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Algebra in superextensions of groups, II: cancelativity and centers

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Abstract. Given a countable group X we study the algebraic structure of its superextension $\lambda(X)$. This is a right-topological semigroup consisting of all maximal linked systems on X endowed with the operation

$$\mathcal{A} \circ \mathcal{B} = \{C \subset X : \{x \in X : x^{-1}C \in \mathcal{B}\} \in \mathcal{A}\}$$

that extends the group operation of X. We show that the subsemigroup $\lambda^\circ(X)$ of free maximal linked systems contains an open dense subset of right cancelable elements. Also we prove that the topological center of $\lambda(X)$ coincides with the subsemigroup $\lambda^\bullet(X)$ of all maximal linked systems with finite support. This result is applied to show that the algebraic center of $\lambda(X)$ coincides with the algebraic center of X provided X is countably infinite. On the other hand, for finite groups X of order $3 \le |X| \le 5$ the algebraic center of $\lambda(X)$ is strictly larger than the algebraic center of X.

Introduction

After the topological proof (see [HS, p.102], [H2]) of Hindman theorem [H1], topological methods become a standard tool in the modern combinatorics of numbers, see [HS], [P]. The crucial point is that any semigroup operation $*$ defined on any discrete space X can be extended to

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a right-topological semigroup operation on $\beta(X)$, the Stone-Čech compactification of X. The extension of the operation from X to $\beta(X)$ can be defined by the simple formula:

$$\mathcal{U} * \mathcal{V} = \Big\{ \bigcup_{x\in U} x * V_x : U \in \mathcal{U},\ \{V_x\}_{x\in U} \subset \mathcal{V}\Big\}, \tag{1}$$

where $\mathcal{U}, \mathcal{V}$ are ultrafilters on X. Endowed with the so-extended operation, the Stone-Čech compactification $\beta(X)$ becomes a compact right-topological semigroup. The algebraic properties of this semigroup (for example, the existence of idempotents or minimal left ideals) have important consequences in combinatorics of numbers, see [HS], [P].

The Stone-Čech compactification $\beta(X)$ of X is the subspace of the double power-set $\mathcal{P}(\mathcal{P}(X))$, which is a complete lattice with respect to the operations of union and intersection. In [G_2] it was observed that the semigroup operation extends not only to $\beta(X)$ but also to the complete sublattice $G(X)$ of $\mathcal{P}(\mathcal{P}(X))$ generated by $\beta(X)$. This complete sublattice consists of all inclusion hyperspaces over X.

By definition, a family $\mathcal{F}$ of non-empty subsets of a discrete space X is called an *inclusion hyperspace* if $\mathcal{F}$ is monotone in the sense that a subset $A \subset X$ belongs to $\mathcal{F}$ provided A contains some set $B \in \mathcal{F}$. On the set $G(X)$ there is an important transversality operation assigning to each inclusion hyperspace $\mathcal{F} \in G(X)$ the inclusion hyperspace

$$\mathcal{F}^\perp = \{A \subset X : \forall F \in \mathcal{F}\ (A \cap F \neq \emptyset)\}.$$

This operation is involutive in the sense that $(\mathcal{F}^\perp)^\perp = \mathcal{F}$.

It is known that the family $G(X)$ of inclusion hyperspaces on X is closed in the double power-set $\mathcal{P}(\mathcal{P}(X)) = \{0,1\}^{\mathcal{P}(X)}$ endowed with the natural product topology.

The extension of a binary operation $*$ from X to $G(X)$ can be defined in the same way as for ultrafilters, i.e., by the formula (1) applied to any two inclusion hyperspaces $\mathcal{U}, \mathcal{V} \in G(X)$. In [$G_2$] it was shown that for an associative binary operation $*$ on X the space $G(X)$ endowed with the extended operation becomes a compact right-topological semigroup. Besides the Stone-Čech extension, the semigroup $G(X)$ contains many important spaces as closed subsemigroups. In particular, the space

$$\lambda(X) = \{\mathcal{F} \in G(X) : \mathcal{F} = \mathcal{F}^\perp\}$$

of maximal linked systems on X is a closed subsemigroup of $G(X)$. The space $\lambda(X)$ is well-known in General and Categorial Topology as the *superextension* of X, see [vM], [TZ]. Endowed with the extended binary operation, the superextension $\lambda(X)$ of a semigroup X is a supercompact right-topological semigroup containing $\beta(X)$ as a subsemigroup.

The thorough study of algebraic properties of the superextensions of groups was started in [BGN] where we described right and left zeros in $\lambda(X)$ and detected all groups X with commutative superextension $\lambda(X)$ (those are groups of cardinality $|X| \le 4$). In [BGN] we also described the structure of the semigroups $\lambda(X)$ for all finite groups X of cardinality $|X| \le 5$. In [BG$_3$] we shall describe the structure of minimal left ideals of the superextensions of groups. In this paper we concentrate at cancellativity and centers (topological and algebraic) in the superextensions $\lambda(X)$ of groups X. Since $\lambda(X)$ is an intermediate subsemigroup between $\beta(X)$ and $G(X)$ the obtained results for $\lambda(X)$ in a sense are intermediate between those for $\beta(X)$ and $G(X)$.

In section 2 we describe cancelable elements of $\lambda(X)$. In particular, we show that for a finite group X all left or right cancelable elements of $\lambda(X)$ are principal ultrafilters. On the other hand, if a group X is countable, then the set of right cancelable elements has open dense intersection with the subsemigroup $\lambda^\circ(X) \subset \lambda(X)$ of free maximal linked systems, see Theorem 2.4. This resembles the situation with the semigroup $\beta(X)\setminus X$ which contains a dense open subset of right cancelable elements (see [HS, 8.10]), and also with the semigroup $G(X)$ whose right cancelable elements form a subset having open dense intersection with the set $G^\circ(X)$ of free inclusion hyperspaces, see [G$_2$].

The section 3 is devoted to describing the topological center of $\lambda(X)$. By definition, the *topological center* of a right-topological semigroup S is the set $\Lambda(S)$ of all elements $a \in S$ such that the left shift $l_a : S \to S$, $l_a : x \mapsto a * x$, is continuous. By [HS], for every group X the topological center of the semigroup $\beta(X)$ coincides with X. On the other hand, the topological center of the semigroup $G(X)$ coincides with the subspace $G^\bullet(X)$ of $G(X)$ consisting of inclusion hyperspaces with finite support, see [G$_2$, 7.1]. A similar results holds also for the semigroup $\lambda(X)$: for any at most countable group X the topological center of $\lambda(X)$ coincides with $\lambda^\bullet(X)$, see Theorem 3.4.

The final section 4 is devoted to describing the algebraic center of $\lambda(X)$. We recall that the *algebraic center* of a semigroup S consists of all elements $s \in S$ that commute with all other elements of S. In Theorem 4.2 we shall prove that for any countable infinite group X the algebraic center of $\lambda(X)$ coincides with the algebraic center of X. It is interesting to note that for any group X the algebraic centers of the semigroups $\beta(X)$ and $G(X)$ also coincide with the center of the group X, see [HS, 6.54] and [G$_2$, 6.2]. In contrast, for finite groups X of cardinality $3 \le |X| \le 5$ the algebraic center of $\lambda(X)$ is strictly larger than the algebraic center of X, see Remark 4.4.

1. Inclusion hyperspaces and superextensions

In this section we recall the necessary definitions and facts.

A family $\mathcal{L}$ of subsets of a set X is called a *linked system* if $A\cap B\neq\emptyset$ for all $A,B\in\mathcal{L}$. Such a linked system $\mathcal{L}$ is *maximal linked* if $\mathcal{L}$ coincides with any linked system $\mathcal{L}'$ on X that contains $\mathcal{L}$. Each (ultra)filter on X is a (maximal) linked system. By $\lambda(X)$ we denote the family of all maximal linked systems on X. Since each ultrafilter on X is a maximal linked system, $\lambda(X)$ contains the Stone-Čech extension $\beta(X)$ of X. It is easy to see that each maximal linked system on X is an inclusion hyperspace on X and hence $\lambda(X)\subset G(X)$. Moreover, it can be shown that $\lambda(X)=\{\mathcal{A}\in G(X):\mathcal{A}=\mathcal{A}^\perp\}$, see [G$_1$].

By [G$_1$] the subspace $\lambda(X)$ is closed in the space $G(X)$ endowed with the topology generated by the sub-base consisting of the sets

$$U^+=\{\mathcal{A}\in G(X):U\in\mathcal{A}\}\text{ and }U^-=\{\mathcal{A}\in G(X):U\in\mathcal{A}^\perp\}$$

where U runs over subsets of X. By [G$_1$] and [vM] the spaces $G(X)$ and $\lambda(X)$ are supercompact in the sense that any their cover by the sub-basic sets contains a two-element subcover. Observe that $U^+\cap\lambda(X)=U^-\cap\lambda(X)$ and hence the topology on $\lambda(X)$ is generated by the sub-basis consisting of the sets

$$U^\pm=\{\mathcal{A}\in\lambda(X):U\in\mathcal{A}\},\ \ U\subset X.$$

We say that an inclusion hyperspace $\mathcal{A}\in G(X)$

- has *finite support* if there is a finite family $\mathcal{F}\subset\mathcal{A}$ of finite subsets of X such that each set $A\in\mathcal{A}$ contains a set $F\in\mathcal{F}$;
- is *free* if for each $A\in\mathcal{A}$ and each finite subset $F\subset X$ the complement $A\setminus F$ belongs to $\mathcal{A}$.

By $G^\bullet(X)$ we denote the subspace of $G(X)$ consisting of inclusion hyperspaces with finite support and $G^\circ(X)$ stands for the subset of free inclusion hyperspaces on X. Those two sets induce the subsets

$$\lambda^\bullet(X)=G^\bullet(X)\cap\lambda(X)\text{ and }\lambda^\circ(X)=G^\circ(X)\cap\lambda(X)$$

in the superextension $\lambda(X)$ of X. By [G$_1$], $\lambda^\bullet(X)$ is an open dense subset of $\lambda(X)$ while $\lambda^\circ(X)$ is closed and nowhere dense in $\lambda(X)$.

Given any semigroup operation $*: X \times X \to X$ on a set X we can extend this operation to $G(X)$ letting

$$\mathcal{U} * \mathcal{V} = \Big\{ \bigcup_{x \in U} x * V_x : U \in \mathcal{U},\ \{V_x\}_{x \in U} \subset \mathcal{V} \Big\}$$

for inclusion hyperspaces $\mathcal{U}, \mathcal{V} \in G(X)$. Equivalently, the product $\mathcal{U} * \mathcal{V}$ can be defined as

$$\mathcal{U} * \mathcal{V} = \{A \subset X : \{x \in X : x^{-1}A \in \mathcal{V}\} \in \mathcal{U}\} \tag{2}$$

where $x^{-1}A = \{z \in X : x * z \in A\}$. By [$G_2$] the so-extended operation turns $G(X)$ into a right-topological semigroup. The structure of this semigroup was studied in details in [G_2]. In this paper we shall concentrate at the study of the algebraic structure of the semigroup $\lambda(X)$ for a group X.

The formula (2) implies that the product $\mathcal{U} * \mathcal{V}$ of two maximal linked systems $\mathcal{U}$ and $\mathcal{V}$ is a principal ultrafilter if and only if both $\mathcal{U}$ and $\mathcal{V}$ are principal ultrafilters. So we get the following

Proposition 1.1. *For any group X the set $\lambda(X) \setminus X$ is a two-sided ideal in $\lambda(X)$.*

2. Cancelable elements of $\lambda(X)$

In this section, given a group X we shall detect cancelable elements of $\lambda(X)$.

We recall that an element x of a semigroup S is *right* (resp. *left*) *cancelable* if for every $a, b \in X$ the equation $x * a = b$ (resp. $a * x = b$) has at most one solution $x \in S$. This is equivalent to saying that the right (resp. left) shift $r_a : S \to S$, $r_a : x \mapsto x * a$, (resp. $l_a : S \to S$, $l_a : x \mapsto a * x$) is injective.

Proposition 2.1. *Let G be a finite group. If $\mathcal{C} \in \lambda(G)$ is left or right cancelable, then $\mathcal{C}$ is a principal ultrafilter.*

Proof. Assume that some maximal linked system $a \in \lambda(G) \setminus G$ is left cancelable. This means that the left shift $l_a : \lambda(G) \to \lambda(G)$, $l_a : x \mapsto a \circ x$, is injective. By Proposition 1.1, the set $\lambda(G) \setminus G$ is an ideal in $\lambda(G)$. Consequently, $l_a(\lambda(G)) = a * \lambda(G) \subset \lambda(G) \setminus G$. Since $\lambda(G)$ is finite, l_a cannot be injective. □

Thus the semigroups $\lambda(X)$ can have non-trivial cancelable elements only for infinite groups X. According to [HS, 8.11] an ultrafilter $\mathcal{U} \in \beta(X)$

is right cancelable if and only if the orbit $\{x\mathcal{U} : x \in X\}$ is discrete in $\beta(X)$ if and only if for every $x \in X$ there is a set $U_x \in \mathcal{U}$ such that the indexed family $\{x * U_x : x \in X\}$ is disjoint.

This characterization admits a partial generalization to the semigroup $G(X)$. According to [G$_2$] if an inclusion hyperspace $\mathcal{A} \in G(X)$ is right cancelable in $G(X)$, then its orbit $\{x * \mathcal{A} : x \in X\}$ is discrete in $G(X)$. On the other hand, $\mathcal{A}$ is cancelable provided for every $x \in X$ there is a set $A_x \in \mathcal{A} \cap \mathcal{A}^\perp$ such that the indexed family $\{x * A_x : x \in X\}$ is disjoint. The latter means that $x * A_x \cap y * A_y = \emptyset$ for any distinct points $x, y \in X$. This result on right cancelable elements in $G(X)$ will help us to prove a similar result on the right cancelable elements in the semigroup $\lambda(X)$.

Theorem 2.2. *Let X be a group and $\mathcal{L} \in \lambda(X)$ be a maximal linked system on X.*

1. *If $\mathcal{L}$ is right cancelable in $\lambda(X)$, then the orbit $\{x\mathcal{L} : x \in X\}$ is discrete in $\lambda(X)$ and $x\mathcal{L} \neq y\mathcal{L}$ for all $x, y \in X$.*
2. *$\mathcal{L}$ is right cancelable in $\lambda(X)$ provided for every $x \in X$ there is a set $S_x \in \mathcal{L}$ such that the family $\{x * S_x : x \in X\}$ is disjoint.*

Proof. 1. First note that the right cancelativity of a maximal linked system $\mathcal{L} \in \lambda(X)$ is equivalent to the injectivity of the map $\mu_X \circ \lambda \bar{R}_{\mathcal{L}} : \lambda(X) \to \lambda(X)$, see [G$_2$]. We recall that $\mu_X : \lambda^2(X) \to \lambda(X)$ is the multiplication of the monad $\lambda = (\lambda, \mu, \eta)$ while $\bar{R}_{\mathcal{L}} : \beta(X) \to \lambda(X)$ is the Stone-Čech extension of the right shift $R_{\mathcal{L}} : X \to \lambda(X)$, $R_{\mathcal{L}} : x \mapsto x * \mathcal{L}$. The map $\bar{R}_{\mathcal{L}}$ certainly is not injective if $R_{\mathcal{L}}$ is not an embedding, which is equivalent to the discreteness of the indexed set $\{x * \mathcal{L} : x \in X\}$ in $\lambda(X)$.

2. Assume that $\{S_x\}_{x \in X} \subset \mathcal{L}$ is a family such that $\{x * S_x : x \in X\}$ is disjoint. To prove that $\mathcal{L}$ is right cancelable, take two maximal linked systems $\mathcal{A}, \mathcal{B} \in \lambda(X)$ with $\mathcal{A} \circ \mathcal{L} = \mathcal{B} \circ \mathcal{L}$. It is sufficient to show that $\mathcal{A} \subset \mathcal{B}$. Take any set $A \in \mathcal{A}$ and observe that the set $\bigcup_{a \in A} aS_a$ belongs to $\mathcal{A} \circ \mathcal{L} = \mathcal{B} \circ \mathcal{L}$. Consequently, there is a set $B \in \mathcal{B}$ and a family of sets $\{L_b\}_{b \in B} \subset \mathcal{L}$ such that

$$\bigcup_{b \in B} bL_b \subset \bigcup_{a \in A} aS_a.$$

It follows from $S_b \in \mathcal{L}$ that $L_b \cap S_b$ is not empty for every $b \in B$.

Since the sets aS_a and bS_b are disjoint for different $a, b \in X$, the inclusion

$$\bigcup_{b \in B} b(L_b \cap S_b) \subset \bigcup_{b \in B} bL_b \subset \bigcup_{a \in A} aS_a$$

implies $B \subset A$ and hence $A \in \mathcal{B}$. □

It is interesting to remark that the first item gives a necessary but not sufficient condition of the right cancelability in $\lambda(X)$ (in contrast to the situation in $\beta(X)$).

Example 2.3. By [BGN, 6.3], the superextension $\lambda(C_4)$ of the 4-element cyclic group C_4 is isomorphic to the direct product $C_4 \times C_2^1$, where $C_2^1 = C_2 \cup \{e\}$ is the 2-element cyclic group with attached external unit e (the latter means that $ex = xe = x$ for all $x \in C_2^1$). Consequently, each element of the ideal $\lambda(C_4) \setminus C_4$ is not cancelative but has the discrete 4-element orbit $\{x\mathcal{L} : x \in C_4\}$. In fact all the (left or right) cancelable elements of $\lambda(C_4)$ are principal ultrafilters, see Proposition 2.1.

According to [HS, 8.10], for each infinite group the semigroup $\beta(X)$ contains many right cancelable elements. In fact, the set of right cancelable elements contains an open dense subset of $\beta(X) \setminus X$. A similar result holds also for the semigroup $G(X)$ over a countable group X: the set of right cancelable elements of $G(X)$ contains an open dense subset of the subsemigroup $G^\circ(X)$. Theorem 2.2 will help us to prove a similar result for the semigroup $\lambda(X)$.

Theorem 2.4. *For each countable group X the subsemigroup $\lambda^\circ(X)$ of free maximal linked systems contains an open dense subset consisting of right cancelable elements in the semigroup $\lambda(X)$.*

Proof. Let $X = \{x_n : n \in \omega\}$ be an injective enumeration of the countable group X. Given a free maximal linked system $\mathcal{L} \in \lambda^\circ(X)$ and a neighborhood $O(\mathcal{L})$ of $\mathcal{L}$ in $\lambda^\circ(X)$, we should find a non-empty open subset of right cancelable elements in $O(\mathcal{L})$. Without loss of generality, the neighborhood $O(\mathcal{L})$ is of basic form:

$$O(\mathcal{L}) = \lambda^\circ(X) \cap U_0^\pm \cap \dots \cap U_{n-1}^\pm$$

for some sets $U_1, \dots, U_{n-1}$ of X. Those sets are infinite because $\mathcal{L}$ is free. We are going to construct an infinite set $C = \{c_n : n \in \omega\} \subset X$ that has infinite intersection with the sets U_i, $i < n$, and such that for any distinct $x, y \in X$ the intersection $xC \cap yC$ is finite. The points c_k, $k \in \omega$, composing the set C will be chosen by induction to satisfy the following conditions:

- $c_k \in U_j$ where $j = k \mod n$;
- c_k does not belong to the finite set

$$F_k = \{z \in X : \exists i, j \le k \; \exists l < k \;\; (x_i z = x_j c_l)\}.$$

It is clear that the so-constructed set $C = \{c_k : k \in \omega\}$ has infinite intersection with each set U_i, $i < n$. The choice of the points c_k for $k > j$ implies that $x_iC \cap x_jC \subset \{x_ic_m : m \le j\}$ is finite.

Now let $\mathcal{C}$ be a free maximal linked system on X enlarging the linked system generated by the sets C and $U_0, \dots, U_{n-1}$. It is clear that $\mathcal{C} \in O(\mathcal{L})$. Consider the open neighborhood

$$O(\mathcal{C}) = O(\mathcal{L}) \cap C^{\pm}$$

of $\mathcal{C}$ in $\lambda^\circ(X)$.

We claim that each maximal linked system $\mathcal{A} \in O(\mathcal{C})$ is right cancelable in $\lambda(X)$. This will follow from Proposition 2.2 as soon as we construct a family of sets $\{A_i\}_{i\in\omega} \in \mathcal{A}$ such that $x_iA_i \cap x_jA_j = \emptyset$ for any numbers $i < j$. Observe that the sets

$$A_i = C \setminus \{x_i^{-1}x_kc_m : k, m \le i\}, \quad i \in \omega,$$

have the required property. □

By [HS, 8.34], the semigroup $\beta(\mathbb{Z}) \setminus \mathbb{Z}$ contains an open dense subset consiting of free ultrafilters that are left cancelable in $\beta(\mathbb{Z})$. On the other hand, by [G_2, 8.1], the only left cancelable elements of the semigroup $G(\mathbb{Z})$ are principal ultrafilters.

Problem 2.5. *Is some free maximal linked system left cancelable in the semigroup $\lambda(\mathbb{Z})$?*

3. The topological center of $\lambda(X)$

In this section we describe the topological center of the superextension $\lambda(X)$ of a group X. By definition, the *topological center* of a right-topological semigroup S is the set $\Lambda(S)$ of all elements $a \in S$ such that the left shift $l_a : S \to S$, $l_a : x \mapsto a * x$, is continuous.

By [HS, 4.24, 6.54], for every group X the topological center of the semigroup $\beta(X)$ coincides with X. On the other hand, the topological center of the semigroup $G(X)$ coincides with $G^\bullet(X)$, see [G_2, 7.1]. A similar results holds also for the semigroup $\lambda(X)$: the topological center of $\lambda(X)$ coincides with $\lambda^\bullet(X)$ (at least for countable groups X).

To prove this result we shall use so-called detecting ultrafilters.

Definition 3.1. A free ultrafilter $\mathcal{D}$ on a group X is called *detecting* if there is an indexed family of sets $\{D_x : x \in X\} \subset \mathcal{D}$ such that for any $A \subset X$

1. the set $U_A = \bigcup_{x\in A} xD_x$ has the property: $U_A \cup yU_A \ne X$ for all $y \in X$;
2. for every $D \in \mathcal{D}$ the set $\{x \in X : xD \subset U_A\}$ is finite and lies in A.

Lemma 3.2. *On each countable group X there is a detecting ultrafilter.*

Proof. Let $X = \{x_n : n \in \omega\}$ be an injective enumeration of the group X such that x_0 is the neutral element of X. For every $n \in \omega$ let $F_n = \{x_i, x_i^{-1} : i \le n\}$. Let $a_0 = x_0$ and inductively, for every $n \in \omega$ choose an element $a_n \in X$ so that

$$a_n \notin F_n^{-1} F_n A_{<n} \quad \text{where} \quad A_{<n} = \{a_i : i < n\}.$$

For every $n \in \omega$ let $A_{\ge n} = \{a_i : i \ge n\}$. Let also $D_0 = \{a_{2i} : i \in \omega\}$.

Let us show that for any distinct numbers n, m the intersection $x_n A_{\ge n} \cap x_m A_{\ge m}$ is empty. Otherwise there would exist two numbers $i \ge n$ and $j \ge m$ such that $x_n a_i = x_m a_j$. It follows from $x_n \ne x_m$ that $i \ne j$. We lose no generality assuming that $j > i$. Then $x_n a_i = x_m a_j$ implies that

$$a_j = x_m^{-1} x_n a_i \in F_j^{-1} F_j A_{<j},$$

which contradicts the choice of a_j.

Let $\mathcal{D} \in \beta(X)$ be any free ultrafilter such that $D_0 \in \mathcal{D}$ and $\mathcal{D}$ is not a P-point. To get such an ultrafilter, take $\mathcal{D}$ to be a cluster point of any countable subset of $D_0^{\pm} \cap \beta(X) \setminus X$. Using the fact that $\mathcal{D}$ fails to be a P-point, we can take a decreasing sequence of sets $\{V_n : n \in \omega\} \subset \mathcal{D}$ having no pseudointersection in $\mathcal{D}$. The latter means that for every $D \in \mathcal{D}$ the almost inclusion $D \subset^* V_n$ (which means that $D \setminus V_n$ is finite) holds only for finitely many numbers n.

For every $n \in \omega$ let $D_n = V_n \cap A_{\ge n} \cap D_0$. We claim that the ultrafilter $\mathcal{D}$ and the family $(D_n)_{n\in\omega}$ satisfy the requirements of Definition 3.1.

Take any subset $A \subset \omega$ and consider the set $U_A = \bigcup_{n\in A} x_n D_n$.

First we verify that $U_A \cup yU_A \ne X$ for each $y \in X$. Find $m \in \omega$ with $y^{-1} = x_m$ and take any odd number $k > m$. We claim that $a_k \notin U_A \cup yU_A$. Otherwise, $a_k \in x_n D_n \cup x_m^{-1} x_n D_n$ for some $n \in A$. It follows that $a_k = x_n a_i$ or $a_k = x_m^{-1} x_n a_i$ for some even $i \ge n$. If $k > i$, then both the equalities are forbidden by the choice of $a_k \notin F_k^{-1} F_k A_{<k} \supset \{x_n a_i, x_m^{-1} x_n a_i\}$. If $k < i$, then those equalities are formidden by the choice of

$$a_i \notin F_i^{-1} F_i A_{<i} \supset \{x_n^{-1} a_k, x_n^{-1} x_m^{-1} a_k\}.$$

Therefore, $U_A \cup yU_A \ne X$.

Next, given arbitrary $D \in \mathcal{D}$ we show that the set $S = \{n \in \omega : x_n D \subset U_A\}$ is finite and lies in A. First we show that $S \subset A$. Assuming the converse, we could find $n \in S \setminus A$. Then $x_n(D \cap D_n) \subset x_n D \subset U_A = \bigcup_{m \in A} x_m D_m$, which is not possible because the set $x_n D_n$ misses the union U_A. Thus $S \subset A$. Next, we show that S is finite. By the choice of the sequence (V_n), the set $F = \{n \in \omega : D \cap D_0 \subset^* V_n\}$ is finite. We claim that $S \subset F$. Indeed, take any $m \in S$. It follows from $x_m D \subset U_A = \bigcup_{n \in A} x_n D_n$ and $x_m A_{\geq m} \cap \bigcap_{n \neq m} x_n D_n = \emptyset$ that

$$x_m(D \cap D_0) \subset^* x_m(D \cap A_{\geq m}) \subset x_m D_m \subset x_m V_m$$

and hence $m \in F$. □

Theorem 3.3. *Let X be a group admitting a detecting ultrafilter $\mathcal{D}$. For a maximal linked system $\mathcal{A} \in \lambda(X)$ the following conditions are equivalent:*

1. *the left shift $L_\mathcal{A} : G(X) \to G(X)$, $L_\mathcal{A} : \mathcal{F} \mapsto \mathcal{A} \circ \mathcal{F}$, is continuous;*
2. *the left shift $l_\mathcal{A} : \lambda(X) \to \lambda(X)$, $l_\mathcal{A} : \mathcal{L} \mapsto \mathcal{A} \circ \mathcal{L}$, is continuous;*
3. *the left shift $l_\mathcal{A} : \lambda(X) \to \lambda(X)$ is continuous at the detecting ultrafilter $\mathcal{D}$;*
4. *$\mathcal{A} \in \lambda^\bullet(X)$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are trivial while (4) ⇒ (1) follows from Theorem 7.1 [G$_2$] asserting that the topological center of the semigroup $G(X)$ coincides with $G^\bullet(X)$. To prove that (3) ⇒ (4), assume that the left shift $l_\mathcal{A} : \lambda(X) \to \lambda(X)$ is continuous at the detecting ultrafilter $\mathcal{D}$.

We need to show that $\mathcal{A} \in \lambda^\bullet(G)$. By Theorem 8.1 of [G$_1$], it suffices to check that each set $A \in \mathcal{A}$ contains a finite set $F \in \mathcal{A}$.

Since $\mathcal{D}$ is a detecting ultrafilter, there is a family of sets $\{D_x : x \in X\} \subset \mathcal{D}$ such that for every $D \in \mathcal{D}$ the set $\{x \in X : xD \subset \bigcup_{x \in A} xD_x\}$ is finite and lies in A.

Consider the set $U_A = \bigcup_{x \in A} xD_x$ belonging to the product $\mathcal{A} \circ \mathcal{D}$. The continuity of the left shift $l_\mathcal{A} : \lambda(X) \to \lambda(X)$ at $\mathcal{D}$ yields us a set $D \in \mathcal{D}$, such that $l_\mathcal{A}(D^\pm) \subset U_A^\pm$. This means that $U_A \in \mathcal{A} \circ \mathcal{L}$ for any maximal linked system $\mathcal{L} \in \lambda(X)$ that contains D.

The choice of $\mathcal{D}$ and $\{D_x\}_{x \in X}$ guarantees that

$$S = \{x \in X : xD \subset U_A\}$$

is a finite subset lying in A. We claim that there is a maximal linked system $\tilde{\mathcal{L}} \in \lambda(X)$ such that $D \in \tilde{\mathcal{L}}$ and $x^{-1}U_A \notin \tilde{\mathcal{L}}$ for all $x \notin S$. Such a system $\tilde{\mathcal{L}}$ can be constructed as an enlargement of the linked system

$$\mathcal{L} = \{D, X \setminus x^{-1}U_A : x \in X \setminus S\}.$$

The latter system is linked because of the definition of $S = \{x \in X : D \subset x^{-1}U_A\}$ and the property (1) of the family $(D_x)_{x\in X}$ from Definition 3.1.

Take any maximal linked system $\tilde{\mathcal{L}}$ containing $\mathcal{L}$ and observe that $D \in \mathcal{L}$ and

$$\{x \in X : x^{-1}U_A \in \tilde{\mathcal{L}}\} = \{x \in X : x^{-1}U_A \in \mathcal{L}\} = S \subset A.$$

Taking into account that $D \in \mathcal{L}$, we conclude that $\mathcal{A} \circ \mathcal{L} = l_{\mathcal{A}}(\mathcal{L}) \in U_A^{\pm}$ and hence the set $S = \{x \in X : x^{-1}U_A \in \mathcal{L}\} \in \mathcal{A}$. This set S is the required finite subset of A belonging to $\mathcal{A}$. □

Combining Theorem 3.3 with Lemma 3.2 we obtain the main result of this section.

Corollary 3.4. *For any countable group X the topological center of the semigroup $\lambda(X)$ coincides with $\lambda^{\bullet}(X)$.*

Question 3.5. *Is Theorem 3.4 true for a group X of arbitrary cardinality?*

4. The algebraic center of $\lambda(X)$

This section is devoted to studying the algebraic center of $\lambda(X)$. We recall that the *algebraic center* of a semigroup S consists of all elements $s \in S$ that commute with all other elements of S. Such elements s are called *central* in S.

Lemma 4.1. *Let X be a group with the neutral element e. A maximal linked system $\mathcal{A} \in \lambda(X)$ is not central in $\lambda(X)$ provided there are sets $S, T \subset X$ such that*

1. *$|T| = 3$;*
2. *for each $A \in \mathcal{A}$ we get $A \cap S \in \mathcal{A}$ and $|A \cap S| \geq 2$;*
3. *there is a finite set $B \in \mathcal{A}$ such that $BS^{-1} \cap T^{-1}T \subset \{e\}$.*

Proof. We claim that $\mathcal{A}$ does not commute with the maximal linked system $\mathcal{T} = \{A \subset X : |A \cap T| \geq 2\}$. By (3), the maximal linked system $\mathcal{A}$ contains a finite set $B \in \mathcal{A}$ such that $BS^{-1} \cap TT^{-1} \subset \{e\}$. By (2), we can assume that $B \subset S$ and B is minimal in the sense that each $B' \subset B$ with $B' \in \mathcal{A}$ is equal to B. By (2), $|B| \geq 2$. Choose a family $\{T_b\}_{b\in B}$ of 2-element subsets of T such that $\bigcup_{b\in B} T_b = T$. Such a choice is possible because $|B| \geq 2$. The union $\bigcup_{b\in B} bT_b$ belongs to $\mathcal{A} \circ \mathcal{T} = \mathcal{T} \circ \mathcal{A}$ and hence we can find a subset $D \in \mathcal{T}$ and a family $\{A_d\}_{d\in D} \subset \mathcal{A}$ with

$\bigcup_{d\in D} dA_d \subset \bigcup_{b\in B} bT_b$. By (2), we can assume that each $A_d \subset S$. Replacing D by a smaller set, if necessary, we can assume that $D \subset T$ and $|D| = 2$. We claim that $A_d = B$ for all $d \in D$ and $T_b = D$ for all $b \in B$.

Indeed, take any $d \in D$ and any $a \in A_d$. Since $da \in \bigcup_{x\in D} xA_x \subset \bigcup_{b\in B} bT_b$, there are $b \in B$ and $t \in T_b$ with $da = bt$. Then $T^{-1}T \ni t^{-1}d = ba^{-1} \in BA_d^{-1} \subset BS^{-1}$. Taking into account that $T^{-1}T \cap BS^{-1} \subset \{e\}$, we conclude that $t^{-1}d = ba^{-1}$ is the neutral element of X. Consequently, $a = b \in B$ and $d = t \in T_b$. Since $a \in A_d$ was arbitrary, we get $\mathcal{A} \ni A_d \subset B$. The minimality of $B \in \mathcal{A}$ implies that $A_d = B$. It follows from $d = t \in T_b$ for $d \in D$ that $D \subset T_b$. Since $|D| = |T_b| = 2$, we get $D = T_b$ for every $b \in B = A_d$. Consequently, $D = \bigcup_{b\in B} T_b = T$ which contradictions (1). □

By [HS, 6.54], for every group X the algebraic center of the semigroups $\beta(X)$ coincides with the center of the group X. Consequently, the semigroup $\beta(X) \setminus X$ contains no central elements. A similar result holds also for the semigroup $\lambda(X)$.

Theorem 4.2. *For any countable infinite group X the algebraic center of $\lambda(X)$ coincides with the algebraic center of X.*

Proof. It is clear that all central elements of X are central in $\lambda(X)$. Now assume that a maximal linked system $\mathcal{C} \in \lambda(X)$ is a central element of the semigroup $\lambda(X)$. Observe that the left shift $l_{\mathcal{C}} : \lambda(X) \to \lambda(X)$, $l_{\mathcal{C}} : \mathcal{X} \mapsto \mathcal{C} \circ \mathcal{X}$, is continuous because it coincides with the right shift $r_{\mathcal{C}} : \lambda(X) \to \lambda(X)$, $r_{\mathcal{C}} : \mathcal{X} \mapsto \mathcal{X} \circ \mathcal{C}$. Consequently, $\mathcal{C}$ belongs to the topological center of $\lambda(X)$. Applying Theorem 3.4, we conclude that $\mathcal{C} \in \lambda^\bullet(X)$. We claim that $\mathcal{C}$ is a principal ultrafilter.

Assuming the converse, consider the family $\mathcal{C}_0$ of minimal finite subsets in $\mathcal{C}$. Since $\mathcal{C} \in \lambda^\bullet(X)$, the family $\mathcal{C}_0$ is finite and hence has finite union $S = \cup\mathcal{C}_0$. Take any set $B \in \mathcal{C}_0$ and observe that $|B| \ge 2$ (because $\mathcal{C}$ is not a principal ultrafilter).

Since the group X is infinite, we can choose a 3-element subset $T \subset X$ such that $T^{-1}T \cap BS^{-1} \subset \{e\}$. Now we see that the maximal linked system $\mathcal{C}$ satisfies the conditions of Lemma 4.1 and hence is not central in $\lambda(X)$, which is a contradiction. □

We do not know if Theorem 4.2 is true for any infinite group X.

Question 4.3. *Let X be an infinite group. Does the algebraic center of $\lambda(X)$ coincides with the algebraic center of X?*

Remark 4.4. Theorem 4.2 certainly is not true for finite groups. According to [BGN, § 6], for any group X of cardinality $3 \le |X| \le 5$ the semigroup $\lambda(X)$ contains a central element, which is not a principal ultrafilter.

Problem 4.5. *Characterize (finite) abelian groups X whose superextensions $\lambda(X)$ have central elements distinct from principal ultrafilters. Have all such groups X cardinality $|X| \le 5$?*

It is interesting to remark that the semigroup $\lambda(X)$ contains many non-principal maximal linked systems that commute with all ultrafilters.

Proposition 4.6. *Let X be a group and $Y, Z \subset X$ be non-empty subsets such that $yz = zy$ for all $y \in Y$, $z \in Z$. Then for any $\mathcal{L} \in \lambda^\bullet(Y) \subset \lambda^\bullet(X)$ and $\mathcal{U} \in \beta(Z) \subset \beta(X)$ we get $\mathcal{L} \circ \mathcal{U} = \mathcal{U} \circ \mathcal{L}$.*

Proof. It is sufficient to prove that $\mathcal{L} \circ \mathcal{U} \subset \mathcal{U} \circ \mathcal{L}$. Let $\bigcup_{x \in L} x * U_x \in \mathcal{L} \circ \mathcal{U}$. Without loss of generality we may assume that $L = \{x_1, \dots, x_n\}$ is finite, $L \subset Y$ and $U_{x_i} \subset Z$. Denote $V = U_{x_1} \cap \dots \cap U_{x_n} \in \mathcal{U}$. Then

$$\bigcup_{x \in L} x * U_x = \bigcup_{x \in L} U_x * x \supset V * L \in \mathcal{U} \circ \mathcal{L}.$$

It follows that $\bigcup_{x \in L} x * U_x \in \mathcal{U} \circ \mathcal{L}$ and the proof is complete. □

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