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RESEARCH ARTICLE

n-ary comodules over n-ary (co)algebras

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Abstract. In the paper we study connections between (co)modules over n-ary and binary (co)algebras.

Introduction

In this paper, the notations of right (left) (co)modules over (co)algebras are generalized from binary to n-ary case Definition 1.1 and 1.2 It is proved that a module M is a right n-ary C-comodule if and only if it is a left n-ary C^*-module Theorem 2.1. A dual statement is proved in Theorem 3.1. Notations of right (left) n-ary (co)module algebras are introduced and it is proved that an n-ary C-comodule algebra is an n-ary C^*-module algebra, Theorem 2.2. Moreover, n-ary C-module algebra is an n-ary C^*-comodule algebra Theorem 3.3.

All necessary notations and definitions can be founded in the papers listed in References.

1. Basic notions

Let k be a ground commutative associative ring with a unit, C and M modules over k. In what follows, $\otimes$ is a tensor product over k. All homomorphisms are k-linear maps. In [Z1], the concept of n-ary algebras (C, m) is defined, where $m : C \otimes \cdots \otimes C \to C$ is n-ary multiplication,

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which is associative. It means that the following diagram is commutative:

$$\begin{array}{ccc} C^{\otimes(2n-1)} & \xrightarrow{m\otimes 1_C^{\otimes(n-1)}} & C^{\otimes n} \\ {\scriptstyle 1_C^{\otimes i}\otimes m\otimes 1_C^{\otimes(n-i-1)}}\downarrow & & \downarrow{\scriptstyle m} \\ C^{\otimes n} & \xrightarrow[m]{} & C \end{array}$$

i.e. for any $i = 1, \ldots, n$ we have

$$m \cdot \left(m \otimes 1_C^{\otimes(n-1)}\right) = m \cdot \left(1_C^{\otimes i} \otimes m \otimes 1_C^{\otimes(n-i-1)}\right).$$

The concept of n-ary coalgebra (C, Δ) is defined in [Z2], where $\Delta : C \to C \otimes \cdots \otimes C$ is n-ary comultiplication, which is coassociative, that is the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C^{\otimes n} \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle 1_C^{\otimes i}\otimes\Delta\otimes 1_C^{\otimes(n-i-1)}} \\ C^{\otimes n} & \xrightarrow[\Delta\otimes 1_C^{\otimes(n-1)}]{} & C^{\otimes(2n-1)} \end{array}$$

i.e. for any $i = 1, \ldots, n$ we have

$$\left(\Delta \otimes 1_C^{\otimes(n-1)}\right) \cdot \Delta = \left(1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-1)}\right) \cdot \Delta.$$

Analogously, the concept of n-ary bialgebra (C, m, Δ) , where m is associative n-ary multiplication and Δ is coassociative n-ary comultiplication. It is shown that Δ is a homomorphism of n-ary algebras. We do not suppose an existence of the unit and counit. In [Z1] the notion of homomorphism $f : (C, m_C) \to (C', m_{C'})$ of n-ary algebras is defined as a morphism $f : C \to C'$, such that the following diagram is commutative:

$$\begin{array}{ccc} C^{\otimes n} & \xrightarrow{f^{\otimes n}} & C'^{\otimes n} \\ {\scriptstyle m_C}\downarrow & & \downarrow{\scriptstyle m_{C'}} \\ C & \xrightarrow[f]{} & C' \end{array}$$

i.e. $f \cdot m_C = m_{C'} \cdot f^{\otimes n}$.

Let C be an n-ary coalgebra and a finitely generated projective k-module. Denote by C^* the k-module $\mathrm{Hom}(C, k)$. Then C^* is an n-ary algebra with multiplication $l_1 * \cdots * l_n$, where for $c \in C$:

$$(l_1 * \cdots * l_n)(c) = \sum_c l_1\left(c_{(1)}\right) \cdots l_n\left(c_{(n)}\right) \tag{1}$$

if

$$\Delta(c) = \sum_c c_{(1)} \otimes \cdots \otimes c_{(n)} \in C^{\otimes n}.$$

Conversely, let C be an n-ary algebra and a finitely generated projective k-module. Define n-ary comultiplication Δ^* in $C^* = \mathrm{Hom}(C, k)$, by the rule:

$$(\Delta^*)(x_1 \otimes \cdots \otimes x_n) = l(x_1 \cdots x_n) \tag{2}$$

where $x_i \in C$. Here we use the isomorphism of k-modules:

$$\left(C^{\otimes n}\right)^* \simeq (C^*)^{\otimes n}$$

because C is a finitely-generated projective k-module. Then, C^* is a n-ary coalgebra. If C is a n-ary (co)algebra, then $(C^*)^* \simeq C$, [B].

Definition 1.1. *Let C be a n-ary coalgebra. We say that k-module M is a* right n-ary C-comodule, *if there exists a map $\rho : M \to M \otimes C^{\otimes(n-1)}$, such that the following diagram is commutative:*

$$\begin{array}{ccc} M & \xrightarrow{\quad\rho\quad} & M \otimes C^{\otimes(n-1)} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle 1_M \otimes 1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-2)}} \\ M \otimes C^{\otimes(n-1)} & \xrightarrow[\rho \otimes 1_C^{\otimes(n-1)}]{} & M \otimes C^{\otimes(2n-1)} \end{array}$$

i.e.

$$\left(1_M \otimes 1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-2)}\right) \cdot \rho = \left(\rho \otimes 1_C^{\otimes(n-1)}\right) \cdot \rho$$

for any i.

Definition 1.2. *Let C be a n-ary algebra. k-module M is called a* left n-ary C-module, *if there exists a map $\gamma : C^{\otimes(n-1)} \otimes M \to M$, such that the following diagram is commutative:*

$$\begin{array}{ccc} C^{\otimes(n-1)} \otimes M & \xrightarrow{\quad\gamma\quad} & M \\ \uparrow{\scriptstyle 1_C^{\otimes(n-1)} \otimes \gamma} & & \uparrow{\scriptstyle\gamma} \\ C^{\otimes(2n-1)} \otimes M & \xrightarrow[1_C^{\otimes i} \otimes m \otimes 1_C^{\otimes(n-i-2)} \otimes 1_M]{} & C^{\otimes(n-1)} \otimes M \end{array}$$

i.e.

$$\gamma \cdot \left(1_C^{\otimes(n-1)} \otimes \gamma\right) = \gamma \cdot \left(1_C^{\otimes i} \otimes m \otimes 1_C^{\otimes(n-i-2)} \otimes 1_M\right)$$

for any $i = 1, \ldots, n$.

Definition 1.3. *Let C be a n-ary bialgebra. n-ary algebra M is called a* left n-ary C-module algebra, *if:*

1) *M is a left n-ary C-module;*

2) *for any $c_1, \dots, c_{n-1} \in C$, $m_1, \dots, m_n \in M$*

$$\begin{aligned}&(c_1 \otimes \cdots \otimes c_{n-1})(m_1 \cdots m_n)\\ &= \sum_{c_1,\dots,c_{n-1}} \left(c_{1(1)} \cdots c_{n-1(1)} m_1\right) \cdots \left(c_{1(n)} \cdots c_{n-1(n)} m_n\right)\end{aligned}$$

Definition 1.4. *Let C be an n-ary bialgebra. n-ary algebra M is called a* right n-ary C-comodule algebra*, if:*

1) *M is a right n-ary C-comodule with the structure morphism $\rho : M \to M \otimes C^{\otimes(n-1)}$;*

2) *ρ is a homomiorphism of n-ary algebras.*

2. The relations between n-ary comodules and modules

Theorem 2.1. *Let C be a n-ary coalgebra. Then M is a right n-ary C-comodule if and only if M is a left n-ary C^*-module.*

Proof. Suppose that M is a right n-ary C-comodule and

$$\rho(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)} \otimes \cdots \otimes m_{(n-1)}, \tag{3}$$

where $m \in M$ and $m_{(0)} \in M$, $m_{(i)} \in C$ for $= 1, \dots, n-1$. If $l_1, \dots, l_{n-1} \in C^*$, $m \in M$, then we put

$$\rho^* \left(l_1 \otimes \cdots \otimes l_{n-1} \otimes m\right) = \sum_{(m)} m_{(0)} l_1 \left(m_{(1)}\right) \cdots s l_{n-1} \left(m_{(n-1)}\right) \in M. \tag{4}$$

Further, if $l_1, \dots, l_{2n-2} \in C^*$ and $m \in M$, then by Definition 1.2 we have:

$$\begin{aligned}&\left(\rho^* \cdot \left(1_{C^*}^{\otimes(n-1)} \otimes \rho^*\right)\right)\left(l_1 \otimes \cdots \otimes l_{2n-2} \otimes m\right)\\ &= \rho^* \left[l_1 \otimes \cdots \otimes l_{n-1} \otimes \sum_{(m)} m_{(0)} l_n \left(m_{(1)}\right) \cdots l_{2n-2} \left(m_{(n-1)}\right)\right]\\ &= \sum_{m(0)} m_{(0)(0)} l_1 \left(m_{(0)(1)}\right) \cdots l_{n-1} \left(m_{(0)(n-1)}\right) l_n \left(m_{(1)}\right) \cdots l_{2n-2} \left(m_{(n-1)}\right).\end{aligned} \tag{5}$$

On the other hand,

$$\left(\rho^* \cdot \left(1_{C^*}^{\otimes(i-1)} \otimes m^* \otimes 1_{C^*}^{\otimes(n-1-i)} \otimes 1_M\right)\right)(l_1 \otimes \cdots \otimes l_{2n-2} \otimes m)$$
$$= \rho^* (l_1 \otimes \cdots \otimes l_{i-1} \otimes (l_i * \cdots * l_{i+n-1}) \otimes l_{n+i} \otimes \cdots l_{2n-2} \otimes m)$$
$$= \sum_m m_{(0)} l_1 \left(m_{(1)}\right) \cdots l_{i-1} \left(m_{(i-1)}\right) (l_1 * \cdots * l_{i+n-1}) \times$$
$$\left(m_{(i)}\right) l_{i+n} \left(m_{(i+1)}\right) \cdots l_{2n-2} \left(m_{(n-1)}\right)$$
$$= \sum_m m_{(0)} l_1 \left(m_{(1)}\right) \cdots l_{i-1} \left(m_{(i-1)}\right) \times$$
$$\left[\sum_{m(i)} l_1 \left(m_{(i)(1)}\right) \cdots l_{i+n-1} \left(m_{(i)(n)}\right)\right] l_{i+n} \left(m_{(i+1)}\right) \cdots l_{2n-2} \left(m_{(n-1)}\right). \tag{6}$$

By Definition 1.2 we have:

$$\sum_m m_{(0)(0)} \otimes m_{(0)(1)} \otimes \cdots \otimes m_{(0)(n-1)} \otimes m_{(1)} \otimes \cdots \otimes m_{(n-1)}$$
$$= \sum_m m_{(0)} \otimes \cdots \otimes m_{(i-1)} \otimes m_{(i)(0)} \otimes \cdots$$
$$\cdots \otimes m_{(i)(n-1)} \otimes m_{(i+1)} \otimes \cdots \otimes m_{(n-1)}.$$

It proves the equallities (5) and (6), i.e. M is a left n-ary C^*-module.

Since, the module C is a finite-generated projective k-module, then $(C^*)^* \simeq C$ and therefore the converse statement follows. □

Theorem 2.2. *Let C be an n-ary bialgebra. If M is a right n-ary C-comodule algebra, then M is a left n-ary C^*-module algebra.*

Proof. Suppose that M is a right n-ary C-comodule algebra and C is an n-ary bialgebra. We shall show that M is left n-ary C^*-module algebra with respect to the action (4). It is necessary to prove the following equality:

$$\rho^* [l_1 \otimes \cdots \otimes l_{n-1} \otimes (m_1 \cdots m_n)]$$
$$= \sum_{l_1,\ldots,l_{n-1}} \rho^* \left(l_{(1)(1)} \otimes \cdots \otimes l_{(n-1)(1)} \otimes m_1\right) \times$$
$$\cdots \times \rho^* \left(l_{(1)(n)} \otimes l_{(n-1)(n)} \otimes m_n\right).$$

By (3) and (4), we have:

$$\rho^* [l_1 \otimes \cdots \otimes l_{n-1} \otimes (m_1 \cdots m_n)]$$
$$= \sum (m_1 \cdots m_n)_{(0)} l_1 \left((m_1 \cdots m_n)_{(1)}\right) \cdots l_{n-1} \left((m_1 \cdots m_n)_{(n-1)}\right). \tag{7}$$

But, the map $\rho : M \to M \otimes C^{\otimes(n-1)}$ is a homomorphism of n-ary algebras, so:

$$\rho(m_1 \cdots m_n) = \sum (m_1 \cdots m_n)_{(0)} \otimes \cdots \otimes (m_1 \cdots m_n)_{(n-1)},$$

$$\rho(m_1) \cdots \rho(m_n) =$$

$$\left(\sum_{m_1} m_{1(0)} \otimes \cdots \otimes m_{1(n-1)}\right) \cdots \left(\sum_{m_n} m_{n(0)} \otimes \cdots \otimes m_{n(n-1)}\right)$$

$$= \sum_{m_1,\dots,m_n} \left(m_{1(0)} \cdots m_{n(0)}\right) \otimes \cdots \otimes \left(m_{1(n-1)} \otimes \cdots \otimes m_{n(n-1)}\right).$$

Consequently,

$$(m_1 \cdots m_n)_{(0)} = \sum m_{1(0)} \cdots m_{n(0)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(m_1 \cdots m_n)_{(n-1)} = \sum m_{1(n-1)} \cdots m_{n(n-1)}.$$

So in (7), we have:

$$\rho^* \left(l_1 \otimes \cdots \otimes l_{n-1} \otimes (m_1 \cdots m_n)\right.$$

$$= \sum \left(m_{1(0)} \cdots m_{n(0)}\right) l_1 \left(m_{1(1)} \cdots m_{n(1)}\right) \cdots l_{n-1} \left(m_{1(n-1)} \cdots m_{n(n-1)}\right)$$

$$= \sum_{m,l} m_{1(0)} \cdots m_{n(0)} l_{1(1)} \left(m_{1(1)}\right) \times$$

$$\cdots l_{1(n)} \left(m_{n(1)}\right) \cdots l_{n-1(1)} \left(m_{1(n-1)}\right) \cdots l_{n-1(n)} \left(m_{n(n-1)}\right).$$

But, it is equal to:

$$\left[\sum_{m_1} m_{1(0)} l_{1(1)} \left(m_{1(1)}\right) \cdots l_{n-1(1)} \left(m_{1(n-1)}\right)\right] \times$$

$$\cdots \left[\sum_{m_1} m_{n(0)} l_{1(n)} \left(m_{n(1)}\right) \cdots l_{n-1(n)} \left(m_{n(n-1)}\right)\right].$$

□

3. Dual situation

Theorem 3.1. *Let C be an n-ary algebra. Then M is a right n-ary C-module M iff M is a left n-ary C^*-comodule.*

Proof. Suppose that $m \in M$ and the submodule $mC^{\otimes(n-1)} \subseteq M$ is contained in the span of linearly indepedent vectors $a_1, \dots, a_t$. Then, for $c_1, \dots, c_{n-1} \in C$, we have:

$$m\left(c_1 \otimes c_{n-1}\right) = f_1\left(c_1 \otimes \cdots \otimes c_{n-1}\right) a_1 + \cdots + f_t\left(c_1 \otimes \cdots \otimes c_{n-1}\right) a_t \quad (8)$$

where $f_1, \dots, f_t \in \left[C^{\otimes(n-1)}\right]^* = (C^*)^{\otimes(n-1)}$, and so C is finitely generated. If the system of vectors $\{a_1, \dots, a_t\}$ can be enlarged by a vector a_{t+1} and then we can put $f_{t+1} = 0$.

Lemma 3.2. *Let $b_1, \dots, b_d \in \langle a_1, \dots, a_t \rangle$ be given and $g_1, \dots, g_d \in (C^*)^{\otimes(n-1)}$, such that for all $c_1, \dots, c_{n-1} \in C$:*

$$m(c_1 \otimes \cdots \otimes c_{n-1}) = \sum_{i=1}^{d} g_i (c_1 \otimes c_{n-1}) b_i.$$

Then $\sum_{i=1}^{d} g_i \otimes b_i = \sum_{j=1}^{t} f_j \otimes a_j$ holds in $(C^*)^{\otimes(n-1)} \otimes M$.

Proof. By the assumption $b_t = \sum_{j=1}^{t} \alpha_{ij} a_j$ where $\alpha_{ij} \in k$. Then

$$m(c_1 \otimes \cdots \otimes c_{n-1}) = \sum_{i=1}^{d} g_i (c_1 \otimes c_{n-1}) \sum_{j=1}^{t} \alpha_{ij} a_j$$
$$= \sum_{j=1}^{t} \left[\sum_{i=1}^{d} g_i (c_1 \otimes \cdots \otimes c_{n-1}) \alpha_{ij} \right] a_j.$$

Since $a_1, \dots, a_t$ are linearly indepedent, for all j by (8) we get:

$$f_j (c_1 \otimes \cdots c_{n-1}) = \sum_{i=1}^{d} g_i (c_1 \otimes \cdots \otimes c_{n-1}) \alpha_{ij}.$$

Thus, $f_j = \sum_{i=1}^{d} g_j \alpha_{ij}$ in $(C^*)^{\otimes(n-1)}$. Then

$$\sum_{j=1}^{t} f_j \otimes a_j = \sum_{j=1}^{t} \sum_{i=1}^{d} g_i \alpha_{ij} \otimes a_j = \sum_{i=1}^{d} \left(g_i \otimes \sum_{j=1}^{t} \alpha_{ij} a_j \right) = \sum_{i=1}^{d} g_i \otimes b_i.$$

□

Define the map $\rho : M \to (C^*)^{\otimes(n-1)} \otimes M$ by the following rule: if (8) holds, then we put

$$\rho(m) = f_1 \otimes a_1 + \cdots + f_t \otimes a_t. \tag{9}$$

By Lemma 3.2 this definition is correct. Let us show now that if M is a right n-ary C-module, then M is a left n-ary C^*-comodule with respect to (9). In fact,

$$\left(1_{C^*}^{\otimes(n-1)} \otimes \rho\right) \rho(m) = f_1 \otimes \rho(m_1) + \cdots + f_t \otimes \rho(m_t).$$

But $\rho(m_i)=\sum_j f_{ij}\otimes m_j$ where $f_{ij}\in (C^*)^{\otimes(n-1)}$. So

$$\left(1_{C^*}^{\otimes(n-1)}\otimes\rho\right)\rho(m)=\sum_{i,j} f_i\otimes f_{ij}\otimes m_j.$$

On the other hand,

$$\begin{aligned}&\left(1_{C^*}^{\otimes i}\otimes\Delta\otimes 1_{C^*}^{\otimes(n-i-2)}\otimes 1_M\right)\rho(m)\\ &=\sum_j\left(1_{C^*}^{\otimes i}\otimes\Delta\otimes 1_{C^*}^{\otimes(n-i-2)}\right) f_j\otimes m_j.\end{aligned}$$

Further, by associativity Definition 1.2, for all $c_1,\dots,c_{2n-2}\in C$ and all $m\in M$, for any i we have

$$\begin{aligned}&[m\,(c_1\otimes\cdots\otimes c_{n-1})]\,(c_n\otimes\cdots\otimes c_{2n-2})\\ &=m\,[c_1\otimes\cdots\otimes c_i\otimes(c_{i+1}\otimes\cdots\otimes c_{i+n})\otimes c_{i+n+1}\otimes\cdots\otimes c_{2n-2}]\end{aligned}\tag{10}$$

Suppose that $m_1,\dots,m_t$ be as above. Then, as in (8)

$$\begin{aligned}&m_j\,(c_n\otimes\cdots\otimes c_{2n-2})\\ &\qquad=g_{j1}\,(c_n\otimes\cdots\otimes c_{2n-2})\,m_1+\cdots+g_{jt}\,(c_n\otimes\cdots\otimes c_{2n-2})\,m_t\end{aligned}$$

where $g_{j1},\dots,g_{jt}\in (C^*)^{\otimes(n-1)}$. So, by (8) and (9) , we have

$$\left(1_{C^*}^{\otimes(n-1)}\otimes\rho\right)\rho(m)=\sum_{i,j} f_i\otimes g_{ij}\otimes m_j.$$

By (10) for all j:

$$\begin{aligned}&\sum_i f_i\,(c_1\otimes\cdots\otimes c_{n-1})\,g_{ij}\,(c_n\otimes\cdots\otimes c_{2n-2})\\ &=f_j\,[c_1\otimes\cdots\otimes c_i\otimes(c_{i+1}\otimes\cdots\otimes c_{i+n})\otimes c_{i+n+1}\otimes\cdots\otimes c_{2n-2}]\,.\end{aligned}$$

In other words we have that in $(C^*)^{\otimes(2n-2)}$ that

$$\sum_i f_i\otimes g_{ij}=\left(1_{C^*}^{\otimes i}\otimes\Delta_{C^*}\otimes 1_{C^*}^{\otimes(n-i-2)}\right) f_j.$$

Tensor-multiplying by m_j and summing on j, we obtain

$$\sum_{i,j} f_i\otimes g_{ij}\otimes m_j=\left(1_{C^*}^{\otimes i}\otimes\Delta_{C^*}\otimes 1_{C^*}^{\otimes(n-i-2)}\right)\sum_j f_j\otimes m_j.$$

But, the left side is equal to $\left(1_{C^*}^{\otimes(n-1)}\otimes\rho\right)\rho(m)$ and right side is equal to $\left(1_{C^*}^{\otimes i}\otimes\Delta_{C^*}\otimes 1_{C^*}^{\otimes(n-i-2)}\right)\rho(m)$. Consequently, M is a left n-ary C^*-comodule. □

Theorem 3.3. *Let C be an n-ary bialgebra, now. If M is a right n-ary C-module algebra, then M is a left n-ary C^*-comodule algebra.*

Proof. Assume that M is a right n-ary C-module algebra. It means that for all $c_1, \ldots, c_{n-1} \in C$ and all $m_1, \ldots, m_n \in M$ we have:

$$\begin{aligned} &(m_1 \cdots m_n)\,(c_1 \otimes \cdots \otimes c_{n-1}) \\ &= \sum_{c_1,\ldots,c_{n-1}} m_1\left(c_{1(1)} \otimes \cdots \otimes c_{n-1(1)}\right) \cdots m_n\left(c_{1(n)} \otimes \cdots \otimes c_{n-1(n)}\right) \end{aligned} \tag{11}$$

Let us show now that M is a left n-ary C^*-comodule algebra, i.e. the map (9) is a homomorphism of n-ary algebras. Suppose that $m_1, \ldots, m_r \in M$ and $a_1, \ldots, a_t$ is a base in $\sum_{j=1}^{r} m_j C^{\otimes(n-1)}$. Assume that $\rho(m_i) = \sum_{j=1}^{t} f_{ij} \otimes a_j$ where $f_{ij} \in (C^*)^{\otimes(n-1)}$ But,

$$\begin{aligned} &\sum_{c_1,\ldots,c_{n-1}} m_1\left(c_{1(1)} \otimes \cdots \otimes c_{n-1(1)}\right) \cdots m_n\left(c_{1(n)} \otimes \cdots \otimes c_{n-1(n)}\right) \\ &\qquad = \sum_{c_1,\ldots,c_{n-1}} \left[\sum_{j_1} f_{1,j_1}\left(c_{1(1)} \otimes \cdots \otimes c_{n-1(1)}\right) a_{j_1}\right] times \\ &\qquad \cdots \times \left[\sum_{j_n} f_{1,j_n}\left(c_{1(n)} \otimes \cdots \otimes c_{n-1(n)}\right) a_{j_n}\right] \\ &\qquad \sum_{j_1,\ldots,j_n} a_{j_1} \cdots a_{j_n} \left[\sum f_{1,j_1}\left(c_{1(1)} \otimes \cdots \otimes c_{n-1(1)}\right) \times \right. \\ &\qquad \left. \cdots \times f_{n,j_n}\left(c_{1(n)} \otimes \cdots \otimes c_{n-1(n)}\right)\right] \end{aligned}$$

For all $j_1, \ldots, j_n$ we have:

$$\begin{aligned} &\sum_{c_1,\ldots,c_{n-1}} f_{1,j_1}\left(c_{1(1)} \otimes \cdots \otimes c_{n-1(1)}\right) \cdots f_{n,j_n}\left(c_{1(n)} \otimes \cdots \otimes c_{n-1(n)}\right) \\ &= (f_{1,j_1} * \cdots * f_{n,j_n})\,(c_1 \otimes \cdots \otimes c_{n-1})\,. \end{aligned}$$

In that way, the right side of (11) is equal to:

$$\sum_{j_1,\ldots,j_n} (f_{1,j_1} * \cdots * f_{n,j_n})\,(c_1 \otimes \cdots \otimes c_{n-1})\, a_{j_1} \cdots a_{j_n}.$$

By Lemma 3.2 $b_* = a_{j_1} \cdots a_{j_n}$ for all $j_1, \ldots, j_n$ and by (11), we obtain that

$$\begin{aligned} \rho\,(m_1 \cdots m_n) &= \sum_{j_1,\ldots,j_n} (f_{1,j_1} * \cdots * f_{n,j_n}) \otimes a_{j_1} \cdots a_{j_n} \\ &= \prod_{s=1}^{n} \left[\sum_{j_s} (f_{s,j_s} \otimes a_{j_s})\right] = \rho(m_1) \cdots \rho(m_n). \end{aligned}$$

□

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