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Idempotent $\mathcal{D}$-cross-sections of the finite inverse symmetric semigroup IS_n

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Abstract. We prove that every finite poset can be embedded in some idempotent $\mathcal{D}$-cross-section of the finite inverse symmetric semigroup $\mathcal{IS}_n$.

The symmetric group $\mathcal{S}_n$ is a central object of study in many branches of mathematics. There exist several "natural" analogues (or generalizations) of $\mathcal{S}_n$ in semigroup theory. The most classical ones are the symmetric semigroup $\mathcal{T}_n$ and the inverse symmetric semigroup $\mathcal{IS}_n$. They arise when one tries to generalize Cayley's Theorem to the classes of all semigroups or all inverse semigroups respectively. A less obvious semigroup generalizations of $\mathcal{S}_n$ is the so-called Brauer semigroup $\mathcal{B}_n$, which appears in the context of centralizer algebras in representation theory, see [1].

Let n be a positive integer. Let us put $N = \{1, \dots, n\}$ and $N' = \{1', \dots, n'\}$. The elements of the Brauer semigroup $\mathcal{B}_n$ are all possible partitions of the set $N \cup N'$ into two-element blocks. Consider the map $' : N \to N'$ as a fixed bijection and denote the inverse bijection by the same symbol, i. e. $(a')' = a$ for all $a \in N$. For $\alpha \in \mathcal{B}_n$ and two different elements a, $b \in N \cup N'$ we set $a \equiv_\alpha b$ provided that $\{a, b\} \in \alpha$. In other words, $\equiv_\alpha$ is the equivalence relation corresponding to the partition α. Let $\alpha = X_1 \cup \dots \cup X_n$ and $\beta = Y_1 \cup \dots \cup Y_n$ be two elements from $\mathcal{B}_n$. Let us define a new equivalence relation, $\equiv$, on $N \cup N'$ as follows:

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- for a, $b \in N$ we have $a \equiv b$ if and only if $a \equiv_\alpha b$ or there is a sequence, $c_1, \ldots, c_{2s}$, $s \geq 1$, of elements of N such that $a \equiv_\alpha c'_1$, $c_1 \equiv_\beta c_2$, $c'_2 \equiv_\alpha c'_3$, $\ldots$, $c_{2s-1} \equiv_\beta c_{2s}$ and $c'_{2s} \equiv_\alpha b$.

- for a, $b \in N$ we have $a' \equiv b'$ if and only if $a' \equiv_\beta b'$ or there is a sequence, $c_1, \ldots, c_{2s}$, $s \geq 1$, of elements of N such that $a' \equiv_\beta c_1$, $c'_1 \equiv_\alpha c'_2$, $c_2 \equiv_\beta c_3$, $\ldots$, $c'_{2s-1} \equiv_\alpha c'_{2s}$ and $c_{2s} \equiv_\beta b'$.

- for a, $b \in N$ we have $a \equiv b'$ if and only if $b' \equiv a$ if and only if there is a sequence, $c_1, \ldots, c_{2s-1}$, $s \geq 1$, of elements of N such that $a \equiv_\alpha c'_1$, $c_1 \equiv_\beta c_2$, $c'_2 \equiv_\alpha c'_3$, $\ldots$, $c'_{2s-2} \equiv_\alpha c'_{2s-1}$ and $c_{2s} \equiv_\beta b'$.

It is easy to see that $\equiv$ determines a partition of $N \cup N'$ into two-element subsets and so belongs to $\mathcal{B}_n$. We define this element to be the product $\alpha\beta$.

Thus, the study of the structure of these semigroups is a natural problem to investigation.

Let ρ be an equivalence relation on a semigroup S. A subsemigroup $T \subset S$ is called a *cross-section* with respect to ρ if T contains exactly one element from every equivalence class. Clearly, the most interesting are the cross-sections with respect to the equivalence relations connected with the semigroup structure on S. The first candidates for such relations are congruences and the Green's relations, which are important tools in the description and decomposition of semigroups.

For any $a \in S$ we denote by $L(a)$ ($R(a)$, $J(a)$) the principal left (right, two-sided) ideal generated by a respectively. The *Green's relations* $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ on semigroup S are defined as binary relations in the following way: $a\mathcal{L}b$ if and only if $L(a) = L(b)$; $a\mathcal{R}b$ if and only if $R(a) = R(b)$; $a\mathcal{J}b$ if and only if $J(a) = J(b)$ for any $a, b \in S$ and the relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, while the relation $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$, where the join is in the lattice of all equivalences on S, that is $\mathcal{D}$ is the least equivalence containing both $\mathcal{L}$ and $\mathcal{R}$.

Cross-sections with respect to congruences are called *retracts*. They are important in study of semigroup endomorphisms.

Cross-sections with respect to the $\mathcal{H}$- ($\mathcal{L}$-, $\mathcal{R}$-, $\mathcal{D}$-, $\mathcal{J}$-) Green's relations are called $\mathcal{H}$- ($\mathcal{L}$-, $\mathcal{R}$-, $\mathcal{D}$-, $\mathcal{J}$-) *cross-sections* in the sequel.

During the last decade cross-sections of Green's relations for some classical semigroups were studied by different authors. In particular, for the inverse symmetric semigroup $\mathcal{IS}_n$ all $\mathcal{H}$-cross-sections were classified in [2] and all $\mathcal{L}$- and $\mathcal{R}$-cross-sections were classified in [3]. For the infinite inverse symmetric semigroup $\mathcal{IS}_X$ all $\mathcal{H}$-, $\mathcal{L}$- and $\mathcal{R}$-cross-sections were classified in [7], and for the symmetric semigroup $\mathcal{T}_X$ all $\mathcal{H}$- and $\mathcal{R}$-cross-sections were classified in [5], [6]. The classification of $\mathcal{H}$-, $\mathcal{L}$- and $\mathcal{R}$-cross-sections for the Brauer semigroup $\mathcal{B}_n$ was obtained in [4].

The problem of classification of $\mathcal{D}$-cross-sections for these semigroups is essentially more difficult, since every $\mathcal{D}$-class has large cardinality and so the semigroups have many different $\mathcal{D}$-cross-sections.

We consider idempotent $\mathcal{D}$-cross-sections of the finite symmetric inverse semigroup $\mathcal{IS}_n$ on the set $N = \{1, \dots, n\}$, that is cross-sections which consist of idempotents. For the first time the problem of classification of these cross-sections appeared in [3] and it is still open. Let us recall that every idempotent of $\mathcal{IS}_n$ has the form id_A, where $A \subseteq N$ and Green's $\mathcal{D}$-classes are $D_k = \{\ a \in \mathcal{IS}_n \mid \mathrm{rk}(a) = k\},\ 0 \le k \le n$. Hence one can naturally construct a partial order on the set of all idempotents of this semigroup: $id_A \le id_B$ if and only if $A \subseteq B$. Thus, one can consider every idempotent $\mathcal{D}$-cross-section of $\mathcal{IS}_n$ as a poset.

Theorem. *The boolean of a set M containing exactly n elements is isomorphic to a some idempotent $\mathcal{D}$-cross-section of the finite symmetric inverse semigroup $\mathcal{IS}_{2^n-1}$.*

Proof. Put $M = \{0, \dots, n-1\}$. Let N be a disjoint union of sets N_i, $i = 0, \dots, n-1$, where $|N_i| = 2^i$ for every i. Then $|N| = 2^n - 1$. Let us define the map $f : 2^M \to 2^N$ by the rule

$$2^M \ni K \mapsto \bigcup_{i \in K} N_i \in 2^N.$$

Clearly, the cardinality of the set $f(K)$ equals the integer which binary representation is the boolean vector of the subset K. Therefore all sets from the image of the map f have pairwise different cardinality. Moreover, for every number l, $0 \le l \le 2^n - 1$ there is exists a set $K \in 2^M$ such that $|f(K)| = l$. Thus, the subset $T = \{id_{f(K)} \mid K \in 2^M\}$ of the semigroup $\mathcal{IS}_N$ contains exactly one element from every $\mathcal{D}$-class of this semigroup. Since from equalities $id_A \cdot id_B = id_{A \cap B}$ and $f(A) \cap f(B) = f(A \cap B)$ we have that the set T is closed under multiplication. Finally, T is an idempotent $\mathcal{D}$-cross-section of $\mathcal{IS}_N$, which is isomorphic (as poset) to the boolean of the set M. □

Remark. The number $2^n - 1$ in the theorem can not be decreased, because every idempotent $\mathcal{D}$-cross-section of the $\mathcal{IS}_n$ contains exactly $n+1$ elements.

Corollary. *Every finite poset can be embedded in some idempotent $\mathcal{D}$-cross-section of the finite symmetric inverse semigroup $\mathcal{IS}_n$.*

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