Algebra and Discrete Mathematics
Number 3. **(2008).** pp. 88 – 97

RESEARCH ARTICLE

On a question of A. N. Skiba about totally saturated formations

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Communicated by L. A. Shemetkov

Abstract. It is proved that the lattice of τ-closed totally saturated formations of finite groups is distributive. This is a solution of Question 4. 2. 15 proposed by A. N. Skiba in his monograph "Algebra of Formations" (1997).

Introduction

All groups under consideration are finite. The notations and definitions we use are borrowed from [1]–[4]. Recall that a formation $\mathfrak{F}$ is called *saturated* if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. It is known that if $\mathfrak{F}$ is a non-empty saturated formation, then $\mathfrak{F} = LF(f)$, i. e., $\mathfrak{F}$ has *a local satellite* f.

In [5] Skiba has offered the following concept of a totally saturated formation. Every group formation is 0-*multiply saturated*. For $n \geq 1$, a formation $\mathfrak{F} \neq \varnothing$ is called n-*multiply saturated*, if it has a local satellite f such that every non-empty value $f(p)$ of f is a $(n-1)$-multiply saturated formation. A formation is called *totally saturated* if it is n-multiply saturated for all natural n. By definition any totally saturated formation is non-empty.

Let τ be a function such that for any group G, $\tau(G)$ is a set of subgroups of G, and $G \in \tau(G)$. Following [3] we say that τ is a *subgroup*

The research is supported by the Grant Φ06MC-017 of BRFFI (Belarus).
2000 Mathematics Subject Classification: *20D10, 20F17.*
Key words and phrases: *formation of finite groups, totally saturated formation, lattice of formations, τ-closed formation.*

functor if for every epimorphism $\varphi : A \rightarrow B$ and any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^{\varphi} \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$.

A group class $\mathfrak{F}$ is called τ-*closed* if $\tau(G) \subseteq \mathfrak{F}$ for all $G \in \mathfrak{F}$. The set l_{∞}^{τ} of all τ-closed totally saturated formations is a complete lattice [3].

In [3] Skiba proved that the lattice of all soluble totally saturated formations is distributive. There were some open questions on modularity or distributivity (see [2], Problem 10.11; [6], Question 14.80; [7], Problem 21; [3], Questions 4.2.14). These questions were solved in [8]–[11]. In [10] we proved that l_{∞}^{τ} is modular.

In this paper we prove that l_{∞}^{τ} is distributive; this is a solution of Question 4.2.15 in [3].

1. Definitions, notations and preliminary results

Every function of the form $f : \mathbb{P} \longrightarrow \{formations\ of\ groups\}$ is called a *local satellite* [12]. A group G is called a *pd-group* if p divides $|G|$; $\pi(G)$ is the set of all primes dividing the order of a group G; $\pi(\mathfrak{F}) = \bigcup_{G\in\mathfrak{F}} \pi(G)$; $F_p(G) = O_{p',p}(G)$.

Following [4], we denote by $LF(f)$ the class of all groups G which satisfy the following condition: for any chief factor H/K of G and for all primes p dividing $|H/K|$, we have $G/C_G(H/K) \in f(p)$. The notation $\mathfrak{F} = LF(f)$ means that f is a *local satellite of* $\mathfrak{F}$. It is well known that a non-empty formation has a local satellite if and only if it is saturated.

For any set $\mathfrak{X}$ of groups, $l_{\infty}^{\tau}\mathrm{form}\mathfrak{X}$ denotes the τ-closed totally saturated formation generated by $\mathfrak{X}$, i.e., $l_{\infty}^{\tau}\mathrm{form}\mathfrak{X}$ is the intersection of all τ-closed totally saturated formations containing $\mathfrak{X}$. For any τ-closed totally saturated formations $\mathfrak{M}$ and $\mathfrak{H}$, we set $\mathfrak{M} \vee_{\infty}^{\tau} \mathfrak{H} = l_{\infty}^{\tau}\mathrm{form}(\mathfrak{M} \cup \mathfrak{H})$. Formations in l_{∞}^{τ} are called l_{∞}^{τ}-*formations.*

A satellite f is called l_{∞}^{τ}-*valued* if all its non-empty values are l_{∞}^{τ}-formations. A local satellite f is called *integrated* if $f(p) \subseteq LF(f)$ for all primes p. If g, h are l_{∞}^{τ}-valued local satellites, then the satellites $f = g \vee_{\infty}^{\tau} h$ and $m = g \cap h$ are defined in the following way:

(1) if either $g(p) \neq \varnothing$ or $h(p) \neq \varnothing$, then $f(p) = l_{\infty}^{\tau}\mathrm{form}(g(p) \cup h(p))$;

(2) if $g(p) = \varnothing$ and $h(p) = \varnothing$, then $f(p) = \varnothing$;

(3) $m(p) = g(p) \cap h(p)$ for all $p \in \mathbb{P}$.

If $\mathfrak{F}$ is an l_{∞}^{τ}-formation, then $\mathfrak{F}_{\infty}^{\tau}$ denotes the minimal l_{∞}^{τ}-valued local satellite of $\mathfrak{F}$, i.e., $\mathfrak{F}_{\infty}^{\tau} = \cap_{i\in I} f_i$, where $\{f_i | i \in I\}$ is the set of all l_{∞}^{τ}-valued local satellites of $\mathfrak{F}$.

For an arbitrary sequence of primes $p_1, p_2, \ldots, p_n$ and for any set $\mathfrak{X}$ of groups, the class $\mathfrak{X}^{p_1 p_2 \cdots p_n}$ is defined recursively in the following way:

(1) $\mathfrak{X}^{p_1} = (A/F_{p_1}(A) | A \in \mathfrak{X})$;

(2) $\mathfrak{X}^{p_1 p_2 \cdots p_n} = (A/F_{p_n}(A) | A \in \mathfrak{X}^{p_1 p_2 \cdots p_{n-1}})$.

For any set $\mathfrak{X}$ of groups, we put $\mathfrak{X}^\tau_\infty(p) = l^\tau_\infty\text{form}\mathfrak{X}^p$, if $p \in \pi(\mathfrak{X})$, and $\mathfrak{X}^\tau_\infty(p) = \varnothing$ if $p \notin \pi(\mathfrak{X})$.

A sequence of primes $p_1, p_2, \ldots, p_n$ is called *suitable for* $\mathfrak{X}$ (or $\mathfrak{X}$-*suitable*) if $p_1 \in \pi(\mathfrak{X})$ and for any $i \in \{2, \ldots, n\}$ we have that $p_i \in \pi(\mathfrak{X}^{p_1 p_2 \cdots p_{i-1}})$.

Let $p_1, p_2, \ldots, p_n$ be an $\mathfrak{F}$-suitable sequence. Then the *totally local satellite* $\mathfrak{F}^\tau_\infty p_1 p_2 \ldots p_n$ is defined recursively as follows:

(1) $\mathfrak{F}^\tau_\infty p_1 = (\mathfrak{F}^\tau_\infty(p_1))^\tau_\infty$;

(2) $\mathfrak{F}^\tau_\infty p_1 \ldots p_n = (\mathfrak{F}^\tau_\infty p_1 \ldots p_{n-1}(p_n))^\tau_\infty$.

If Θ is a complete lattice, then Θ^l denotes the complete lattice of all saturated formations $\mathfrak{F} = LF(f)$ such that $f(p) \in \Theta$ for every $p \in \pi(\mathfrak{F})$.

A complete lattice Θ is called *inductive* if for any set

$$\{\mathfrak{F}_i = LF(f_i) | \mathfrak{F}_i \in \Theta^l, f_i \text{ is integrated}, i \in I\},$$

the following equality holds:

$$\vee_{\Theta^l}(\mathfrak{F}_i | i \in I) = LF(\vee_\Theta(f_i | i \in I)).$$

For any group G and a non-empty formation $\mathfrak{F}$, by $G^\mathfrak{F}$ we denote the $\mathfrak{F}$-*residual* of G, i.e., the intersection of all normal subgroups N of G such that $G/N \in \mathfrak{F}$. $\mathfrak{F}\mathfrak{H} = \{G | G^\mathfrak{H} \in \mathfrak{F}\}$ is *the product* of formations $\mathfrak{F}$ and $\mathfrak{H}$.

$\mathfrak{S}_\pi$ is the class of all soluble π-groups (π is a non-empty set of primes); $\mathfrak{N}_p$ is the class of all p-groups (p is a prime).

The next lemma is a special case of Lemma 4.1.2 in [3, p. 152].

Lemma 1. *Let $\mathfrak{F}_i$ be a τ-closed totally saturated formation, $i \in I$. Then $f = \vee^\tau_\infty(\mathfrak{F}_{i\,\infty} | i \in I)$ is the minimal l^τ_∞-valued local satellite of $\mathfrak{F} = \vee^\tau_\infty(\mathfrak{F}_i | i \in I)$.*

Lemma 2. *[3, p. 33]. Let $\mathfrak{X}$ be a non-empty class of group and $\mathfrak{F} = l^\tau_\infty\text{form}\mathfrak{X}$. If f is the minimal l^τ_∞-valued local satellite of $\mathfrak{F}$, then*

(a) $\pi(\mathfrak{X}) = \pi(\mathfrak{F})$;

(b) $f(p) = l^\tau_\infty\text{form}(G/F_p(G) | G \in \mathfrak{X}) = l^\tau_\infty\text{form}(G/F_p(G) | G \in \mathfrak{F})$ for all $p \in \pi(\mathfrak{X})$;

(c) $f(p) = \varnothing$ for all $p \notin \pi(\mathfrak{X})$;

(d) if $\mathfrak{F} = LF(h)$, where h is an l^τ_∞-valued local satellite, then

$$f(p) = l^\tau_\infty\text{form}(A | A \in h(p) \cap \mathfrak{F},\, O_p(A) = 1).$$

Lemma 3. *[13]. The lattice of τ-closed totally saturated formations is inductive.*

Lemma 4. *[14]. Let $\mathfrak{F}$ be a non-empty τ-closed formation and π be a set of primes such that $\pi(\mathfrak{F}) \subseteq \pi$. Then the formation $\mathfrak{S}_\pi\mathfrak{F}$ is τ-closed totally saturated.*

Lemma 5. *[3, p. 41]. Let A be a monolithic group and let $\mathrm{Soc}(A)$ be a non-abelian group. Let $\mathfrak{M}$ be a τ-closed homomorph. If $A \in l_n^\tau \mathrm{form}\mathfrak{M}$, then $A \in \mathfrak{M}$.*

Lemma 6. *[2, p. 168]. Let $\mathfrak{F}$ be a formation, $\mathfrak{M}$ a saturated formation, and let G be a group of minimal oder in $\mathfrak{F} \setminus \mathfrak{M}$. Then*
(a) G is a monolithic group and $\mathrm{Soc}(G) = G^{\mathfrak{M}}$;
(b) if $G^{\mathfrak{M}}$ is a p-group, then $G^{\mathfrak{M}} = C_G(G^{\mathfrak{M}}) = F_p(G)$.

Lemma 7. *[2, p. 78]. Let $\mathfrak{F} = LF(f)$. If $G/O_p(G) \in f(p) \cap \mathfrak{F}$, then $G \in \mathfrak{F}$.*

Lemma 8. *[1, p. 38]. Let $\mathfrak{F} = LF(f)$. Then the following statements are equivalent:*
(a) $G \in \mathfrak{F}$;
(b) $G/F_p(G) \in f(p)$ for all $p \in \pi(G)$.

Lemma 9. *[3, p. 152]. Let G be a group such that $O_p(G) = 1$, let $N_1 \times \ldots \times N_k = \mathrm{Soc}(G)$, where N_i is a minimal normal subgroup of G $(k \geq 2)$. Let M_i denote a maximal normal subgroup of G, which contains $N_1 \times \ldots \times N_{i-1} \times N_{i+1} \times \ldots \times N_k$ and does not contain N_i, $i \in \{1, \ldots, k\}$. Then*
(a) G/M_i is a monolithic and $\mathrm{Soc}(G/M_i) = N_i M_i/N_i$ for any $i \in \{1, \ldots, k\}$;
(b) $N_i M_i/N_i$ is G-isomorphic to N_i;
(c) $O_p(G/M_i) = 1$;
(d) $M_1 \cap \ldots \cap M_k = 1$.

2. Main result

The following lemma can be proved by direct calculations.

Lemma 10. *Let $\mathfrak{F}_i = LF(f_i)$, where f_i is an integrated l_∞^τ-valued local satellite of $\mathfrak{F}_i$, $i \in I$. If $\mathfrak{F} = \cap(\mathfrak{F}_i | i \in I)$, then $f = \cap(f_i | i \in I)$ is an integrated l_∞^τ-valued local satellite of $\mathfrak{F}$.*

Let $\mathfrak{F}$, $\mathfrak{M}$, and $\mathfrak{X}$ be τ-closed totally saturated formations. Let $p_1, \ldots, p_n$ be some suitable sequence for $\mathfrak{F}$, $\mathfrak{M}$, and $\mathfrak{X}$. Then by $\widehat{\mathfrak{L}}_\infty^\tau$, $\widehat{\mathfrak{H}}_\infty^\tau$, $\widehat{\mathfrak{L}}_\infty^\tau p_1$, $\widehat{\mathfrak{H}}_\infty^\tau p_1, \ldots, \widehat{\mathfrak{L}}_\infty^\tau p \ldots p_n, \widehat{\mathfrak{H}}_\infty^\tau p \ldots p_n$ we denote a l_∞^τ-valued local satellites such that

$$\widehat{\mathfrak{L}}_\infty^\tau = (\mathfrak{X}_\infty^\tau \cap \mathfrak{F}_\infty^\tau) \vee_\infty^\tau (\mathfrak{M}_\infty^\tau \cap \mathfrak{F}_\infty^\tau), \quad \widehat{\mathfrak{H}}_\infty^\tau = (\mathfrak{X}_\infty^\tau \vee_\infty^\tau \mathfrak{M}_\infty^\tau) \cap \mathfrak{F}_\infty^\tau,$$

$$\widehat{\mathfrak{L}}_\infty^\tau p_1 = (\mathfrak{X}_\infty^\tau p_1 \cap \mathfrak{F}_\infty^\tau p_1) \vee_\infty^\tau (\mathfrak{M}_\infty^\tau p_1 \cap \mathfrak{F}_\infty^\tau p_1),$$

$$\widehat{\mathfrak{H}}^{\tau}_{\infty}p_1 = (\mathfrak{X}^{\tau}_{\infty}p_1 \vee^{\tau}_{\infty} \mathfrak{M}^{\tau}_{\infty}p_1) \cap \mathfrak{F}^{\tau}_{\infty}p_1, \quad \dots ,$$

$$\widehat{\mathfrak{L}}^{\tau}_{\infty}p_1p_2 \, \dots \, p_n =$$

$$= (\mathfrak{X}^{\tau}_{\infty}p_1p_2 \, \dots \, p_n \cap \mathfrak{F}^{\tau}_{\infty}p_1p_2 \, \dots \, p_n) \vee^{\tau}_{\infty} (\mathfrak{M}^{\tau}_{\infty}p_1p_2 \, \dots \, p_n \cap \mathfrak{F}^{\tau}_{\infty}p_1p_2 \, \dots \, p_n),$$

$$\widehat{\mathfrak{H}}^{\tau}_{\infty}p_1p_2 \, \dots \, p_n = (\mathfrak{X}^{\tau}_{\infty}p_1p_2 \, \dots \, p_n \vee^{\tau}_{\infty} \mathfrak{M}^{\tau}_{\infty}p_1p_2 \, \dots \, p_n) \cap \mathfrak{F}^{\tau}_{\infty}p_1p_2 \, \dots \, p_n.$$

Lemma 11. *Let $\mathfrak{M}$, $\mathfrak{X}$, and $\mathfrak{F}$ be τ-closed totally saturated formations. If $\mathfrak{L} = (\mathfrak{X} \cap \mathfrak{F}) \vee^{\tau}_{\infty} (\mathfrak{M} \cap \mathfrak{F})$ and $\mathfrak{H} = (\mathfrak{X} \vee^{\tau}_{\infty} \mathfrak{M}) \cap \mathfrak{F}$, then*

(a) $\pi(\mathfrak{L}) = \pi(\mathfrak{H})$;

(b) for any suitable sequence $p_1, \dots , p_n$ for $\mathfrak{X}$, $\mathfrak{M}$ and $\mathfrak{F}$, the satellites $\widehat{\mathfrak{L}}^{\tau}_{\infty}$, $\widehat{\mathfrak{H}}^{\tau}_{\infty}$, $\widehat{\mathfrak{L}}^{\tau}_{\infty}p_1$, $\widehat{\mathfrak{H}}^{\tau}_{\infty}p_1$, ... , $\widehat{\mathfrak{L}}^{\tau}_{\infty}p_1 \, \dots \, p_n$, $\widehat{\mathfrak{H}}^{\tau}_{\infty}p_1 \, \dots \, p_n$ are integrated l^{τ}_{∞}-valued local satellites of the formations $\mathfrak{L}$, $\mathfrak{H}$,

$$\widehat{\mathfrak{L}}^{\tau}_{\infty}(p_1), \quad \widehat{\mathfrak{H}}^{\tau}_{\infty}(p_1), \quad \dots, \quad \widehat{\mathfrak{L}}^{\tau}_{\infty}p_1 \, \dots \, p_{n-1}(p_n), \quad \widehat{\mathfrak{H}}^{\tau}_{\infty}p_1 \, \dots \, p_{n-1}(p_n),$$

respectively.

Proof. Put $\mathfrak{L}_1 = \mathfrak{X} \cap \mathfrak{F}$, $\mathfrak{L}_2 = \mathfrak{M} \cap \mathfrak{F}$, $\mathfrak{H}_1 = \mathfrak{X} \vee^{\tau}_{\infty} \mathfrak{M}$, $l_1 = \mathfrak{X}^{\tau}_{\infty} \cap \mathfrak{F}^{\tau}_{\infty}$, $l_2 = \mathfrak{M}^{\tau}_{\infty} \cap \mathfrak{F}^{\tau}_{\infty}$, and $h_1 = \mathfrak{X}^{\tau}_{\infty} \vee^{\tau}_{\infty} \mathfrak{M}^{\tau}_{\infty}$. By Lemmas 10 and 1, it follows that $\mathfrak{L}_1 = LF(l_1)$, $\mathfrak{L}_1 = LF(l_1)$, $\mathfrak{H}_1 = LF(h_1)$, and l_1, l_2, h_1 are integrated l^{τ}_{∞}-valued local satellites.

(a) Since the inclusion $\mathfrak{L} \subseteq \mathfrak{H}$ is obvious, we obtain $\pi(\mathfrak{L}) \subseteq \pi(\mathfrak{H})$. Let $p \in \pi(\mathfrak{H}) \setminus \pi(\mathfrak{L})$. Since $p \in \pi(\mathfrak{H}_1)$, we have $h_1(p) \neq \varnothing$, by Lemma 2. If $p \notin \pi(\mathfrak{X}) \cup \pi(\mathfrak{M})$, then it follows from Lemma 2 that $\mathfrak{X}^{\tau}_{\infty}(p) = \varnothing$ and $\mathfrak{M}^{\tau}_{\infty}(p) = \varnothing$. Consequently, $h_1(p) = \varnothing$, a contradiction. Hence $p \in \pi(\mathfrak{X}) \cup \pi(\mathfrak{M})$. But in this case $p \in \pi(\mathfrak{X} \cap \mathfrak{F}) \cup \pi(\mathfrak{M} \cap \mathfrak{F}) = \pi(\mathfrak{L})$. Thus, $\pi(\mathfrak{L}) = \pi(\mathfrak{H})$.

(b) Since $\mathfrak{L} = \mathfrak{L}_1 \vee^{\tau}_{\infty} \mathfrak{L}_2$ and $\mathfrak{H} = \mathfrak{H}_1 \cap \mathfrak{F}$, it follows from Lemmas 3 and 10 that

$$\widehat{\mathfrak{L}}^{\tau}_{\infty} = l_1 \vee^{\tau}_{\infty} l_2 = (\mathfrak{X}^{\tau}_{\infty} \cap \mathfrak{F}^{\tau}_{\infty}) \vee^{\tau}_{\infty} (\mathfrak{M}^{\tau}_{\infty} \cap \mathfrak{F}^{\tau}_{\infty}),$$

$$\widehat{\mathfrak{H}}^{\tau}_{\infty} = h_1 \cap \mathfrak{F}^{\tau}_{\infty} = (\mathfrak{X}^{\tau}_{\infty} \vee^{\tau}_{\infty} \mathfrak{M}^{\tau}_{\infty}) \cap \mathfrak{F}^{\tau}_{\infty}$$

are integrated l^{τ}_{∞}-valued local satellites of the formations $\mathfrak{L}$ and $\mathfrak{H}$, respectively.

Let $p_1, \dots , p_n$ be a suitable sequence for $\mathfrak{X}$, $\mathfrak{M}$, and $\mathfrak{F}$.

Since by the definition $\mathfrak{X}^{\tau}_{\infty}p_1 \dots p_i$, $\mathfrak{M}^{\tau}_{\infty}p_1 \dots p_i$, $\mathfrak{F}^{\tau}_{\infty}p_1 \dots p_i$ are minimal l^{τ}_{∞}-valued local satellites of the formations

$$\mathfrak{X}^{\tau}_{\infty}p_1 \dots p_{i-1}(p_i), \quad \mathfrak{M}^{\tau}_{\infty}p_1 \dots p_{i-1}(p_i), \quad \text{and} \quad \mathfrak{F}^{\tau}_{\infty}p_1 \dots p_{i-1}(p_i)$$

$(i = \overline{1, n})$, respectively, it follows from Lemmas 1, 3, and 10 that

$$\widehat{\mathfrak{L}}^{\tau}_{\infty}p_1p_2 \, \dots \, p_i =$$

$$= (\mathfrak{X}^{\tau}_{\infty} p_1 p_2 \, \dots \, p_i \cap \mathfrak{F}^{\tau}_{\infty} p_1 p_2 \, \dots \, p_i) \vee^{\tau}_{\infty} (\mathfrak{M}^{\tau}_{\infty} p_1 p_2 \, \dots \, p_i \cap \mathfrak{F}^{\tau}_{\infty} p_1 p_2 \, \dots \, p_i),$$

$$\widehat{\mathfrak{H}}^{\tau}_{\infty} p_1 p_2 \, \dots \, p_i = (\mathfrak{X}^{\tau}_{\infty} p_1 p_2 \, \dots \, p_i \vee^{\tau}_{\infty} \mathfrak{M}^{\tau}_{\infty} p_1 p_2 \, \dots \, p_i) \cap \mathfrak{F}^{\tau}_{\infty} p_1 p_2 \, \dots \, p_i$$

are integrated l^{τ}_{∞}-valued local satellites of the formations

$$\widehat{\mathfrak{L}}^{\tau}_{\infty} p_1 \dots p_{i-1}(p_i) \ \text{ and } \ \widehat{\mathfrak{H}}^{\tau}_{\infty} p_1 \dots p_{i-1}(p_i),$$

respectively. □

Lemma 12. *Let $\mathfrak{M}$, $\mathfrak{X}$, and $\mathfrak{F}$ be τ-closed totally saturated formations. Let A be a monolithic group with a non-abelian socle. If $A \in \mathfrak{F} \cap (\mathfrak{X} \vee^{\tau}_{\infty} \mathfrak{M})$, then $A \in (\mathfrak{X} \cap \mathfrak{F}) \vee^{\tau}_{\infty} (\mathfrak{M} \cap \mathfrak{F})$.*

Proof. Let $A \in \mathfrak{F} \cap (\mathfrak{X} \vee^{\tau}_{\infty} \mathfrak{M})$ and $\pi = \pi(\mathfrak{X} \vee^{\tau}_{\infty} \mathfrak{M})$. It follows from Lemma 4 that $\mathfrak{S}_{\pi}\tau\mathrm{form}(\mathfrak{X} \cup \mathfrak{M})$ is a τ-closed totally saturated formation. Therefore

$$l^{\tau}_{\infty}\mathrm{form}(\mathfrak{X} \cup \mathfrak{M}) \subseteq \mathfrak{S}_{\pi}\tau\mathrm{form}(\mathfrak{X} \cup \mathfrak{M}).$$

Since A is a monolithic group and $\mathrm{Soc}(A)$ is a non-abelian group, we have $A \in \tau\mathrm{form}(\mathfrak{X} \cup \mathfrak{M})$. But then by Lemma 5, it follows that $A \in \mathfrak{X} \cup \mathfrak{M}$. Since $A \in \mathfrak{F}$, we obtain $A \in (\mathfrak{X} \cap \mathfrak{F}) \cup (\mathfrak{M} \cap \mathfrak{F})$. Hence

$$A \in l^{\tau}_{\infty}\mathrm{form}((\mathfrak{X} \cap \mathfrak{F}) \cup (\mathfrak{M} \cap \mathfrak{F})) = (\mathfrak{X} \cap \mathfrak{F}) \vee^{\tau}_{\infty} (\mathfrak{M} \cap \mathfrak{F}).$$

□

Theorem 1. *The lattice l^{τ}_{∞} of all τ-closed totally saturated formations is distributive.*

Proof. Assume that there exist τ-closed totally saturated formations $\mathfrak{M}$, $\mathfrak{X}$, and $\mathfrak{F}$ such that

$$(\mathfrak{X} \cap \mathfrak{F}) \vee^{\tau}_{\infty} (\mathfrak{M} \cap \mathfrak{F}) \neq (\mathfrak{X} \vee^{\tau}_{\infty} \mathfrak{M}) \cap \mathfrak{F}.$$

Put $\mathfrak{L} = (\mathfrak{X} \cap \mathfrak{F}) \vee^{\tau}_{\infty} (\mathfrak{M} \cap \mathfrak{F})$ and $\mathfrak{H} = (\mathfrak{X} \vee^{\tau}_{\infty} \mathfrak{M}) \cap \mathfrak{F}$. Since the inclusion $\mathfrak{L} \subseteq \mathfrak{H}$ is obvious, we obtain $\mathfrak{H} \not\subseteq \mathfrak{L}$. Let A be a group of minimal order in $\mathfrak{H} \setminus \mathfrak{L}$. Since $\mathfrak{H}$ and $\mathfrak{L}$ are τ-closed saturated formations, we see that A is a τ-minimal non-$\mathfrak{L}$-group with a unique minimal normal subgroup P and $P = A^{\mathfrak{L}} \not\subseteq \Phi(A)$.

If P is a non-abelian group, then $A \in \mathfrak{L}$, by Lemma 12. This contradiction shows that P is an abelian p-group for some prime $p \in \pi(\mathfrak{H})$. It follows from Lemma 11 that $p \in \pi(\mathfrak{L})$, $\mathfrak{L} = LF(\widehat{\mathfrak{L}}^{\tau}_{\infty})$, $\mathfrak{H} = LF(\widehat{\mathfrak{H}}^{\tau}_{\infty})$, and $\widehat{\mathfrak{L}}^{\tau}_{\infty}$, $\widehat{\mathfrak{H}}^{\tau}_{\infty}$ are integrated l^{τ}_{∞}-valued satellites such that

$$\widehat{\mathfrak{L}}^{\tau}_{\infty}(p) = (\mathfrak{X}^{\tau}_{\infty}(p) \cap \mathfrak{F}^{\tau}_{\infty}(p)) \vee^{\tau}_{\infty} (\mathfrak{M}^{\tau}_{\infty}(p) \cap \mathfrak{F}^{\tau}_{\infty}(p)),$$

$$\widehat{\mathfrak{H}}^{\tau}_{\infty}(p) = (\mathfrak{X}^{\tau}_{\infty}(p) \vee^{\tau}_{\infty} \mathfrak{M}^{\tau}_{\infty}(p)) \cap \mathfrak{F}^{\tau}_{\infty}(p).$$

Since $\mathfrak{L}$ is a saturated formation and $p \in \pi(\mathfrak{L})$, we see that A is not a p-group and $\widehat{\mathfrak{L}}^{\tau}_{\infty}(p) \neq \varnothing$. By Lemma 6, it follows that $P = C_A(P) = F_p(A)$. Then $P = O_p(A)$. Since $P \not\subseteq \Phi(A)$, we have $A = [P]A_1$, where A_1 is a maximal subgroup of A such that $P \not\subseteq A_1$.

The inclusion $\widehat{\mathfrak{L}}^{\tau}_{\infty}(p) \subseteq \widehat{\mathfrak{H}}^{\tau}_{\infty}(p)$ is obvious. Since $A \in \mathfrak{H} \setminus \mathfrak{L}$, we claim that $\widehat{\mathfrak{L}}^{\tau}_{\infty}(p) \subset \widehat{\mathfrak{H}}^{\tau}_{\infty}(p)$. Indeed, by Lemma 2, it follows that $A/F_p(A) \in \mathfrak{H}^{\tau}_{\infty}(p)$. If $\widehat{\mathfrak{L}}^{\tau}_{\infty}(p) = \widehat{\mathfrak{H}}^{\tau}_{\infty}(p)$, then

$$A/O_p(A) = A/F_p(A) \in \mathfrak{H}^{\tau}_{\infty}(p) \subseteq \widehat{\mathfrak{H}}^{\tau}_{\infty}(p) = \widehat{\mathfrak{L}}^{\tau}_{\infty}(p)$$

and $A \in \mathfrak{L}$, by Lemma 7. It is a contradiction. Therefore $\widehat{\mathfrak{L}}^{\tau}_{\infty}(p) \subset \widehat{\mathfrak{H}}^{\tau}_{\infty}(p)$. Note also that the condition $\widehat{\mathfrak{L}}^{\tau}_{\infty}(p) \subset \widehat{\mathfrak{H}}^{\tau}_{\infty}(p)$ implies $\mathfrak{X}^{\tau}_{\infty}(p) \neq \varnothing$ and $\mathfrak{M}^{\tau}_{\infty}(p) \neq \varnothing$, otherwise $\widehat{\mathfrak{L}}^{\tau}_{\infty}(p) = \widehat{\mathfrak{H}}^{\tau}_{\infty}(p)$. Hence $p \in \pi(\mathfrak{X}) \cap \pi(\mathfrak{M})$. Thus

$$A_1 \simeq A/F_p(A) \in \widehat{\mathfrak{H}}^{\tau}_{\infty}(p) \setminus \widehat{\mathfrak{L}}^{\tau}_{\infty}(p), \quad \widehat{\mathfrak{L}}^{\tau}_{\infty}(p) \neq \varnothing.$$

It follows from Lemma 11 that $\pi(\widehat{\mathfrak{L}}^{\tau}_{\infty}(p)) = \pi(\widehat{\mathfrak{H}}^{\tau}_{\infty}(p))$, $\widehat{\mathfrak{L}}^{\tau}_{\infty}(p) = LF(\widehat{\mathfrak{L}}^{\tau}_{\infty}p)$, $\widehat{\mathfrak{H}}^{\tau}_{\infty}(p) = LF(\widehat{\mathfrak{H}}^{\tau}_{\infty}p)$, and $\widehat{\mathfrak{L}}^{\tau}_{\infty}p$, $\widehat{\mathfrak{H}}^{\tau}_{\infty}p$ are integrated l^{τ}_{∞}-valued local satellites such that

$$\widehat{\mathfrak{L}}^{\tau}_{\infty}p = (\mathfrak{X}^{\tau}_{\infty}p \cap \mathfrak{F}^{\tau}_{\infty}p) \vee^{\tau}_{\infty} (\mathfrak{M}^{\tau}_{\infty}p \cap \mathfrak{F}^{\tau}_{\infty}p),$$

$$\widehat{\mathfrak{H}}^{\tau}_{\infty}p = (\mathfrak{X}^{\tau}_{\infty}p \vee^{\tau}_{\infty} \mathfrak{M}^{\tau}_{\infty}p) \cap \mathfrak{F}^{\tau}_{\infty}p.$$

Since $A_1 \notin \widehat{\mathfrak{L}}^{\tau}_{\infty}(p)$, by Lemma 8(b), there exists a prime $p_1 \in \pi(A_1)$ such that $A_1/F_{p_1}(A_1) \notin \widehat{\mathfrak{L}}^{\tau}_{\infty}p(p_1)$. Since $p_1 \in \pi(\widehat{\mathfrak{H}}^{\tau}_{\infty}(p))$, we have $p_1 \in \pi(\widehat{\mathfrak{L}}^{\tau}_{\infty}(p))$ and $\widehat{\mathfrak{L}}^{\tau}_{\infty}p(p_1) \neq \varnothing$. Obviously,

$$\widehat{\mathfrak{L}}^{\tau}_{\infty}p(p_1) = (\mathfrak{X}^{\tau}_{\infty}p(p_1) \cap \mathfrak{F}^{\tau}_{\infty}p(p_1)) \vee^{\tau}_{\infty} (\mathfrak{M}^{\tau}_{\infty}p(p_1) \cap \mathfrak{F}^{\tau}_{\infty}p(p_1)) \subseteq$$

$$\subseteq \widehat{\mathfrak{H}}^{\tau}_{\infty}p(p_1) = (\mathfrak{X}^{\tau}_{\infty}p(p_1) \vee^{\tau}_{\infty} \mathfrak{M}^{\tau}_{\infty}p(p_1)) \cap \mathfrak{F}^{\tau}_{\infty}p(p_1).$$

Besides, since $A_1/F_{p_1}(A_1) \in \widehat{\mathfrak{H}}^{\tau}_{\infty}p(p_1) \setminus \widehat{\mathfrak{L}}^{\tau}_{\infty}p(p_1)$, we have $\widehat{\mathfrak{L}}^{\tau}_{\infty}p(p_1) \subset \widehat{\mathfrak{H}}^{\tau}_{\infty}p(p_1)$. Therefore $\mathfrak{X}^{\tau}_{\infty}p(p_1) \neq \varnothing$ and $\mathfrak{M}^{\tau}_{\infty}p(p_1) \neq \varnothing$. Hence $p_1 \in \pi(\mathfrak{X}^{\tau}_{\infty}(p)) \cap \pi(\mathfrak{M}^{\tau}_{\infty}(p))$.

Suppose that $F_{p_1}(A_1) = 1$ and let N be a minimal normal subgroup of A. Then N is a non-abelian p_1d-group. If A_1 is a monolithic group, then since

$$A_1 \in \widehat{\mathfrak{H}}^{\tau}_{\infty}(p) = (\mathfrak{X}^{\tau}_{\infty}(p) \vee^{\tau}_{\infty} \mathfrak{M}^{\tau}_{\infty}(p)) \cap \mathfrak{F}^{\tau}_{\infty}(p),$$

by Lemma 12, it follows that

$$A_1 \in (\mathfrak{X}^{\tau}_{\infty}(p) \cap \mathfrak{F}^{\tau}_{\infty}(p)) \vee^{\tau}_{\infty} (\mathfrak{M}^{\tau}_{\infty}(p) \cap \mathfrak{F}^{\tau}_{\infty}(p)) = \widehat{\mathfrak{L}}^{\tau}_{\infty}(p),$$

a contradiction. Therefore the group A_1 is not monolithic.

Let $\mathrm{Soc}(A_1) = N_1 \times \ldots \times N_k$, where N_i is a minimal normal subgroup of A_1, and let M_i denotes a maximal normal subgroup of A_1 such that M_i contains $N_1 \times \ldots \times N_{i-1} \times N_{i+1} \times \ldots \times N_k$ and does not contain N_i, $i = 1, 2 \ldots, k$. By Lemma 9, it follows that A_1/M_i is a monolithic group with a non-abelian minimal normal subgroup N_iM_i/N_i, and N_iM_i/N_i is A_1-isomorphic to N_i. Set $B_i = A_1/M_i$, $i = 1, 2 \ldots, k$. Since

$$B_i \in \widehat{\mathfrak{H}}^{\tau}_{\infty}(p) = (\mathfrak{X}^{\tau}_{\infty}(p) \vee^{\tau}_{\infty} \mathfrak{M}^{\tau}_{\infty}(p)) \cap \mathfrak{F}^{\tau}_{\infty}(p),$$

we have $B_i \in \widehat{\mathfrak{L}}^{\tau}_{\infty}(p)$, by Lemma 13. It follows from Lemma 9(d) that A_1 is a subdirect product of $B_1, \ldots, B_k$. Hence $A_1 \in \widehat{\mathfrak{L}}^{\tau}_{\infty}(p)$, a contradiction. Therefore $F_{p_1}(A_1) \neq 1$.

On the other hand, $F_{p_1}(A_1) \neq A_1$, otherwise $A_1/F_{p_1}(A_1) \simeq 1 \in \mathfrak{L}^{\tau}_{\infty}p(p_1) \neq \varnothing$.

Thus

$$A_1/F_{p_1}(A_1) \in \widehat{\mathfrak{H}}^{\tau}_{\infty}p(p_1) \setminus \widehat{\mathfrak{L}}^{\tau}_{\infty}p(p_1),\ \ \widehat{\mathfrak{L}}^{\tau}_{\infty}p(p_1) \neq \varnothing,\ 1 \neq F_{p_1}(A_1) \subset A_1.$$

Put $A_2 = A_1/F_{p_1}(A_1)$. It follows from Lemma 11 that

$$\pi(\widehat{\mathfrak{L}}^{\tau}_{\infty}p(p_1)) = \pi(\widehat{\mathfrak{H}}^{\tau}_{\infty}p(p_1)),$$

$$\widehat{\mathfrak{L}}^{\tau}_{\infty}p(p_1) = LF(\widehat{\mathfrak{L}}^{\tau}_{\infty}pp_1),\quad \widehat{\mathfrak{H}}^{\tau}_{\infty}p(p_1) = LF(\widehat{\mathfrak{H}}^{\tau}_{\infty}pp_1),$$

and $\widehat{\mathfrak{L}}^{\tau}_{\infty}pp_1$, $\widehat{\mathfrak{H}}^{\tau}_{\infty}pp_1$ are integrated l^{τ}_{∞}-valued local satellites such that

$$\widehat{\mathfrak{L}}^{\tau}_{\infty}pp_1 = (\mathfrak{X}^{\tau}_{\infty}pp_1 \cap \mathfrak{F}^{\tau}_{\infty}pp_1) \vee^{\tau}_{\infty} (\mathfrak{M}^{\tau}_{\infty}pp_1 \cap \mathfrak{F}^{\tau}_{\infty}pp_1),$$

$$\widehat{\mathfrak{H}}^{\tau}_{\infty}pp_1 = (\mathfrak{X}^{\tau}_{\infty}pp_1 \vee^{\tau}_{\infty} \mathfrak{M}^{\tau}_{\infty}pp_1) \cap \mathfrak{F}^{\tau}_{\infty}pp_1.$$

Since $A_2 \notin \widehat{\mathfrak{L}}^{\tau}_{\infty}p(p_1)$, by Lemma 8(b), there exists $p_2 \in \pi(A_2)$ such that $A_2/F_{p_2}(A_2) \notin \widehat{\mathfrak{L}}^{\tau}_{\infty}pp_1(p_2)$. Hence

$$A_2/F_{p_2}(A_2) \in \widehat{\mathfrak{H}}^{\tau}_{\infty}pp_1(p_2) \setminus \widehat{\mathfrak{L}}^{\tau}_{\infty}pp_1(p_2).$$

Considering A_2 in the same way as the group A_1, we obtain

$$p_2 \in \pi(\mathfrak{X}^{\tau}_{\infty}p(p_1)) \cap \pi(\mathfrak{M}^{\tau}_{\infty}p(p_1)),$$

$$A_2/F_{p_2}(A_2) \in \widehat{\mathfrak{H}}^{\tau}_{\infty}pp_1(p_2) \setminus \widehat{\mathfrak{L}}^{\tau}_{\infty}pp_1(p_2),$$

$$\widehat{\mathfrak{L}}^{\tau}_{\infty}pp_1(p_2) \neq \varnothing, \quad \text{and} \quad 1 \neq F_{p_2}(A_2) \subset A_2.$$

Put $A_3 = A_2/F_{p_2}(A_2)$. According to the same argument, we see that the group A_3 satisfies the analogous conditions: there exists

$$p_3 \in \pi(\mathfrak{X}^{\tau}_{\infty}pp_1(p_2)) \cap \pi(\mathfrak{M}^{\tau}_{\infty}pp_1(p_2))$$

such that

$$A_3/F_{p_3}(A_3) \in \widehat{\mathfrak{H}}^{\tau}_{\infty} pp_1p_2(p_3) \setminus \widehat{\mathfrak{L}}^{\tau}_{\infty} pp_1p_2(p_3),$$

$$\widehat{\mathfrak{L}}^{\tau}_{\infty} pp_1p_2(p_3) \neq \varnothing, \text{ and } 1 \neq F_{p_3}(A_3) \subset A_3.$$

Continuing this line of reasoning, we construct the groups

$$A_4 = A_3/F_{p_3}(A_3), \quad \ldots, \quad A_n = A_{n-1}/F_{p_{n-1}}(A_{n-1}), \ldots$$

such that for any i, the following conditions are satisfied:

$$p_{i-1} \in \pi(\mathfrak{X}^{\tau}_{\infty} pp_1 \ldots p_{i-3}(p_{i-2})) \cap \pi(\mathfrak{M}^{\tau}_{\infty} pp_1 \ldots p_{i-3}(p_{i-2})),$$

$$A_i = A_{i-1}/F_{p_{i-1}}(A_{i-1}) \in \widehat{\mathfrak{H}}^{\tau}_{\infty} pp_1 \ldots p_{i-2}(p_{i-1}) \setminus \widehat{\mathfrak{L}}^{\tau}_{\infty} pp_1 \ldots p_{i-2}(p_{i-1}),$$

$$\widehat{\mathfrak{L}}^{\tau}_{\infty} pp_1 \ldots p_{i-2}(p_{i-1}) \neq \varnothing, \text{ and } 1 \neq F_{p_{i-1}}(A_{i-1}) \subset A_{i-1}.$$

Since $F_{p_{i-1}}(A_{i-1}) \neq 1$, we see that for the constructed sequence $A,\ A_1,\ A_2,\ A_3,\ \ldots,\ A_n,\ \ldots$ of groups, it follows that

$$|A| > |A_1| > |A_2| > |A_3| > \ldots > |A_n| > \ldots.$$

Since A is finite, we obtain $A_m = 1$ for some number m. But $A_m = A_{m-1}/F_{p_{m-1}}(A_{m-1})$. This implies that $F_{p_{m-1}}(A_{m-1}) = A_{m-1}$, a contradiction.

Thus, our assumption is not true and $\mathfrak{H} \subseteq \mathfrak{L}$. Hence $\mathfrak{H} = \mathfrak{L}$. □

Let τ be the trivial subgroup functor. Then from Theorem 1 we obtain

Corollary 1. *[11]. The lattice l_∞ of all totally saturated formations is distributive.*

In the case when $\tau(G) = S(G)$ is the set of all subgroups of G, from Theorem 1 we have the following.

Corollary 2. *The lattice of hereditary totally saturated formations is distributive.*

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Received by the editors: 17.12.2007
and in final form 14.10.2008.