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RESEARCH ARTICLE

Combinatorics of partial wreath power of finite inverse symmetric semigroup $\mathcal{IS}_d$

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Abstract. We study some combinatorial properties of $\wr_p^k \mathcal{IS}_d$. In particular, we calculate its order, the number of idempotents and the number of $\mathcal{D}$-classes. For a given based graph $\Gamma \subset T$ we compute the number of elements in its $\mathcal{D}$-class D_Γ and the number of $\mathcal{R}$- and $\mathcal{L}$-classes in D_Γ.

Introduction

The wreath product of semigroups has appeared as a generalization to semigroups of the corresponding construction for groups. Firstly transformation wreath product of transformation semigroups has appeared as a natural generalization of the wreath product of permutation groups [1]. Later different modifications have been introduced, for instance, partial wreath product of arbitrary semigroup and semigroup of partial transformation was defined in [2] and construction related to this one, namely inverse wreath product of inverse semigroups, was proposed in [3]. Wreath products provide means to construct a semigroup with certain properties. They also appear in certain natural settings, that allows to lighten the study of known semigroups presenting them if possible as a wreath product of appropriate semigroups

The article discusses the partial wreath product of two finite symmetric semigroup $\mathcal{IS}_d$ and a generalization of this construction to the case of more then two factors. It is proved that the partial wreath k-th power

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of the semigroup $\mathcal{IS}_d$ is isomorphic to the appropriate subsemigroup of semigroup of partial automorphisms of the rooted k-level d-regular tree. We study some combinatorial properties of $\wr_p^k \mathcal{IS}_d$, in particular, we calculate its order and the number of idempotents and the number of $\mathcal{D}$-classes. Also, we describe Green's relations of the partial wreath power of $\mathcal{IS}_d$ and calculate the number of $\mathcal{D}$-classes, the number of elements in a given $\mathcal{D}$-class and the number of $\mathcal{R}$- and $\mathcal{L}$-classes in this $\mathcal{D}$-class.

1. The partial wreath power of semigroup $\mathcal{IS}_d$

Let $\mathcal{N}_d = \{1, \ldots, d\}$. Define $S^{P\mathcal{N}_d}$ by

$$S^{P\mathcal{N}_d} = \{f : \mathcal{N}_d \to \mathcal{IS}_d |\, \mathrm{dom}(f) \subseteq \mathcal{N}_d\}.$$

as the set of functions from subsets of $\mathcal{N}_d$ to $\mathcal{IS}_d$. If $f, g \in S^{P\mathcal{N}_d}$, we define the product fg by:

$$\mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g), (fg)(x) = f(x)g(x) \text{for all } x \in \mathrm{dom}(fg).$$

If $a \in \mathcal{IS}_d, f \in S^{P\mathcal{N}_d}$, we define f^a by:

$$\mathrm{dom}(f^a) = \{x \in \mathrm{dom}(a); xa \in \mathrm{dom}(f)\} = (\mathrm{ran}(a) \cap \mathrm{dom}(f))a^{-1}$$
$$(f^a)(x) = f(xa).$$

Definition. The partial wreath square of semigroup $\mathcal{IS}_d$ is defined as the set $\{(f, a) \in S^{P\mathcal{N}_d} \times \mathcal{IS}_d \,|\, \mathrm{dom}(f) = \mathrm{dom}(a)\}$ with composition defined by

$$(f, a) \cdot (g, b) = (fg^a, ab)$$

Denote it by $\mathcal{IS}_d \wr_p \mathcal{IS}_d$.

The partial wreath square of $\mathcal{IS}_d$ is a semigroup, moreover, it is an inverse semigroup [1, Lemmas 2.22 and 4.6]. We may recursively define any partial wreath power of the finite inverse symmetric semigroup.

Definition. The partial wreath k-th power of semigroup $\mathcal{IS}_d$ is defined as semigroup $\wr_p^k \mathcal{IS}_d = \left(\wr_p^{k-1} \mathcal{IS}_d\right) \wr_p \mathcal{IS}_d = \{(f, a) \subset S_{k-1}^{P\mathcal{N}_d} \times \mathcal{IS}_d \,|\, \mathrm{dom}(f) = dom(a)\}$ with composition defined by

$$(f, a) \cdot (g, b) = (fg^a, ab),$$

where $S_{k-1}^{P\mathcal{N}_d} = \{f : \mathcal{N}_d \to \wr_p^{k-1} \mathcal{IS}_d, \mathrm{dom}(f) \subseteq \mathcal{N}_d\}$, $\wr_p^{k-1} \mathcal{IS}_d$ is the partial wreath $(k-1)$-th power of semigroup $\mathcal{IS}_d$

For an arbitrary function F we denote $F^k(x) = \underbrace{F(F \ldots (F}_{k}(x)) \ldots)$.

Proposition 1. *$|\wr_p^k \mathcal{IS}_d| = S^k(1)$, where $S(x) = \sum_{i=1}^{d} \binom{d}{i}^2 i! x^i$*

Proof. We provide the proof by induction on k.

Let $k = 1$, then $|\mathcal{IS}_d| = \sum_{i=1}^{d} \binom{n}{i}^2 i! = S(1)$ (cf. [4]).

Assume that we know the order of the partial wreath $(k-1)$-th power of semigroup $\mathcal{IS}_d$: $|\wr_p^{k-1} \mathcal{IS}_d| = S^{k-1}(1)$. Prove that $|\wr_p^k \mathcal{IS}_d| = S^k(1)$. The elements of semigroup $\wr_p^{k-1} \mathcal{IS}_d$ are pairs $(f, a) \in S_{k-1}^{P\mathcal{N}_d} \times \mathcal{IS}_d$ with $\mathrm{dom}(f) = \mathrm{dom}(a)$. Let $P_A = \{a \in \mathcal{IS}_d | \operatorname{dom}(a) = A\}$. Then the number of all such pairs (f, a) is equal to

$$\sum_{A \subset \mathcal{N}_d} \Big| \mathop{\wr}_{p}^{k-1} \mathcal{IS}_d \Big|^{|A|} \cdot \big| P_A \big| = \sum_{i=1}^{d} \Big| \mathop{\wr}_{p}^{k-1} \mathcal{IS}_d \Big|^{i} \binom{d}{i}^2 i!$$

$$= S(\big| \mathop{\wr}_{p}^{k-1} \mathcal{IS}_d \big|) = S(S^{k-1}(1)) = S^k(1). \quad (1)$$

□

Let $E(\mathcal{IS}_d)$ be the set of idempotents of semigroup $\mathcal{IS}_d$.

Proposition 2. *An element $(f, a) \in \mathcal{IS}_d \wr_p \mathcal{IS}_d$ is an idempotent if and only if $a \in E(\mathcal{IS}_d)$ and $f(\mathrm{dom}(a)) \subseteq E(\mathcal{IS}_d)$.*

Proof. Let (f, a) be idempotent, then $(f, a)(f, a) = (ff^a, a^2) = (f, a)$. Hence, $ff^a = f$, $a^2 = a$, i.e., $a \in \mathcal{IS}_d$ is an idempotent. It follows from the equality $ff^a = f$ that for any $c \in \mathrm{dom}(a)$ $ff^a(ca) = f(ca)f^a(ca) = f(ca)f(ca^2) = f(ca)f(ca)$.

Conversely, let $(f, a) \in \wr_p^k \mathcal{IS}_d$ be such an element that $a \in E(\mathcal{IS}_d)$ and $f(\mathrm{dom}(a)) \subseteq E(\mathcal{IS}_d)$. Then for any $c \in \mathrm{dom}(a)$ $f(ca) = f(ca)f(ca)$. So $f(ca) = f(ca)f(ca) = f(ca)f(ca^2) = ff^a(ca)$. Since it holds for all $c \in \mathrm{dom}(a)$, we have $(f, a)(f, a) = (f, a)$. □

Let $T_k^{(d)}$ be a rooted k-level d-regular tree. The partial automorphism of the tree $T_k^{(d)}$ is such partial (i.e. not necessarily completely defined) injective map $\varphi : VT_k^{(d)} \to VT_k^{(d)}$ that subgraphs generated by domain of φ and range of φ are isomorphic (i.e. φ maps isomorphically certain subgraph of the tree $T_k^{(d)}$ on another subgraph of the same tree). Partial automorphisms form a semigroup under composition $ab(x) = b(a(x))$, we will denote it by $\mathrm{PAut}\, T_k^{(d)}$. Evidently, this semigroup is an inverse semigroup. Let $\mathrm{ConPAut}\, T$ be the semigroup of partial automorphisms of the tree T, defined on a connected graph containing root and preserving the level of vertices. Further we will consider only partial automorphisms of this type.

Theorem 1. *Let $T_k^{(d)}$ be a rooted k-level d-regular tree. Then*

$$\operatorname{ConPAut} T_k^{(d)} \cong \overset{k}{\underset{p}{\wr}} \mathcal{IS}_d.$$

Proof. We provide the proof by induction on k.

Let $T_1^{(d)}$ be one-level tree, $\operatorname{ConPAut} T_1^{(d)}$ be the semigroup of partial automorphisms of this tree defined as above. By definition, $\operatorname{ConPAut} T_1^{(d)}$ contains partial automorphisms defined on a connected subgraph and that fix the root vertex and preserve the level of vertices, then every partial automorphism $\varphi \in \operatorname{ConPAut} T_1^{(d)}$ is determined only by the vertices permutation satisfying condition

$$\varphi(i) = \begin{cases} a_i, & \text{if } i \in \operatorname{dom}(\varphi)\ ; \\ \emptyset, & \text{otherwise.} \end{cases}$$

In other words, φ is the partial permutation from $\mathcal{IS}_d$. So, every partial automorphism $\varphi \in \operatorname{ConPAut} T_1^{(d)}$ is uniquely defined by partial permutation $\sigma \in \mathcal{IS}_d$. Thus, we have one-to-one correspondence between $\operatorname{ConPAut} T_1^{(d)}$ and $\mathcal{IS}_d$. Hence $\operatorname{ConPAut} T_1^{(d)} \cong \mathcal{IS}_d$.

Assume that $\wr_p^{k-1} \mathcal{IS}_d \cong \operatorname{ConPAut}_{k-1}$.

Prove that $\wr_p^{k} \mathcal{IS}_d \cong \operatorname{ConPAut}_k$. Let $\varphi \in \operatorname{ConPAut}_k$ and V_i be the i-th level of the tree $T_k^{(d)}$. Define a map $\psi : \operatorname{ConPAut}_k \to \wr_p^{k} \mathcal{IS}_d$ by: $\varphi \mapsto (\varphi|_{T_{k-1}}, \varphi|_{V_1})$, where $\varphi|_{T_{k-1}}$ is a partial automorphism that acts on the rooted subtrees, which root vertices lie on the first level of the tree $T_k^{(d)}$ and belong to $\operatorname{dom}(\varphi|_{V_1})$. Hence $\varphi|_{V_1} \in \mathcal{IS}_d$ and $\varphi|_{T_{k-1}}$: $\operatorname{dom}(\varphi|_{V_1}) \to \wr_p^{k-1} \mathcal{IS}_d$. Thus we may establish correspondence between given partial automorphism $\varphi \in \operatorname{ConPAut}_k$ and a unique pair (σ, f), where $\sigma \in \mathcal{IS}_d$, $f : \mathcal{N}_d \to \wr_p^{k-1} \mathcal{IS}_d, \operatorname{dom}(f) = \operatorname{dom}(\sigma)$. And we have $\operatorname{ConPAut} T_k^{(d)} \cong \wr_p^{k} \mathcal{IS}_d$.

□

Proposition 3. *Let $E(\wr_p^k \mathcal{IS}_d)$ be the set of idempotents of semigroup $\wr_p^k \mathcal{IS}_d$. Then $\left|E(\wr_p^k \mathcal{IS}_d)\right| = F^k(1) = \underbrace{(((1+1)^d+1)^d \ldots + 1)^d}_{k}$, where*

$F(x) = (x+1)^d$.

Proof. It follows from the Theorem 1 that there exists bijection between set of idempotents of semigroup $\wr_p^k \mathcal{IS}_d$ and set of connected subgraphs of the tree $T_k^{(d)}$ with different domains. We calculate number of idempotents as a number of such subgraphs of the tree $T_k^{(d)}$, because idempotents of $ConPAutT_k^{(d)}$ are identity maps: $id_\Gamma : \Gamma \to \Gamma$, $\Gamma \subset T_k^{(d)}$.

We compute their number by induction on k. Let $k = 1$, then $\wr_p^1 \mathcal{IS}_d = \mathcal{IS}_d$, consequently $|E(\wr_p^1 \mathcal{IS}_d)| = |E(\mathcal{IS}_d)| = 2^d = F(1)$.

Assume that $|E(\wr_p^{k-1} \mathcal{IS}_d)| = F^{k-1}(1) = |E(ConPAutT_k^{(d)})|$.

Find now the number of idempotents of semigroup $ConPAutT_k^{(d)}$. For all $i = 1, \ldots, d$ we can choose i-element subset among the first level vertices in $\binom{d}{i}$ ways. Denote these subsets A_i^j, $i = 1, \ldots, d$, $j = 1, \ldots, \binom{d}{i}$ Each vertex from A_i^j is the root vertex of $(k-1)$-level tree. We know the number of idempotents of the semigroup $ConPAutT_k^{(d)}$, then

$$|E(\mathop{\wr}\limits_p^k \mathcal{IS}_d)| = |E(ConPAutT_k^{(d)})| = \sum_{i=1}^{d} \binom{d}{i} (F^{k-1}(1))^i$$
$$= (F^{k-1}(1) + \ 1)^d = F(F^{k-1}) = F^k(1).$$

□

2. Combinatorics of Green's relations

Theorem 2. *Let $(f,a),(g,b) \in \wr_p^k \mathcal{IS}_d$. Then*

1. $(f,a)\, \mathcal{L}\, (g,b)$ if and only if $\mathrm{ran}(a) = \mathrm{ran}(b)$ and $g^{b^{-1}}(z)\, \mathcal{L}\, f^{a^{-1}}(z)$ for all $z \in \mathrm{ran}(a)$, where a^{-1} is the inverse element for a;

2. $(f,a)\, \mathcal{R}\, (g,b)$ if and only if $\mathrm{dom}(a) = \mathrm{dom}(b)$ and $f(z)\, \mathcal{R}\, g(z)$ for all $z \in \mathrm{dom}(a)$;

3. $(f,a)\, \mathcal{H}\, (g,b)$ if and only if $\mathrm{ran}(a) = \mathrm{ran}(b)$ and $\mathrm{dom}(a) = \mathrm{dom}(b)$, $g^{b^{-1}}(z)\ \ \mathcal{L}\ \ f^{a^{-1}}(z)$ and $f(z)\, \mathcal{R}\, g(z)$ for $z \in \mathrm{dom}(a) \cap \mathrm{ran}(a)$;

4. $(f,a)\mathcal{D}(g,b)$ if and only if there exists a bijection map $x : \mathrm{dom}(b) \to \mathrm{dom}(a)$ such that $f(zx)\ \mathcal{D}\ g(z)$.

5. $\mathcal{D}{=}\mathcal{J}$.

Proof. Green's relations on semigroup $\mathcal{IS}_d$ are described in [4].

1. Let $(f,a)\, \mathcal{L}\, (g,b)$, then there exist $(u,x),(v,y) \in \wr_p^k \mathcal{IS}_d$ such that $(u,x)(f,a) = (g,b)$ and $(v,y)(g,b) = (f,a)$, i.e.

$$(u,x)(f,a) = (uf^x, xa) = (g,b),$$
$$(v,y)(g,b) = (vg^y, yb) = (f,a).$$

We get from these equalities that $xa = b, yb = a$, and therefore $a\, \mathcal{L}\, b$ and then $\mathrm{ran}(a) = \mathrm{ran}(b)$, and also we get $uf^x = g, vg^y = f$.

Multiplying the both sides of the equality $xa = b$ by a^{-1} from the left and by b^{-1} from the right we obtain $b^{-1}x = a^{-1}$. Analogously we obtain $a^{-1}y = b$. Put $t = zb^{-1}$ for any $z \in \mathrm{dom}(b^{-1}) = \mathrm{ran}(b)$, then

$$\begin{gathered} uf^x(t) = g(t), \\ u(t)f(tx) = g(t), \\ u(zb^{-1})f(zb^{-1}x) = u(zb^{-1})f(za^{-1}) = g(zb^{-1}), \\ u(zb^{-1})f^{a^{-1}}(z) = g^{b^{-1}}(z). \end{gathered}$$

Putting $t = za^{-1}$ for any $z \in \mathrm{ran}(a) = \mathrm{ran}(b)$, we analogously get $v(za^{-1})g^{b^{-1}}(z) = f^{a^{-1}}(z)$. We have $u(zb^{-1})f^{a^{-1}}(z) = g^{b^{-1}}(z)$ and $v(za^{-1})g^{b^{-1}}(z) = f^{a^{-1}}(z)$. This implies $f^{a^{-1}}(z)\ \mathcal{L}\ g^{b^{-1}}(z)$, $z \in \mathrm{ran}(a) = \mathrm{ran}(b)$.

Conversely, let $\mathrm{ran}(a) = \mathrm{ran}(b)$ and $f^{a^{-1}}(z)\mathcal{L}g^{b^{-1}}(z)\ \forall z \in \mathrm{ran}(a) = \mathrm{ran}(b)$. From the first condition we get $a\,\mathcal{L}\,b$, and hence there exist $x, y \in \mathcal{IS}_d$ such that $xa = b, yb = a$. From the second condition it follows that there exist functions $u, v \in S_k^{P\mathcal{N}_d}$ such that $u(z)f^{a^{-1}}(z) = g^{b^{-1}}(z)$ and $v(z)g^{b^{-1}}(z) = f^{a^{-1}}(z)$, $z \in \mathrm{ran}(a) = \mathrm{ran}(b)$. Consider $(u, x), (v, y) \in \wr_p^k \mathcal{IS}_d$, where x, y, u, v are defined as above. Then

$$(u, x)(f, a) = (uf^x, xa) = (uf^{ba^{-1}}, b) = (g^{bb^{-1}}, b) = (g, b)$$

and in the same way we get $(v, y)(g, b) = (f, a)$. Therefore $(f, a)\,\mathcal{L}\,(g, b)$.

2. Let $(f, a)\ \mathcal{R}\ (g, b)$, then there exist $(u, x), (v, y) \in \wr_p^k \mathcal{IS}_d$ such that $(f, a)(u, x) = (g, b)$, $(g, b)(v, y) = (f, a)$. This is equivalent to $ax = b, by = a, fu^a = g, gv^b = f$. This gives us the conditions $a\ \mathcal{R}\ b$, and hence $\mathrm{dom}(a) = \mathrm{dom}(b)$, and $fu^a = g, gv^b = f$. Consequently, $f(z)\,\mathcal{R}\,g(z)\ \forall z \in \mathrm{dom}(a)$.

   Conversely, let $(f, a), (g, b) \in \wr_p^k \mathcal{IS}_d$ and $\mathrm{dom}(a) = \mathrm{dom}(b)$, $f(z)\,\mathcal{R}\,g(z)\ \forall z \in \mathrm{dom}(a)$. From $\mathrm{dom}(a) = \mathrm{dom}(b)$ it follows $a\,\mathcal{R}\,b$, then there exists $x, y \in \mathcal{IS}_d$ such that $ax = b, by = a$, and from $f(z)\,\mathcal{R}\,g(z)\ \forall z \in \mathrm{dom}(a)$ it follows that there exist $u', v' \in S_{k-1}^{P\mathcal{N}_d}$ such that for any $z \in \mathrm{dom}(a)$ $fu'(z) = g(z), gv'(z) = f(z)$. Define $u, v \in S_{k-1}^{P\mathcal{N}_d}$ by $u(za) = u'(z)$, $v(zb) = v'(z)$. Then for $t \in \mathrm{dom}(a)$ it holds $fu^a(t) = f(t)u(ta) = f(t)u'(t) = g(t)$ and $gv^b(t) = f(t)$, then

$$\begin{gathered} (f, a)(u, x) = (fu^a, ax) = (g, b), \\ (g, b)(v, y) = (f, a). \end{gathered}$$

Therefore, $(f,a)\,\mathcal{R}\,(g,b)$.

3. As $\mathcal{H} = \mathcal{L} \wedge \mathcal{R}$, this statement follows from the first and second ones.

4. Let $(f,a)\,\mathcal{D}\,(g,b)$. Then there exist $(h,c) \in \wr_p^k \mathcal{IS}_d$ such that $(f,a)\,\mathcal{L}\,(h,c)$ and $(h,c)\,\mathcal{R}\,(g,b)$. From $(f,a)\,\mathcal{L}\,(h,c)$ we get that $\mathrm{ran}(a) = \mathrm{ran}(c)$ and for $z \in \mathrm{ran}(a)$ $f^{a^{-1}}(z)\,\mathcal{L}\,h^{c^{-1}}(z)$. Then there exist functions u and v such that $u(z)f^{a^{-1}}(z) = h^{c^{-1}}(z)$ and $v(z)h^{c^{-1}}(z) = f^{a^{-1}}(z)$. Put $x = a^{-1}c$. By definition of $\mathcal{IS}_d$ x is a partial bijection map. We now obtain $f(zx)\,\mathcal{L}\,h(z)$, and $x : \mathrm{dom}(c) \to \mathrm{dom}(a)$. From $(h,c)\,\mathcal{R}\,(g,b)$ we have that for $z \in \mathrm{dom}(b)$: $h(z)\,\mathcal{R}\,g(z)$ and $\mathrm{dom}(b) = \mathrm{dom}(c)$. From $\mathrm{ran}(a) = \mathrm{ran}(c)$ and $\mathrm{dom}(b) = \mathrm{dom}(c)$ we get $|\,\mathrm{dom}(a)| = |\,\mathrm{dom}(b)|$. Thus there a exists bijection $x : \mathrm{dom}(b) \to \mathrm{dom}(a)$ such that $f(zx)\,\mathcal{D}\,h(z), z \in \mathrm{dom}(b) \cap \mathrm{ran}(a)$.

   Conversely, assume that there exists a bijection map $x : \mathrm{dom}(b) \to \mathrm{dom}(a)$ such that $f(zx)\mathcal{D}g(z)$, i.e. there exists a function $h(z)$ such that $f(zx)\,\mathcal{L}\,h(z)$ and $h(z)\,\mathcal{R}\,g(z)$. Let $u'(z), v'(z) \in S^{P\mathcal{N}_d}$ satisfy conditions $u'(z)f(zx) = u'f^x(z) = h(z)$ and $v'(z)h(z) = f(zx)$. Put $c = xa$, then c is partial bijection $c : \mathrm{dom}(b) \to \mathrm{ran}(a)$ exists. Define $u(z)$ by $u(z) = u'(z)$ and $v(z)$ by $v(z) = v'(zx^{-1})$. Then

   $$(u,x)(f,a) = (uf^x, xa) = (h,c),$$
   $$(v,x^{-1})(h,c) = (f,a).$$

   Hence $(f,a)\,\mathcal{L}\,(h,c)$. As $h(z)\,\mathcal{R}\,g(z)$ and $\mathrm{dom}(c) = \mathrm{dom}(b)$, then $(h,c)\,\mathcal{R}\,(g,b)$. It implies $(f,a)\,\mathcal{D}\,(g,b)$.

5. As $\wr_p^k \mathcal{IS}_d$ is finite then $\mathcal{D} = \mathcal{J}$.

□

Corollary. If $(f,a),(g,b) \in \mathcal{IS}_d \wr_p \mathcal{IS}_d$, then

1. $(f,a)\,\mathcal{L}\,(g,b)$ if and only if $\mathrm{ran}(a) = \mathrm{ran}(b)$ and $\mathrm{ran}(g^{a^{-1}}(z)) = \mathrm{ran}(f^{b^{-1}}(z))$ for all $z \in \mathrm{ran}(a)$;

2. $(f,a)\,\mathcal{R}\,(g,b)$ if and only if $\mathrm{dom}(a) = \mathrm{dom}(b)$ and $\mathrm{dom}(f(z)) = \mathrm{dom}(g(z))$ for all $z \in \mathrm{dom}(a)$;

3. $(f,a)\,\mathcal{H}\,(g,b)$ if and only if $\mathrm{ran}(a) = \mathrm{ran}(b)$, $\mathrm{dom}(a) = \mathrm{dom}(b)$, $\mathrm{ran}(g^{a^{-1}}(z)) = \mathrm{ran}(f^{b^{-1}}(z))$ for $z \in \mathrm{ran}(a)$, and $\mathrm{dom}(f(z)) = \mathrm{dom}(g(z))$ for $z \in \mathrm{dom}(a)$.

Lemma 1. *Let* $\sigma, \tau \in \mathrm{PAut}\, T_k^{(d)}$*. Then* $\sigma \,\mathcal{D}\, \tau$ *if and only if* $\mathrm{dom}(\sigma) \cong \mathrm{dom}(\tau)$*.*

Proof. Let $\sigma\mathcal{D}\tau$, then there exists $\gamma \in \mathrm{PAut}\, T_k^{(d)}$ such that $\sigma\mathcal{L}\gamma$ and $\gamma\mathcal{R}\tau$. Thus, $\mathrm{ran}(\sigma) = \mathrm{ran}(\gamma)$, $\mathrm{dom}(\gamma) = \mathrm{dom}(\tau)$. By definition of semigroup $\mathrm{PAut}\, T_k^{(d)}$ all these maps are isomorphisms between their domains and ranges. It immediately follows that map $\varphi = \gamma\sigma^{-1} : \mathrm{dom}(\tau) \to \mathrm{dom}(\sigma)$ is isomorphism from $\mathrm{dom}(\tau)$ to $\mathrm{dom}(\sigma)$, so $\mathrm{dom}(\sigma) \cong \mathrm{dom}(\tau)$.

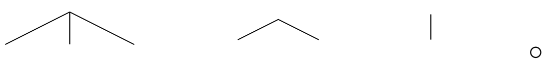
Let now $\mathrm{dom}(\sigma) \cong \mathrm{dom}(\tau)$. As before by definition of semigroup $\mathrm{PAut}\, T_k^{(d)}$ it follows $\mathrm{dom}(\sigma) \cong \mathrm{ran}(\sigma)$, hence isomorphism $\gamma : \mathrm{ran}(\sigma) \to \mathrm{dom}(\tau)$ exists. Therefore, $\sigma \,\mathcal{L}\, \gamma$ and $\gamma \,\mathcal{R}\, \tau$. It implies $\sigma \,\mathcal{D}\, \tau$. □

Proposition 4. *The number of* $\mathcal{D}$*-classes of semigroup* $\wr_p^k \mathcal{IS}_d$ *equals* $P^k(1)$*, where* $P(x) = \binom{x+d}{d}$*.*

Proof. By Theorem 1 and Lemma 1 the calculation of the number of $\mathcal{D}$-classes of semigroup $\wr_p^k \mathcal{IS}_d$ is equivalent to that of the number of non-isomorphic connected subgraphs of the tree $T_k^{(d)}$ containing root vertex. Later on all subgraphs are supposed to be connected and to contain root vertex.

Partition the set of all connected subgraphs of the tree $T_k^{(d)}$ into the classes of isomorphic subgraphs. Define the set of graphs-representatives denoted by $GRep_k$ in the following way.

Consider firstly one-level d-regular tree. It is clear that the set of all connected subgraphs is divided into d+1 class. We choose a representative from each class and number them with integers from 0 to d in decreasing order of root vertices degree. For example, if $d = 3$ we have:

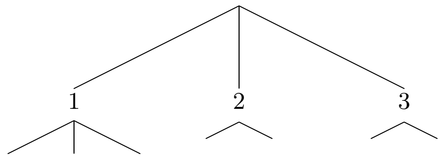


0 1 2 3

Define the following order relation on the set of graphs-representatives $GRep_1$. Let i_1, i_2 be the numbers of graphs Γ_1 and Γ_2 respectively. Then $\Gamma_1 > \Gamma_2 \Leftrightarrow i_1 < i_2$.

Consider now 2-level tree $T_2^{(d)}$. Partition again the set of connected subgraphs into classes of isomorphic subgraphs. Notice that each vertex of the first level of $T_2^{(d)}$ is a root vertex of a one-level subgraph, which is isomorphic to a ceratin subgraph from the set $GRep_1$. Attach a number sequence $(i_1, i_2, \ldots, i_l)$ to each subgraph by, where l is a degree of the root vertex and i_j is the number of subgraph from $GRep_1$ subgraph of

$T_2^{(d)}$ with root vertex labelled by j is isomorphic to. For example, the corresponding sequence for subgraph



is $(0,1,1)$. It is evident that connected subgraphs of $T_2^{(d)}$ are isomorphic if and only if corresponding sequences are equal up to the permutation of sequences members. Choose a subgraph described by non-decreasing corresponding sequence from each class of isomorphic subgraphs. We call these subgraphs graphs-representatives and define a linear order relation on the set of graphs-representatives $GRep_2$ in the following way: let $\Gamma_1, \Gamma_1 \in GRep_2$ and $a_1 = (i_1, i_2, \dots, i_m)$, $a_2 = (j_1, j_2, \dots, j_n)$ be corresponding sequences. Then $\Gamma_1 > \Gamma_2$ if and only if $a_1 < a_2$, set of sequences is lexicographically ordered. For instance, if the number sequence related to subgraph Γ_1 is $(0,0,0)$ and the number sequence related to subgraph Γ_2 is $(0,0,1)$, then $\Gamma_1 > \Gamma_2$. We have linearly ordered set and we may arrange graphs-representatives in decreasing order and number them in such a way that 0 corresponds to the "biggest" graph.

Let now $\Gamma_0 > \Gamma_1 > \dots > \Gamma_N$ be ordered set $GRep_{k-1}$ of graphs-representatives of $(k-1)$-level tree $T_{k-1}^{(d)}$. Partition again the set of all connected subgraph of the tree $T_k^{(d)}$ into classes of isomorphic subgraphs. Attach again a number sequence $(i_1, i_2, \dots, i_l)$ to each subgraph, where i_j is the number of corresponding graph from $GRep_{k-1}$, $i_j \in \{0, 1, \dots, N\}, j = \overline{1, l},\ l \le d$, and construct the set of graphs-representatives $GRep_k$ of the k-level tree $T_k^{(d)}$ as above. It is easy to check that set $GRep_k$ has following properties:

1. For all subgraph $\Gamma \subset T_k^{(d)}$ there exists subgraph $\tilde{\Gamma}$ from the set $GRep_k$ such that $\Gamma \cong \tilde{\Gamma}$;

2. If $\Gamma_1 \cong \Gamma_2$, where $\Gamma_1, \Gamma_2 \subset GRep$, then $\Gamma_1 = \Gamma_2$.

Therefore, we have to compute the cardinality of the set of graphs-representatives $GRep_k$ to find the number of connected non-isomorphic subgraphs of the tree $T_k^{(d)}$ that gives us the number of $\mathcal{D}$-classes of semigroup $\wr_p^k \mathcal{IS}_d$.

We use induction on k to calculate the cardinality of the set of graphs-representatives.

If $k=1$, then $|GRep_1| = d+1 = \binom{d+1}{d} = P(1)$.

Assume that $N = P^{k-1}(1)$ is the cardinality of the set of graphs-representatives $GRep_{k-1}$ of $(k-1)$-level tree.

Each vertex of the first level of the tree $T_k^{(d)}$ is the root vertex of $(k-1)$-level tree that is isomorphic to a certain graph from $GRep_{k-1}$. Assume that all vertices of the first level are labelled with integers from 1 to l, $l \le d$. As set $GRep_k$ contains no equal graphs, then corresponding sequences are all different. Hence, there exists one-to-one correspondence between set $\{1,2,\dots,l\}$ and set $\{0,1,\dots,N-1\}$. Consider all non-decreasing functions $f : \{1,2,\dots,l\} \to \{0,1,\dots,N-1\}$. The number of all such functions is equal to the cardinality of the set $GRep_k$. Define $x_0 = f(1), x_1 = f(2)-f(1), \dots, x_{l-1} = f(l)-f(l-1), x_l = N-1-f(l)$. Then $f(k) = x_0+x_1+x_2+\dots+x_k$. Since f is non-decreasing, then for all $i = 1,\dots,l$ $x_i \ge 0$. So the number of non-decreasing functions is equal to the number of integer solutions of the equation $N-1 = x_0+x_1+\dots+x_l$ for $l = 1,\dots,d$. Thus, we get the number of $\mathcal{D}$-classes of semigroup $\wr_p^k \mathcal{IS}_d$:

$$\sum_{l=0}^{d} \binom{N+l-1}{l} = \binom{N+d}{d} = P(N) = P(P^{k-1}(1)) = P^k(1)$$

□

Let Γ be a subtree of the tree $T_k^{(d)}$, $St_{T_k^{(d)}}(\Gamma)$ be the stabilizer of the subtree Γ, $Fix_{T_k^{(d)}}(\Gamma)$ be the fixator of the subtree Γ and let D_Γ be $\mathcal{D}$-class such that for any $\sigma \in D_\Gamma$ $\operatorname{dom}(\sigma) \cong \Gamma$. Let $\{\Gamma_1,\dots,\Gamma_i\}$ be the set of all pairwise non-isomorphic subtrees of Γ with root vertices in the first level of Γ. Let α_j be the number of isomorphic to Γ_j subtrees of Γ with root vertices in the first level of Γ, $j = 1,\dots,i$. The type of Γ is a set $\{(\Gamma_1,\alpha_1),(\Gamma_2,\alpha_2),\dots,(\Gamma_i,\alpha_i)\}$ such that disjoint union of vertices sets of all subtrees and root vertex gives vertices of Γ. Notice that $\sum_{j=1}^{i} \alpha_j = l$, where l is the degree of root vertex of Γ.

Proposition 5. *Let Γ be a subtree of the tree $T_k^{(d)}$ and its type be $\{(\Gamma_1,\alpha_1),(\Gamma_2,\alpha_2),\dots,(\Gamma_i,\alpha_i)\}$, and degree of the root vertex of Γ be l, $l \le d$. Then*

1. $|\operatorname{Aut}\Gamma| = \prod_{j=1}^{i}(\alpha_j)!|\operatorname{Aut}(\Gamma_j)|^{\alpha_j}$, $l \le d$,
2. $|St_{T_k^{(d)}}(\Gamma)| = (d-l)!|\operatorname{Aut} T_{k-1}^{(d)}|^{d-l} \prod_{j=1}^{i}(\alpha_j)!|St_{T_{k-1}^{(d)}}(\Gamma_j)|^{\alpha_j}$,
3. $|Fix_{T_k^{(d)}}(\Gamma) = (d-l)!|\operatorname{Aut} T_{k-1}^{(d)}|^{d-l} \prod_{j=1}^{i}|Fix_{T_{k-1}^{(d)}}(\Gamma_j)|^{\alpha_j}$.

Proof.

1. Prove the proposition by induction on k. Let Γ be one-level tree and root vertex degree be $l \leq d$, then $|\operatorname{Aut}\Gamma| = l!$.

   Assume we know the orders of groups $\operatorname{Aut}\Gamma_j$ for all $j = 1, \ldots, i$. Find the order of $\operatorname{Aut}\Gamma$. Degrees of root vertices of isomorphic trees are equal. Let them be l. It is clear that types of all trees isomorphic to Γ are equal up to the permutation of items. Thus only permutation of the first level vertices, and consequently permutation of subtrees of Γ, distinguishes graph Γ from isomorphic one. All the vertices of the first level may permute, but with several restrictions, namely, roots of non-isomorphic subtrees stay roots of non-isomorphic subtrees. Since the orders of $\operatorname{Aut}\Gamma_j$ for all $j = 1, \ldots, i$ are known, we can derive the order of $\operatorname{Aut}\Gamma$:

$$|\operatorname{Aut}\Gamma| = \prod_{j=1}^{i} (\alpha_j)! |Aut\Gamma_j|^{\alpha_j}.$$

2. The proof is analogous to the proof of the previous statement.

   Consider the stabilizer of subtree Γ in the automorphisms group of the rooted tree $T_1^{(d)}$. Let the degree of the root of Γ be l. Then it is obvious that $|St_{T_1^{(d)}}(\Gamma)| = l!(d-l)!$.

   Assume now that we know the order of $St_{T_{k-1}^{(d)}}(\Gamma)$. Let the degree of the root of Γ be l. Then $(d-l)$ vertices of the first level of Γ may permute and each of them is the root of $(k-1)$-level tree $T_{k-1}^{(d)}$. Among l vertices, as in proof of the previous statement, distinguish only vertices that are roots of isomorphic subtrees. Then

$$|St_{T_k^{(d)}}(\Gamma)| = (d-l)! |\operatorname{Aut} T_{k-1}^{(d)}|^{d-i} \prod_{j=1}^{i} (\alpha_j)! |St_{T_{k-1}^{(d)}}|^{\alpha_j}(\Gamma_j).$$

3. Taking into account that fixator of the subtree does not allow vertices permutation of this subtree, the proof is analogous to the proof of point 2.

□

Proposition 6. *The cardinality of the set of idempotents $E(D_\Gamma)$ of class D_Γ equals*

$$|E(D_\Gamma)| = \frac{(d!)^{\frac{1-d^k}{1-d}}}{|St_{T_k^{(d)}}(\Gamma)|}.$$

Proof follows from one-to-one correspondence between the set of ranges of idempotents of D_Γ and the set $\operatorname{Aut} T_k^{(d)}/St(\Gamma)$, and $|\operatorname{Aut} T_k^{(d)}| = (d!)^{\frac{1-d^k}{1-d}}$. □

Corollary 1. The number of $\mathcal{R}$-classes and the number of $\mathcal{L}$-classes containing in $\mathcal{D}$-class D_Γ is equal to

$$\frac{(d!)^{\frac{1-d^k}{1-d}}}{|St_{T_k^{(d)}}(\Gamma)|}.$$

Proof follows from the fact that in inverse semigroup every $\mathcal{L}$-class and every $\mathcal{R}$-class contains exactly one idempotent. □

Corollary 2. The cardinality of $\mathcal{H}$-class containing in $\mathcal{D}$-class D_Γ is equal to $|\operatorname{Aut}\Gamma|$.

Proof. Let $\sigma, \tau \in \operatorname{PAut} T_k^{(d)}$. Then $\sigma\,\mathcal{H}\,\tau$ if and only if $\operatorname{dom}(\sigma) = \operatorname{dom}(\tau)$ and $\operatorname{ran}(\sigma) = \operatorname{ran}(\tau)$. The statement is now obvious. □

Corollary 3. $|D_\Gamma| = |E(D_\Gamma)|^2 |Aut\Gamma|$.

Proof follows from corollaries 1 and 2. □

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