is the set of the prime divisors of n, we can compute the free algebras with a finite number of generators. We denote by $\prod X$ the product of the elements of the finite set of integers X.

Proposition 4.3. *For every positive integer k, we have*

$$\mathcal{F}_{\mathcal{A}}(k) \cong \prod_{q \in \bigcup_{1 \leq i \leq r} \operatorname{div}(n_i)} \underline{\text{Ł}}_q^{f(k,q)},$$

where

$$f(k,q) = \sum_{X \subseteq P(q)} (-1)^{|X|} \cdot \left(\frac{q}{\prod X} + 1\right)^k.$$

Proof. By Proposition 3.5, we know that

$$f(k,q) = f_{\mathcal{A}}(k, \underline{\text{Ł}}_q) = \sum_{\underline{\text{Ł}}_s \in \mathbb{S}(\underline{\text{Ł}}_q)} \mu_{\mathbf{Sub}(\underline{\text{Ł}}_m)}(\underline{\text{Ł}}_s, \underline{\text{Ł}}_q) \cdot |\underline{\text{Ł}}_s|^k.$$

Since $\mathbf{Sub}(\underline{\text{Ł}}_q)$ is isomorphic to the lattice of divisors of q, we can write successively

$$\mu_{\mathbf{Sub}(\underline{\text{Ł}}_m)}(\underline{\text{Ł}}_s, \underline{\text{Ł}}_q) = \mu_{\operatorname{div}(q)}(s,q) = \mu_{\operatorname{div}(\frac{q}{s})}(1, \frac{q}{s}) = \mu(\frac{q}{s}),$$

and since $|\underline{\text{Ł}}_s| = s+1$, it follows that

$$f(k,q) = \sum_{s \in \operatorname{div}(q)} \mu(\frac{q}{s})(s+1)^k.$$

The integer $\frac{q}{s}$ is not square free if and only if there is a subset X of $P(q)$ such that $\frac{q}{s} = \prod X$. In this case, we have $\mu(\frac{q}{s}) = (-1)^{|X|}$ and $s = \frac{q}{\prod X}$, which allows us to draw the desired conclusion. □

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