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Quantum Boolean algebras

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Abstract. We introduce quantum Boolean algebras which are the analogue of the Weyl algebras for Boolean affine spaces. We study quantum Boolean algebras from the logical and the set theoretical viewpoints.

1. Introduction

After Stone [20] and Zhegalkin [23], Boole's main contribution to science [5] can be understood as the realization that the mathematics of logical phenomena is controlled — to a large extend — by the field $\mathbb{Z}_2 = \{0, 1\}$ with two elements; in contrast the mathematics of classical physical phenomena is controlled — to a large extend — by the field $\mathbb{R}$ of real numbers. The switch from $\mathbb{Z}_2$ to $\mathbb{R}$ corresponds with a deep ontological jump from logical to physical phenomena. The switch from $\mathbb{R}$ to $\mathbb{C}$ corresponds to the jump from classical to quantum physics.

What makes the logic/physics jump possible is the fact that $\mathbb{Z}_2$ may be regarded as an object of two different categories. On the one hand, it is a field $(\mathbb{Z}_2, +, .)$ with sum and product defined by making 0 the neutral element and 1 the product unit. On the other hand, it is a set of truth values with 0 and 1 representing falsity and truth, respectively. Indeed, $(\mathbb{Z}_2, \vee, \wedge, \overline{(\cdot)})$ is a Boolean algebra: a complemented distributive lattice with minimum 0 and maximum 1. The operations $\vee$, $\wedge$, and $\overline{(\cdot)}$ correspond with the logical connectives OR, AND, and NOT. The two viewpoints are

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related by the identities: $a \vee b = a + b + ab, a \wedge b = ab, \overline{a} = a + 1$. These identities, together with the inverse relation $a + b = (a \wedge \overline{b}) \vee (\overline{a} \wedge b)$, allow us to switch back and forth from the algebraic to the logical viewpoint.

By and large, the logical and algebraic viewpoints have remained separated. In this work, in order to explore quantum-like phenomena in characteristic 2, we place ourselves at the jump. Our algebraic viewpoint is, in a sense, complementary to the quantum logic approach initiated by Birkhoff and von Neumann [3] based on the theory of lattices. For example, while the meet in quantum logic is a commutative connective, we propose in this work a quantum analogue for the meet which turns out to be non-commutative. The appearance of non-commutative operations is an essential feature of quantum mechanics [1, 7, 22].

We take as our guide the well-known fact that the quantization of canonical phase space may be identified with the algebra of differential operators on configuration space. In analogy with the real/complex case, we introduce the algebra BDO_n of Boolean differential operators on $\mathbb{Z}_2^n$. We provide a couple of presentations by generators and relations of BDO_n, giving rise to the Boole-Weyl algebras BA_n and the shifted Boole-Weyl algebras SBA_n. We call these algebras the quantum Boolean algebras. We study the structural coefficients of BA_n and SBA_n in various bases.

Having introduced quantum Boolean algebras, we proceed to study them from the logical and set theoretical viewpoints. For us, the main difference between classical and quantum logic rest on the fact that classical observations, propositions, can be measured without, in principle, modifying the state of the system; quantum observations, in contrast, are quantum operators: the measuring process changes the state of the system. Indeed, regardless of the actual state of the system, after measurement the system will be an eigenstate of the observable. Quantum observables are operators acting on the states of the system, and thus quite different to classical observables which are descriptions of the state of the system.

This work is organized as follows. In Section 2 we review some standard facts on regular functions on affine spaces over $\mathbb{Z}_2$. In Section 3 we introduce BDO_n, the algebra of Boolean differential operators on $\mathbb{Z}_2^n$. In Section 4 we introduce the Boole-Weyl algebra BA_n which is a presentation by generators and relations of BDO_n. We describe the structural coefficients of BA_n in several bases. In Section 5 we introduce the shifted Boole-Weyl algebra SBA_n which is another presentation by generators and relations of BDO_n, and describe the structural coefficients of SBA_n in several bases. In Section 6 we discuss the logical aspects of our constructions: we introduce a quantum operational logic that generalizes classical propositional logic, and for which Boolean differential operators play a semantic role akin to that played by truth functions in classical propositional logic. We use the theory of operads and props to describe our results. In Section 7 we adopt a set theoretical viewpoint and show that just as classical propositional logic is intimately related with $\mathrm{PP}(x)$, the Boolean algebra of sets of subsets of x, quantum operational logic is intimately related with $\mathrm{PP}(x \sqcup x)$ the quantum Boolean algebra of sets of subsets of two disjoint copies of x. In the final Section 8 we make some closing remarks and mention a few topics for future research.

2. Regular functions on Boolean affine spaces

Our main goal in this work is to study the Boolean analogue for the Weyl algebras, and to describe those algebras from a logical and a set theoretical viewpoints. Fixing a field k, the Weyl algebra W_n over k can be identified with the k-algebra of algebraic differential operators on the affine space $\mathbb{A}^n(k) = k^n$. By definition [14, 19] the k-algebra $k[\mathbb{A}^n]$ of regular functions on k^n is the k-algebra of maps

$$f\colon k^n \to k$$

such that there exists a polynomial $F \in k[x_1, \ldots, x_n]$ with $f(a) = F(a)$ for all $a \in k^n$. If k is a field of characteristic zero, then the k-algebra of regular functions on k^n can be identified with $k[x_1, \ldots, x_n]$ the polynomial ring of over k. Let $\partial_1, \ldots, \partial_n$ be the derivations of $k[x_1, \ldots, x_n]$ given by $\partial_i x_j = \delta_{i,j}$ for $i, j \in [n] = \{1, \ldots, n\}$. The k-algebra DO_n of differential operators on k^n is the subalgebra of

$$\mathrm{End}_k(k[x_1, \ldots, x_n])$$

generated by ∂_i and the operators of multiplication by x_i for $i \in [n]$.

By definition, the Weyl algebra A_n is the k-algebra defined via generators and relations as

$$k\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle / \langle x_i x_j - x_j x_i, y_i y_j - y_j y_i, y_i x_j - x_j y_i, y_i x_i - x_i y_i - 1\rangle.$$

where $k\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$ is the free associative k-algebra generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$, and $\langle x_i x_j - x_j x_i, y_i y_j - y_j y_i, y_i x_j - x_j y_i, y_i x_i - x_i y_i - 1\rangle$ is the ideal generated by the relations $x_i x_j = x_j x_i$ and $y_i y_j = y_j y_i$ for $i, j \in [n]$, $y_i x_j = x_j y_i$ for $i \neq j \in [n]$, $y_i x_i = x_i y_i + 1$ for $i \in [n]$. The Weyl algebra A_n comes with a natural representation

$\mathrm{A}_n \to \mathrm{End}_k(k[x_1, \dots, x_n])$ sending y_i to ∂_i and x_i to the operator of multiplication by x_i. This representation induces an isomorphism of algebras $\mathrm{A}_n \to \mathrm{DO}_n$.

We proceed to study the analogue of the Weyl algebras for the Boolean affine spaces $\mathbb{A}^n(\mathbb{Z}_2) = \mathbb{Z}_2^n$. First, we review some basic facts on regular functions on $\mathbb{Z}_2^n$. Let $\mathrm{M}(\mathbb{Z}_2^n, \mathbb{Z}_2)$ be the $\mathbb{Z}_2$-algebra of all maps from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2$ with pointwise addition and multiplication. The $\mathbb{Z}_2$-algebra $\mathbb{Z}_2[\mathbb{A}^n]$ of regular functions on $\mathbb{Z}_2^n$ is the sub-algebra of $\mathrm{M}(\mathbb{Z}_2^n, \mathbb{Z}_2)$ consisting of the maps $f\colon \mathbb{Z}_2^n \to \mathbb{Z}_2$ for which there exists a polynomial $F \in \mathbb{Z}_2[x_1, \dots, x_n]$ such that $f(a) = F(a)$ for all $a \in \mathbb{Z}_2^n$. In this case $\mathbb{Z}_2[\mathbb{A}^n]$ is not a polynomial ring; instead we have the following result.

Lemma 1. *There is an exact sequence of $\mathbb{Z}_2$-algebras*

$$0 \to \langle x_1^2 + x_1, \dots, x_n^2 + x_n \rangle \to \mathbb{Z}_2[x_1, \dots, x_n] \to \mathbb{Z}_2[\mathbb{A}^n] \to 0$$

where $\langle x_1^2 + x_1, \dots, x_n^2 + x_n \rangle$ is the ideal generated by the relations $x_i^2 = x_i$ for $i \in [n]$.

Therefore the ring $\mathbb{Z}_2[\mathbb{A}^n]$ of regular functions on $\mathbb{Z}_2^n$ can be identified with the quotient ring

$$\mathbb{Z}_2[\mathbb{A}^n] = \mathbb{Z}_2[x_1, \dots, x_n]/\langle x_1^2 + x_1, \dots, x_n^2 + x_n \rangle.$$

Often we think of $\mathbb{Z}_2^n$ as a ring, with coordinate-wise sum and product. We identify $\mathbb{Z}_2^n$ with $\mathrm{P}[n]$, the set of subsets of $[n]$, via characteristic functions: $a \in \mathbb{Z}_2^n$ is identified with the subset $a \subseteq [n]$ such that $i \in a$ if and only if $a_i = 1$. With this identification the product ab of elements in $\mathbb{Z}_2^n$ agrees with the intersection $a \cap b$ of the sets a and b; the sum $a + b$ corresponds with the symmetric difference $a + b = (a \cup b) \setminus (a \cap b)$; the element $a + (1, \dots, 1)$ is identified with the complement $\overline{a}$ of a. Note that $a \cup b = a + b + ab$. We let $\mathrm{PP}[n]$ be the set of families of subsets of $[n]$. For $a \in \mathrm{P}[n]$, let $m^a\colon \mathbb{Z}_2^n \to \mathbb{Z}_2$ be the characteristic function of the set $\{a\} \subseteq \mathrm{P}[n]$. For $a \in \mathrm{P}[n]$ non-empty, let $x^a \in \mathbb{Z}_2[x_1, \dots, x_n]$ be the monomial $x^a = \prod_{i \in a} x_i$. Also set $x^{\varnothing} = 1$. The monomial x^a defines the characteristic function $x^a\colon \mathbb{Z}_2^n \to \mathbb{Z}_2$ of the set $\{b \mid a \subseteq b\} \subseteq \mathrm{P}[n]$. For $a \in \mathrm{P}[n]$ non-empty, let $w^a \in \mathbb{Z}_2[x_1, \dots, x_n]$ be given by $w^a = \prod_{i \in a}(x_i + 1)$. Also set $w^{\varnothing} = 1$. The monomial w^a defines the characteristic function $w^a\colon \mathbb{Z}_2^n \to \mathbb{Z}_2$ of the set $\{b \mid b \subseteq \overline{a}\} \subseteq \mathrm{P}[n]$.

Lemma 2 below follows from the definitions above and the Möbius inversion formula [18], which can be stated as follows. Given maps

$f, g\colon \mathrm{P}[n] \to R$, with R a ring of characteristic 2, then

$$f(b) = \sum_{a \subseteq b} g(a) \quad \text{if and only if} \quad g(b) = \sum_{a \subseteq b} f(a).$$

Lemma 2. *The following identities hold in* $\mathbb{Z}_2[\mathbb{A}^n]$*:*

1) $m^a = x^a w^{\overline{a}}$. 2) $x^a = \sum_{a \subseteq b} m^b$. 3) $m^a = \sum_{a \subseteq b} x^b$. 4) $w^b = \sum_{a \subseteq \overline{b}} m^a$.

5) $m^a = \sum_{\overline{a} \subseteq b} w^b$. 6) $w^b = \sum_{a \subseteq b} x^a$. 7) $x^b = \sum_{a \subseteq b} w^a$. 8) $m^a m^b = \delta_{ab} m^a$.

9) $x^a x^b = x^{a \cup b}$. 10) $w^a w^b = w^{a \cup b}$.

Note that $\mathbb{Z}_2[\mathbb{A}^n] = \mathrm{M}(\mathbb{Z}_2^n, \mathbb{Z}_2)$, indeed a map $f\colon \mathbb{Z}_2^n \to \mathbb{Z}_2$ can be written as

$$\begin{aligned} f &= \sum_{f(a)=1} m^a = \sum_{f(a)=1} x^a w^{\overline{a}} = \sum_{f(a)=1} \prod_{i \in a} x_i \prod_{i \in \overline{a}} (x_i + 1) \\ &= \sum_{f(a)=1, b \subseteq \overline{a}} x^{a \cup b} = \sum_{f(a)=1, a \subseteq b} x^b. \end{aligned}$$

From Lemma 2 we see that there are several natural bases for the $\mathbb{Z}_2$-vector space

$$\mathbb{Z}_2[\mathbb{A}^n] = \mathbb{Z}_2[x_1, \ldots, x_n] / \langle x_1^2 + x_1, \ldots, x_n^2 + x_n \rangle,$$

namely we can pick $\{m^a \mid a \in \mathrm{P}[n]\}$, $\{x^a \mid a \in \mathrm{P}[n]\}$, or $\{w^a \mid a \in \mathrm{P}[n]\}$. We use the following notation to write the coordinates of $f \in \mathbb{Z}_2[\mathbb{A}^n]$ in each one of these bases

$$f = \sum_{a \in \mathrm{P}[n]} f(a) m^a = \sum_{a \in \mathrm{P}[n]} f_x(a) x^a = \sum_{a \in \mathrm{P}[n]} f_w(a) w^a.$$

We obtain three linear maps $f \to f$, $f \to f_x$ and $f \to f_w$ from $\mathbb{Z}_2[\mathbb{A}^n]$ to $\mathrm{M}(\mathbb{Z}_2^n, \mathbb{Z}_2)$. The coordinates f, f_x and f_w are connected, via the Möbius inversion formula, by the relations:

$$f_x(b) = \sum_{a \subseteq b} f(a), f(b) = \sum_{a \subseteq b} f_x(a), f_w(b) = \sum_{a \subseteq b} f(\overline{a}),$$

$$f(b) = \sum_{a \subseteq \overline{b}} f_w(a), f_x(a) = \sum_{a \subseteq b} f_w(b), f_w(a) = \sum_{a \subseteq b} f_x(b).$$

The maps $f \to f_x$ and $f \to f_w$ fail to be ring morphisms. Instead we have the identities:

$$(fg)_x(c) = \sum_{a \cup b = c} f_x(a) g_x(b) \quad \text{and} \quad (fg)_w(c) = \sum_{a \cup b = c} f_w(a) g_w(b).$$

We define a predicate O on finite sets as follows: given a finite set a, then Oa holds if and only if the cardinality of a is an odd number. In other words, O is the map from finite sets to $\mathbb{Z}_2$ such that $Oa = 1$ if and only if the cardinality of a is odd.

Example 3. Let $C \in \mathrm{PP}[n]$. An ordered k-covering of $a \in \mathrm{P}[n]$ by elements of C is a tuple $c_1, \ldots, c_k \in C$ such that $c_1 \cup \cdots \cup c_k = a$. Let $k\text{-}Cov_C(a)$ be the set of k-coverings of a by elements of C. Then $a \in \mathrm{P}[n]$ belongs to C if and only if $|k\text{-}Cov_C(a)|$ is odd for every $k \geqslant 1$. Indeed, let $f \in \mathbb{Z}_2[\mathbb{A}^n]$ be given by

$$f = \sum_{c \in C} x^c = \sum_{a \in \mathrm{P}[n]} 1_C(a) x^a,$$

where $1_C \colon \mathrm{P}[n] \to \mathbb{Z}_2$ is the characteristic function of C. Since $f^k = f$ for every $f \in \mathbb{Z}_2[\mathbb{A}^n]$, we have

$$\begin{aligned} \sum_{a \in \mathrm{P}[n]} 1_C(a) x^a = f = f^k &= \sum_{a \in \mathrm{P}[n]} \Bigg(\sum_{a_1 \cup \cdots \cup a_k = a} \prod_{i=1}^{k} 1_C(a_i) \Bigg) x^a \\ &= \sum_{a \in \mathrm{P}[n]} O(k\text{-}Cov_C(a)) x^a. \end{aligned}$$

We conclude that $1_C(a) = O(k\text{-}Cov_C(a))$, and thus $a \in C$ if and only if $|k\text{-}Cov_C(a)|$ is odd.

3. Differential operators on Boolean affine spaces

Next we consider the algebra of differential operators on affine Boolean spaces. Note that the partial derivatives ∂_i on $\mathbb{Z}_2[x_1, \ldots, x_n]$ do not descent to well-defined operators on $\mathbb{Z}_2[\mathbb{A}^n]$; indeed if we had such an operator, then $0 = x_i + x_i = \partial_i x_i^2 = \partial_i x_i = 1$. The Boolean partial derivative $\partial_i f \colon \mathbb{Z}_2^n \to \mathbb{Z}_2$ of a map $f \colon \mathbb{Z}_2^n \to \mathbb{Z}_2$ is given [6, 17] by

$$\partial_i f(x) = f(x + e_i) + f(x)$$

where $e_i \in \mathbb{Z}_2^n$ is the vector with vanishing entries except at position i. This definition yields well-defined operators $\partial_i \colon \mathbb{Z}_2[\mathbb{A}^n] \to \mathbb{Z}_2[\mathbb{A}^n]$. The operators ∂_i are skew derivations; indeed they satisfy the twisted Leibnitz identity

$$\partial_i(fg) = (\partial_i f)g + (s_i f)(\partial_i g)$$

where the shift operators $s_i \colon \mathbb{Z}_2[\mathbb{A}^n] \to \mathbb{Z}_2[\mathbb{A}^n]$ are given by $s_i f(x) = f(x+e_i)$. Indeed:

$$\begin{aligned}\partial_i(fg)(x) &= f(x+e_i)g(x+e_i) + f(x)g(x)\\ &= [f(x+e_i)+f(x)]g(x) + f(x+e_i)[g(x+e_i)+g(x)]\\ &= \partial_i f(x)g(x) + s_i f(x)\partial_i g(x).\end{aligned}$$

The operators ∂_i are nilpotent:

$$\partial_i^2 f(x) = \partial_i f(x+e_i) + \partial_i f(x) = f(x) + f(x+e_i) + f(x+e_i) + f(x) = 0.$$

Definition 4. The $\mathbb{Z}_2$-algebra BDO_n of Boolean differential operators on $\mathbb{Z}_2^n$ is the $\mathbb{Z}_2$-subalgebra of $\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ generated by ∂_i and the operators of multiplication by x_i for $i \in [n]$.

Theorem 5. *The following identities hold for $x_i, \partial_i, s_i \in \mathrm{BDO}_n$ and $i \in [n]$:*

$$1.\ x_i^2 = x_i; \quad 2.\ \partial_i^2 = 0; \quad 3.\ s_i^2 = 1; \quad 4.\ \partial_i = s_i + 1;$$
$$5.\ \partial_i s_i = s_i\partial_i = \partial_i; \quad 6.\ s_i = \partial_i + 1; \quad 7.\ s_i x_i = x_i s_i + s_i = (x_i+1)s_i;$$
$$8.\ \partial_i x_i = x_i\partial_i + s_i = x_i\partial_i + \partial_i + 1.$$

Proof. We have already shown that $x_i^2 = x_i$ and $\partial_i^2 = 0$. For the other identities we have

- $s_i^2 f(x) = s_i f(x+e_i) = f(x+e_i+e_i) = f(x)$;
- $\partial_i f(x) = f(x+e_i) + f(x) = s_i f(x) + f(x) = (s_i+1)f(x)$;
- $s_i\partial_i f(x) = \partial_i f(x+e_i) = f(x+e_i+e_i) + f(x+e_i) = f(x) + f(x+e_i) = \partial_i f(x)$;
- $\partial_i s_i f(x) = s_i f(x+e_i) + s_i f(x) = f(x+e_i+e_i) + f(x+e_i) = f(x) + f(x+e_i) = \partial_i f(x)$;
- $s_i x_i f(x) = (x_i+1)f(x+e_i) = x_i f(x+e_i) + f(x+e_i) = x_i s_i f(x) + s_i f(x) = (x_i s_i + s_i)f(x)$;
- $s_i f(x) = f(x+e_i) = f(x+e_i) + f(x) + f(x) = \partial_i f(x) + f(x) = (\partial_i + 1)f(x)$;
- $\partial_i(x_i f)(x) = x_i f(x+e_i) + f(x+e_i) + x_i f(x) = x_i(f(x+e_i) + f(x)) + f(x+e_i)$, thus
- $\partial_i(x_i f) = x_i\partial_i f + f(x+e_i) = (x_i\partial_i + s_i)f = (x_i\partial_i + \partial_i + 1)f$. $\square$

The operator ∂_i acts on the bases m^a, x^a and w^a as follows:

$$\partial_i m^a = m^{a+e_i} + m^a,$$
$$\partial_i x^a = \begin{cases} x^{a\setminus i} & \text{if } i \in a \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \partial_i w^a = \begin{cases} w^{a\setminus i} & \text{if } i \in a \\ 0 & \text{otherwise.} \end{cases}$$

From these expressions we obtain that:

- $\partial_i f(a) = 1$ if and only if $f(a) \neq f(a+e_i)$, that is,

$$\partial_i f = \sum_{a\in \mathrm{P}[n], f(a)\neq f(a+e_i)} m^a.$$

- $(\partial_i f)_x(a) = f_x(a\cup i)$ if $i \notin a$, and $(\partial_i f)_x(a) = 0$ if $i \in a$, that is,

$$\partial_i f = \sum_{i\in a\in \mathrm{P}[n]} f_x(a) x^{a-i}.$$

- $(\partial_i f)_w(a) = f_w(a\cup i)$ if $i \notin a$, and $(\partial_i f)_w(a) = 0$ if $i \in a$, that is,

$$\partial_i f = \sum_{i\in a\in \mathrm{P}[n]} f_w(a) w^{a-i}.$$

More generally one can show by induction, for $a, b \in \mathrm{P}[n]$, that:

$$\partial^b m^a = \sum_{c\subseteq b} m^{a+c},$$
$$\partial^b x^a = \begin{cases} x^{a\setminus b} & \text{if } b \subseteq a \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \partial^b w^a = \begin{cases} w^{a\setminus b} & \text{if } b \subseteq a \\ 0 & \text{otherwise.} \end{cases}$$

By definition $\mathrm{BDO}_n \subseteq \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ acts naturally on $\mathbb{Z}_2[\mathbb{A}^n]$, so we get a map

$$\mathrm{BDO}_n \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\mathbb{A}^n] \to \mathbb{Z}_2[\mathbb{A}^n].$$

Proposition 6. *Consider maps $D\colon \mathrm{P}[n]\times \mathrm{P}[n] \to \mathbb{Z}_2$ and $f\colon \mathrm{P}[n] \to \mathbb{Z}_2$.*

1. *Let $D = \sum_{a,b\in \mathrm{P}[n]} D(a,b) m^a \partial^b \in \mathrm{BDO}_n$, $f = \sum_{c\in \mathrm{P}[n]} f(c) m^c \in \mathbb{Z}_2[\mathbb{A}^n]$, and $Df = \sum_{a\in \mathrm{P}[n]} Df(a) m^a$. Then we have*

$$Df(a) = \sum_{e\subseteq b} D(a,b) f(a+e).$$

2. *Let* $D = \sum\limits_{a,b\in \mathrm{P}[n]} D_x(a,b)x^a\partial^b \in \mathrm{BDO}_n,\ f = \sum\limits_{c\in\mathrm{P}[n]} f_x(c)x^c \in \mathbb{Z}_2[\mathbb{A}^n],$ *and* $Df = \sum\limits_{e\in\mathrm{P}[n]} Df_x(e)x^e$. *Then*

$$Df_x(e) = \sum_{\substack{a,b\subseteq c\\ a\cup(c\setminus b)=e}} D_x(a,b)f_x(c).$$

Proof.

1. $Df = \sum\limits_{a,b,c\in\mathrm{P}[n]} D(a,b)f(c)m^a\partial^b m^c = \sum\limits_{a,e\subseteq b,c} D(a,b)f(c)m^a m^{c+e}$

$$= \sum_{a,e\subseteq b} D(a,b)f(a+e)m^a = \sum_{a\in\mathrm{P}[n]}\Big(\sum_{e\subseteq b} D(a,b)f(a+e)\Big)m^a.$$

2. $Df = \sum\limits_{a,b,c\in\mathrm{P}[n]} D_x(a,b)f_x(c)x^a\partial^b x^c = \sum\limits_{a,b\subseteq c} D_x(a,b)f_x(c)x^{a\cup c\setminus b}$

$$= \sum_{e\in\mathrm{P}[n]}\Bigg(\sum_{\substack{a,b\subseteq c\\ a\cup(c\setminus b)=e}} D_x(a,b)f_x(c)\Bigg)x^e. \qquad \square$$

Theorem 7. *For $n \geqslant 1$ we have* $\mathrm{BDO}_n = \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$.

Proof. Note that

$$\dim\Big(\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])\Big) = \dim(\mathbb{Z}_2[\mathbb{A}^n])\dim(\mathbb{Z}_2[\mathbb{A}^n]) = 2^n2^n = 2^{2n}.$$

The set $\{x^a\partial^b \mid a,b\in\mathrm{P}[n]\}$ has 2^{2n} elements and generates BDO_n as a vector space over $\mathbb{Z}_2$; thus it is enough to show that it is a linearly independent set. Suppose that

$$\sum_{a,b\in\mathrm{P}[n]} f(a,b)x^a\partial^b = \sum_{b\in\mathrm{P}[n]}\Big(\sum_{a\in\mathrm{P}[n]} f(a,b)x^a\Big)\partial^b = 0.$$

Pick a minimal set $c\in\mathrm{P}[n]$ such that $\sum\limits_{a\in\mathrm{P}[n]} f(a,c)x^a \neq 0$. We have

$$\begin{aligned}\Big(\sum_{a,b\in\mathrm{P}[n]} f(a,b)x^a\partial^b\Big)(x^c) &= \sum_{b\in\mathrm{P}[n]}\Big(\sum_{a\in\mathrm{P}[n]} f(a,b)x^a\Big)\partial^b(x^c)\\ &= \sum_{a\in\mathrm{P}[n]} f(a,c)x^a = 0.\end{aligned}$$

Therefore, since $\{x^a \mid a \in \mathrm{P}[n]\}$ is a basis for $\mathbb{Z}_2[\mathbb{A}^n]$, we have $f(a,c) = 0$ in contradiction with the fact $\sum_{a\in\mathrm{P}[n]} f(a,c)x^a \neq 0$. We conclude that $\dim(\mathrm{BDO}_n) = 2^{2n}$ yielding the desired result. $\square$

Putting together Proposition 6 and Theorem 7 we get a couple of explicit ways of identifying BDO_n with $\mathrm{M}_{2^n}(\mathbb{Z}_2)$, the algebra of square matrices of size 2^n with coefficients in $\mathbb{Z}_2$. Note that $\mathrm{M}_{2^n}(\mathbb{Z}_2)$ may be identified with $\mathrm{M}(\mathrm{P}[n] \times \mathrm{P}[n], \mathbb{Z}_2)$. Moreover, we can identify $\mathrm{M}(\mathrm{P}[n] \times \mathrm{P}[n], \mathbb{Z}_2)$ with the set of directed graphs with vertex set $\mathrm{P}[n]$ and without multiple edges as follows: given a matrix $M \in \mathrm{M}_{2^n}(\mathbb{Z}_2)$ its associated graph has an edge from b to a if and only if $M_{a,b} = 1$. Let $\mathrm{R}\colon \mathrm{BDO}_n \to \mathrm{M}_{2^n}(\mathbb{Z}_2)$ be the $\mathbb{Z}_2$-linear map constructed as follows. Consider the bases

$$\{m^a\partial^b \mid a,b \in \mathrm{P}[n]\} \text{ for } \mathrm{BDO}_n \quad \text{and} \quad \{m^a \mid a \in \mathrm{P}[n]\} \text{ for } \mathbb{Z}_2[\mathbb{A}^n].$$

For $a, b \in \mathrm{P}[n]$, let $\mathrm{R}(m^a\partial^b)$ be the matrix of $m^a\partial^b$ on the basis m^a. The action of $m^a\partial^b$ on m^c is given by $m^a\partial^b m^c = m^a \sum_{e\subseteq b} m^{c+e} = \sum_{e\subseteq b} m^a m^{c+e} = m^a \mathrm{if} c + a \subseteq b$ and zero otherwise. Therefore, the matrix $\mathrm{R}(m^a\partial^b)$ is given for $c, d \in \mathrm{P}[n]$ by the rule

$$\mathrm{R}(m^a\partial^b)_{c,d} = \begin{cases} 1 & \text{if } c = a \text{ and } d + a \subseteq b, \\ 0 & \text{otherwise.} \end{cases}$$

Example 8. The graph of $\mathrm{R}(m^{\{1,2\}}\partial^{\{2,3\}})$ is show in Figure 1.

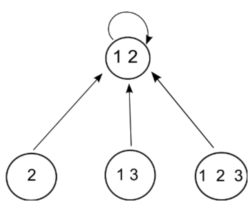

Figure 1. Graph of the matrix $\mathrm{R}(m^{\{1,2\}}\partial^{\{2,3\}})$.

For a second representation consider the $\mathbb{Z}_2$-linear map $\mathrm{S}\colon \mathrm{BDO}_n \to \mathrm{M}_{2^n}(\mathbb{Z}_2)$ constructed as follows. Consider the bases

$$\{x^a\partial^b \mid a,b \in \mathrm{P}[n]\} \text{ for } \mathrm{BDO}_n \quad \text{and} \quad \{x^a \mid a \in \mathrm{P}[n]\} \text{ for } \mathbb{Z}_2[\mathbb{A}^n].$$

For $a, b \in \mathrm{P}[n]$ let $\mathrm{S}(x^a\partial^b)$ be the matrix of $x^a\partial^b$ on the basis x^a. The action of $x^a\partial^b$ on x^c is given by

$$x^a\partial^b x^c = \begin{cases} x^{a\cup c\setminus b} & \text{if } b \subseteq c, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the matrix $\mathrm{S}(x^a\partial^b)$ is given for $c, d \in \mathrm{P}[n]$ by the rule

$$\mathrm{S}(x^a\partial^b)_{c,d} = \begin{cases} 1 & \text{if } c = a \cup d \setminus b \text{ and } b \subseteq d, \\ 0 & \text{otherwise.} \end{cases}$$

Example 9. The graph associated to the matrix $\mathrm{S}(m^{\{1\}}\partial^{\{3\}})$ is shown in Figure 2.

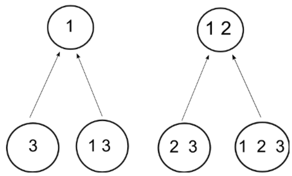

FIGURE 2. Graph of the matrix $\mathrm{S}(m^{\{1\}}\partial^{\{3\}})$.

4. Boole-Weyl Algebras

First we motivate, from the viewpoint of canonical quantization, our definition of Boole-Weyl algebras. Canonical phase space for a field k of characteristic zero can be identified with the affine space $k^n \times k^n$. The Poisson bracket on $k[x_1, \dots, x_n, y_1, \dots, y_n]$ in canonical coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ on $k^n \times k^n$ is given by

$$\{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0, \quad \{x_i, y_j\} = \delta_{i,j}.$$

Equivalently, the Poisson bracket is given for $f, g \in k[x_1, \dots, x_n, y_1, \dots, y_n]$ by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial f}{\partial x_i}.$$

Canonical quantization may be formulated as the problem of promoting the commutative variables x_i and y_j into non-commutative operators $\widehat{x}_i$ and $\widehat{y}_j$ satisfying the commutation relations:

$$[\widehat{x}_i, \widehat{x}_j] = 0, \quad [\widehat{y}_i, \widehat{y}_j] = 0, \quad [\widehat{y}_i, \widehat{x}_j] = \delta_{i,j}.$$

Note that the free algebra generated by $\widehat{x}_i$ and $\widehat{y}_j$ subject to the above relations is precisely what is called the Weyl algebra and its usually denoted by A_n. Now let $k = \mathbb{Z}_2$ and consider the affine phase spaces

$\mathbb{Z}_2^n \times \mathbb{Z}_2^n$. Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be canonical coordinates on $\mathbb{Z}_2^n \times \mathbb{Z}_2^n$. The analogue of the Poisson bracket

$$\{\,,\}\colon \mathbb{Z}_2[\mathbb{A}^{2n}] \otimes \mathbb{Z}_2[\mathbb{A}^{2n}] \to \mathbb{Z}_2[\mathbb{A}^{2n}]$$

can be expressed for $f, g \in \mathbb{Z}_2[\mathbb{A}^{2n}]$ as

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} + \frac{\partial f}{\partial y_i}\frac{\partial f}{\partial x_i},$$

where $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$ are the Boolean partial derivatives along the coordinates x_i and y_i. Clearly, the full set of axioms for a Poisson bracket will not longer hold, e.g. Boolean derivatives are skew derivations. Nevertheless, the bracket is still determined by its values on the canonical coordinates: $\{x_i, x_j\} = 0, \{y_i, y_j\} = 0, \{x_i, y_j\} = \delta_{i,j}$. Canonical quantization consists in promoting the commutative variables x_i and y_j to non-commutative operators $\widehat{x}_i$ and $\widehat{y}_j$ satisfying the commutation relations:

$$[\widehat{x}_i, \widehat{x}_j] = 0,\ [\widehat{y}_i, \widehat{y}_j] = 0,\ [\widehat{y}_i, \widehat{x}_j] = 0 \text{ for } i \neq j, \quad \text{and} \quad [\widehat{y}_i, \widehat{x}_i]_{s_i} = 1.$$

Note that in the last relation we use the twisted commutator

$$[f, g]_{s_i} = fg + (s_i f)g;$$

this choice is expected since the operators $\widehat{y}_i$ are skew derivations instead of usual derivations. The relation $[\widehat{y}_i, \widehat{x}_i]_{s_i} = 1$ can be equivalently written using commutators as $[\widehat{y}_i, \widehat{x}_i] = \widehat{y}_i + 1$.

Definition 10. The Boole-Weyl algebra BA_n is the quotient of $\mathbb{Z}_2\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$, the free associative $\mathbb{Z}_2$-algebra generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$, by the ideal

$$\langle x_i^2 + x_i, x_i x_j + x_j x_i, y_i y_j + y_j y_i, y_i^2, y_i x_j + x_j y_i, y_i x_i + x_i y_i + y_i + 1\rangle.$$

Theorem 11. *The map $\mathbb{Z}_2\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ sending x_i to the operator of multiplication by x_i, and y_i to ∂_i, descends to an isomorphism $\mathrm{BA}_n \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ of $\mathbb{Z}_2$-algebras.*

Proof. By Theorem 5 the given map descends. By definition it is a surjective map $\mathrm{BA}_n \to \mathrm{BDO}_n = \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$. Moreover, this map is an isomorphisms since $\dim(\mathrm{BA}_n) = \dim(\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n]))$. Indeed using the commutation relations it is easy to check that the natural map

$$\mathbb{Z}_2[x_1, \ldots, x_n]/\langle x_i^2 + x_i\rangle \otimes \mathbb{Z}_2[y_1, \ldots, y_n]/\langle y_i^2\rangle \to \mathrm{BA}_n$$

is surjective. If $\sum\limits_{a,b\in\mathrm{P}[n]} f(a,b)x^a \otimes y^b$ is in the kernel of the latter map, then the Boolean differential operator $\sum\limits_{a,b\in\mathrm{P}[n]} f(a,b)x^a\partial^b$ would vanish, and therefore the coefficients $f(a,b)$ must vanish as well. Thus

$$\begin{aligned}\dim(\mathrm{BA}_n) &= \dim(\mathbb{Z}_2[x_1,\ldots,x_n]/\langle x_i^2+x_i\rangle)\dim(\mathbb{Z}_2[y_1,\ldots,y_n]/\langle y_i^2\rangle)\\ &= 2^n2^n = \dim(\mathbb{Z}_2[\mathbb{A}^n])\dim(\mathbb{Z}_2[\mathbb{A}^n]) = \dim(\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])).\square\end{aligned}$$

Theorem 12. The map $\mathbb{Z}_2\langle x_1,\ldots,x_n,y_1,\ldots,y_n\rangle \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ sending x_i to the operator of multiplication by $w_i = x_i+1$, and y_i to the operator ∂_i, descends to an isomorphism $\mathrm{BA}_n \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ of $\mathbb{Z}_2$-algebras.

Proof. Follows from the fact that w_i and ∂_j satisfy exactly the same relation as x_i and ∂_j. $\square$

Corollary 13. Any identity in BA_n involving x_i and ∂_j has an associated identity involving w_i and ∂_j obtained by replacing x_i by w_i.

Lemma 14. *For $a,b,c,d \in \mathrm{P}[n]$ the following identities hold in BW_n:*

1. $y^b m^c = \sum\limits_{b_1\subseteq b_2\subseteq b} m^{c+b_2}y^{b_1}$.
2. $m^a y^b m^c y^d = \sum\limits_{\substack{d\subseteq e\\ e\setminus d\subseteq a+c\subseteq b}} m^a y^e$.
3. $y^b x^c = \sum\limits_{k_1\subseteq k_2\subseteq b\cap c} x^{c\setminus k_2}y^{b\setminus k_1}$.
4. $x^a y^b x^c y^d = \sum\limits_{a\subseteq e, d\subseteq f} c(a,b,c,d,e,f)x^e y^f$, *where* $c(a,b,c,d,e,f) = O\{k_1\subseteq k_2\subseteq b\cap c \mid a\cup(c\setminus k_2) = e, b\setminus k_1 = f\setminus d\}$.

Proof. 1. By Theorem 11 it is enough to show that the differential operators associated with both sides of the equation are equal. Consider the operator of multiplication by $f\colon \mathbb{Z}_2^n \to \mathbb{Z}_2$ and let $g\colon \mathbb{Z}_2^n \to \mathbb{Z}_2$ be any other map. The twisted Leibnitz rule $\partial_i(fg) = (\partial_i f)g + (s_i f)\partial_i g$ can be extended, since s_i and ∂_i commute, to the identity:

$$\partial^b(fg) = \sum_{b_1\sqcup b_2=b}(s^{b_2}\partial^{b_1}f)\partial^{b_2}g,$$

thus the following identity holds in BA_n:

$$y^b f = \sum_{b_1\sqcup b_2=b} s^{b_2}(\partial^{b_1}f)y^{b_2} \quad \text{for } f\in\mathbb{Z}_2[x_1,\ldots,x_n].$$

In particular we obtain that

$$y^b m^c = \sum_{b_1 \sqcup b_2 \subseteq b} s^{b_2} s^{b_1}(m^c) y^{b_2} = \sum_{b_1 \sqcup b_2 \subseteq b} m^{c+b_1+b_2} y^{b_2} = \sum_{b_1 \subseteq b_2 \subseteq b} m^{c+b_2} y^{b_1}.$$

2. We have

$$m^a y^b m^c y^d = \sum_{b_1 \subseteq b_2 \subseteq b} m^a m^{c+b_2} y^{b_1} y^d = \sum_{b_1 \subseteq b_2 \subseteq b} \delta_{a,c+b_2} m^a y^{b_1 \sqcup d}$$

$$= \sum_{b_1 \subseteq a+c \subseteq b} m^a y^{b_1 \sqcup d} = \sum_{\substack{d \subseteq e \\ e \setminus d \subseteq a+c \subseteq b}} m^a y^e.$$

where the last identity follows from the fact that $b_2 = a+c$ and $e = b_1 \sqcup d$.

3. From the relations $y_i x_j = x_j y_i$ for $i \neq j$ and $y_i x_i = x_i y_i + y_i + 1$ we can argue as follows. If a letter y_i is placed just to the left of a x_j we can move it to the right, since these letters commute. If instead we have a product $y_i x_i$, then three options arises:

a) y_i moves to the right of x_i;

b) y_i absorbs x_i;

c) x_i and y_i annihilate each other leaving an 1.

Call k_1 the set of indices for which c) occurs, and k_2 the set of indices for which either b) or c) occur. Then $k_1 \subseteq k_2 \subseteq b \cap c$ and the set for which option a) occurs is $b \cap c \setminus k_2$. Thus the desired identity is obtained. 4. We have

$$x^a y^b x^c y^d = \sum_{k_1 \subseteq k_2 \subseteq b \cap c} x^{a \cup c \setminus k_2} y^{(b \setminus k_1) \sqcup d} = \sum_{a \subseteq e, d \subseteq f} c(a,b,c,d,e,f) x^e y^f,$$

where $c(a,b,c,d,e,f) = O\{k_1 \subseteq k_2 \subseteq b \cap c \mid a \cup (c \setminus k_2) = e,\ b \setminus k_1 = f \setminus d\}$. $\square$

Example 15. $y^{\{1\}} m^{\{1\}} = m^{\{1\}} + m^{\varnothing} + m^{\{\varnothing\}} y^{\{1\}}$; $m^{\{1\}} y^{\{1\}} m^{\{1\}} y^{\{1\}} = m^{\{1\}} y^{\{1\}}$; $y^{\{1\}} m^{\{1,2\}} = m^{\{1,2\}} + m^{\{2\}} + m^{\{2\}} y^{\{1\}}$; $m^{\{2\}} y^{\{1\}} m^{\{1,2\}} y^{\{1\}} = m^{\{2\}} y^{\{1\}}$; $y^{\{1,2\}} m^{\{1,2,3\}} = m^{\{1,2,3\}} + m^{\{2,3\}} + m^{\{1,3\}} + m^{\{1\}} + m^{\{2,3\}} y^{\{1\}} + m^{\{3\}} y^{\{1\}} + m^{\{1,3\}} y^{\{2\}} + m^{\{3\}} y^{\{2\}} + m^{\{3\}} y^{\{1,2\}}$; $m^{\{3\}} y^{\{1,2\}} m^{\{1,2,3\}} y^{\{1\}} = m^{\{3\}} y^{\{1,2\}}$.

Example 16. For $i \in [k]$ assume given $A_i \in \mathrm{PP}[n]$ and $f_i = \sum_{a \in A_i} y^a$. Then

$$f_1 \cdots f_k = \sum_{b \in \mathrm{P}[n]} O\{(a_1, \ldots, a_k) \in A_1 \times \ldots \times A_k \mid a_1 \sqcup \cdots \sqcup a_k = b\} y^b.$$

In particular, for $A \in \mathrm{PP}[n]$ and $f = \sum_{a\in A} y^a$, we get that

$$f^k = \sum_{b\in \mathrm{P}[n]} O\{a_1, \ldots, a_k \in A \mid a_1 \sqcup \cdots \sqcup a_k = b\} y^b.$$

For example, if $A = \mathrm{P}[n]$ then for $k \geqslant 2$ we have

$$f^k = \sum_{b\in \mathrm{P}[n]} O\{a_1, \ldots, a_k \in \mathrm{P}[n] \mid a_1 \sqcup \cdots \sqcup a_k = b\} y^b = \sum_{b\in \mathrm{P}[n]} (k^{|b|} \bmod 2) y^b,$$

thus $f^k = f$ if k is odd and $f^k = 1$ if k is even.

From Lemma 2 we see that there are several natural basis for BA_n, namely:

$$\{m^a y^b | a, b \in \mathrm{P}[n]\}, \quad \{x^a y^b \mid a, b \in \mathrm{P}[n]\}, \quad \{w^a y^b \mid a, b \in \mathrm{P}[n]\}.$$

We write the coordinates of $f \in \mathrm{BA}_n$ in these bases as:

$$f = \sum_{a,b\in \mathrm{P}[n]} f_m(a,b) m^a y^b = \sum_{a,b\in \mathrm{P}[n]} f_x(a,b) x^a y^b = \sum_{a,b\in \mathrm{P}[n]} f_w(a,b) w^a y^b.$$

These coordinates systems are connected by the relations:

$$f_x(b,c) = \sum_{a\subseteq b} f_m(a,c), f_m(b,c) = \sum_{a\subseteq b} f_x(a,c), f_w(b,c) = \sum_{a\subseteq b} f_m(\overline{a},c),$$

$$f_m(b,c) = \sum_{a\subseteq \overline{b}} f_w(a,c), f_x(a,c) = \sum_{a\subseteq b} f_w(b,c), f_w(a) = \sum_{a\subseteq b} f_x(b,c).$$

Theorem 17. *For $f, g \in \mathrm{BA}_n$ the following identities hold for $a, e, h \in \mathrm{P}[n]$:*

1) $(fg)_m(a,e) = \sum\limits_{\substack{b,c,d\subseteq e \\ e\setminus d\subseteq a+c\subseteq b}} f_m(a,b) g_m(c,d).$

2) $(fg)_x(e,h) = \sum\limits_{a\subseteq e, b,c,d\subseteq h} c(a,b,c,d,e,h) f_x(a,b) g_x(c,d)$, *where*

$$c(a,b,c,d,e,h) = O\Big\{k_1 \subseteq k_2 \subseteq b\cap c \mid a \cup (c\setminus k_2) = e, b\setminus k_1 = h\setminus d\Big\}.$$

Proof. 1. Let $f = \sum_{a,b\in \mathrm{P}[n]} f_m(a,b) m^a y^b$, $g = \sum_{c,d\in \mathrm{P}[n]} g_m(c,d) m^c y^d$, then

$$\begin{aligned} fg &= \sum_{a,b,c,d\in \mathrm{P}[n]} f_m(a,b) g_m(c,d) m^a y^b m^c y^d \\ &= \sum_{a,b,c,d,e\in \mathrm{P}[n]} f_m(a,b) g_m(c,d) \sum_{\substack{d\subseteq e \\ e\setminus d\subseteq a+c\subseteq b}} m^a y^e \\ &= \sum_{d\subseteq e, e\setminus d\subseteq a+c\subseteq b} f_m(a,b) g_m(c,d) m^a y^e. \end{aligned}$$

2. Let $f=\sum_{a,b\in \mathrm{P}[n]} f_x(a,b)x^ay^b$ and $g=\sum_{c,d\in \mathrm{P}[n]} g_x(c,d)x^cy^d$, then

$$\begin{aligned} fg &= \sum_{a,b,c,d\in \mathrm{P}[n]} f_x(a,b)g_x(c,d)x^ay^bx^cy^d \\ &= \sum_{a,b,c,d\in \mathrm{P}[n]} \sum_{a\subseteq e, d\subseteq h} f_x(a,b)g_x(c,d)c(a,b,c,d,e,h)x^ey^h, \end{aligned}$$

where $c(a,b,c,d,e,h)=O\{k_1\subseteq k_2\subseteq b\cap c \,|\, a\cup(c\setminus k_2)=e,\ b\setminus k_1=h\setminus d\}$. $\square$

Example 18. Let $x^ry^r=\sum\limits_{a,b\in \mathrm{P}[n]} f_m(a,b)m^ay^b$ and $x^sy^s=\sum\limits_{a,b\in \mathrm{P}[n]} g_m(a,b)m^ay^b$. Then

$$(fg)^2_m(a,e)=\sum_{\substack{b,c,d\subseteq e\\ e\setminus d\subseteq a+c\subseteq b}} f_m(a,b)g_m(c,d).$$

For a non-vanishing summand we must have that $a=b=r$, $c=d=s$, and $s\subseteq e$. The conditions $e\setminus s\subseteq r+s\subseteq r$ implies that $s\subseteq r$ and $e\setminus s\subseteq r\setminus s$, thus $e\subseteq r$. We conclude that $(fg)^2_m(a,e)=1$ if and only if $s\subseteq r$, $a=r$ and $s\subseteq e\subseteq r$. Thus $x^ry^rx^sy^s=0$ if $s\nsubseteq r$. For $s\subseteq r$ we get

$$x^ry^rx^sy^s=\sum_{s\subseteq e\subseteq r} x^ry^e.$$

In particular we get that $(x^ry^r)^n=x^ry^r$.

Example 19. Let $f=\sum_{a,b\in \mathrm{P}[n]} m^ay^b=\sum_{a,b\in \mathrm{P}[n]} f_m(a,b)m^ay^b$. We have

$$f^2_m(a,e)=\sum_{\substack{b,c,d\subseteq e\\ e\setminus d\subseteq a+c\subseteq b}} 1 = O\Big\{b,c,d \mid d\subseteq e, e\setminus d\subseteq a+c\subseteq b\Big\}.$$

Note that if $a+c$ is not equal to $[n]$, then there are an even number of choices for b, thus we can assume that $c=\overline{a}$ and $b=[n]$. The condition $e\setminus d\subseteq a+c=[n]$ becomes trivial, and therefore $f^2(a,e)=O\mathrm{P}[|e|]=0$ if $e\neq\varnothing$ and $f^2(a,e)=1$ if $e=\varnothing$. Therefore we have $f^2=\sum_{a\in \mathrm{P}[n]} m^a$.

Example 20. Let $r=\sum_{i\in[n]} x^{\{i\}}y^{\{i\}}=\sum_{a,b\in \mathrm{P}[n]} r_x(a,b)x^ay^b\in \mathrm{BA}_n$, then $r^2_x(e,f)=\sum\limits_{a\subseteq e,b,c,d\subseteq f} c(a,b,c,d,e,f)r_x(a,b)r_x(c,d)$, where $c(a,b,c,d,e,f)=O\{k_1\subseteq k_2\subseteq b\cap c \mid a\cup(c\setminus k_2)=e, b\setminus k_1=f\setminus d\}$. Clearly $|a|=|b|=|c|=|d|=1$, $a=b$, and $c=d$. Moreover, we have $|b\cap c|\leqslant 1$. If $|b\cap c|=1$, then $a=b=c=d=e=\{i\}$ for some $i\in[n]$. If $k_1=\varnothing$, then there are two options for k_2 leading to a vanishing coefficient. Thus we may assume that $k_1 = k_2 = \{i\}$ and then necessarily $f = \{i\}$. Thus we conclude that $r_x^2(\{i\},\{i\}) = 1$. If instead $|b \cap c| = 0$, then $k_1 = k_2 = \varnothing$, $a \cup c = e$ and $b = f \setminus d$. Let $i \neq j$ and suppose that $a = b = \{i\}$ and $c = d = \{j\}$. Then $e = f = \{i,j\}$ and $r_x^2(\{i,j\},\{i,j\}) = 1$. All together we conclude that

$$r^2 = \sum_{i\in[n]} x^{\{i\}}y^{\{i\}} + \sum_{i\neq j\in[n]} x^{\{i,j\}}y^{\{i,j\}}.$$

Example 21. From Corollary 13 we see that if $s = \sum_{i\in[n]} w^{\{i\}}y^{\{i\}}$ then

$$s^2 = \sum_{i\in[n]} w^{\{i\}}y^{\{i\}} + \sum_{i\neq j\in[n]} w^{\{i,j\}}y^{\{i,j\}}.$$

Equivalently, if $s = \sum_{i\in[n]} y^{\{i\}} + \sum_{i\in[n]} x^{\{i\}}y^{\{i\}}$ then $s^2 = \sum_{i\in[n]} y^{\{i\}} + \sum_{i\in[n]} x^{\{i\}}y^{\{i\}} + \sum_{i\neq j\in[n]} y^{\{i,j\}} + \sum_{i\neq j\in[n]} x^{\{i,j\}}y^{\{i,j\}} + \sum_{i\neq j\in[n]}(x^{\{i\}} + x^{\{j\}})y^{\{i,j\}}$.

5. A shifted presentation

So far, the operators ∂_i have played the main role. In this section we take an alternative viewpoint and let the operators s_i be the main characters. Recall that the Boolean partial derivatives and the Boolean shift operators are related by the identities $y_i = s_i + 1$ and $s_i = y_i + 1$. For $a, b \in \mathrm{P}[n]$, set $y^a = \prod\limits_{i\in a} y_i$ and $s^a = \prod\limits_{i\in a} s_i$. We have

$$y^b = \prod_{i\in b} y_i = \prod_{i\in b}(s_i + 1) = \sum_{a\subseteq b} s^a,$$

and by the Möbius inversion formula $s^b = \sum_{a\subseteq b} y^a$.

Proposition 22. *Consider maps $D\colon \mathrm{P}[n]\times\mathrm{P}[n] \to \mathbb{Z}_2$ and $f\colon \mathrm{P}[n] \to \mathbb{Z}_2$.*

1. Let $D = \sum\limits_{a,b\in\mathrm{P}[n]} D(a,b)m^a s^b \in \mathrm{BDO}_n$, $f = \sum\limits_{c\in\mathrm{P}[n]} f(c)m^c \in \mathbb{Z}_2[\mathbb{A}^n]$, and $Df = \sum\limits_{a\in\mathrm{P}[n]} Df(a)m^a$. Then we have

$$Df(a) = \sum_{b\in\mathrm{P}[n]} D(a,b)f(a+b).$$

2. Let $D = \sum\limits_{a,b \in \mathrm{P}[n]} D_x(a,b) x^a s^b \in \mathrm{BDO}_n$*,* $f = \sum\limits_{c \in \mathrm{P}[n]} f_x(c) x^c \in \mathbb{Z}_2[\mathbb{A}^n]$*, and* $Df = \sum\limits_{d \in \mathrm{P}[n]} Df_x(d) x^d$*. Then*

$$Df_x(d) = \sum_{\substack{a, e \subseteq b \cap c \\ a \cup (c \setminus e) = d}} D_x(a,b) f_x(c).$$

Proof. 1. $Df = \sum\limits_{a,b,c \in \mathrm{P}[n]} D(a,b) f(c) m^a s^b m^c = \sum\limits_{a,b,c} D(a,b) f(c) m^a m^{b+c} = \sum\limits_{a,b} D(a,b) f(a+b) m^a = \sum\limits_{a \in \mathrm{P}[n]} \Bigg(\sum\limits_{b \in \mathrm{P}[n]} D(a,b) f(a+b) \Bigg) m^a.$

2. $Df = \sum\limits_{a,b,c \in \mathrm{P}[n]} D_x(a,b) f_x(c) x^a s^b x^c = \sum\limits_{a, e \subseteq b \cap c} D_x(a,b) f_x(c) x^{a \cup c \setminus e} = \sum\limits_{d \in \mathrm{P}[n]} \Bigg(\sum\limits_{\substack{a, e \subseteq b \cap c \\ a \cup (c \setminus e) = d}} D_x(a,b) f_x(c) \Bigg) x^d.$ □

Proposition 22 and Theorem 7 provide a couple of explicit ways of identifying BDO_n with $\mathrm{M}_{2^n}(\mathbb{Z}_2)$ the algebra of square matrices of size 2^n with coefficients in $\mathbb{Z}_2$. Consider the $\mathbb{Z}_2$-linear map $\mathrm{R} \colon \mathrm{BDO}_n \to \mathrm{M}_{2^n}(\mathbb{Z}_2)$ sending $m^a s^b$ to $\mathrm{R}(m^a s^b)$ the matrix of $m^a s^b$ on the basis m^a. The action of $m^a s^b$ on m^c is given by $m^a s^b m^c = m^a$ if $c = a + b$ and 0 otherwise. Therefore, the matrix $\mathrm{R}(m^a s^b)$ is given for $c, d \in \mathrm{P}[n]$ by the rule

$$\mathrm{R}(m^a s^b)_{c,d} = \begin{cases} 1 & \text{if } c = a \text{ and } d = a + b \\ 0 & \text{otherwise.} \end{cases}$$

Example 23. The graph of the matrix $\mathrm{R}(m^a s^b)_{c,d}$ is shown in Figure 3.

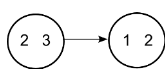

FIGURE 3. Graph of the matrix $\mathrm{R}(m^{\{1,2\}} \partial^{\{2,3\}})$.

For a second representation consider $\mathbb{Z}_2$-linear the map $\mathrm{S} \colon \mathrm{BDO}_n \to \mathrm{M}_{2^n}(\mathbb{Z}_2)$ sending $x^a s^b$ to $\mathrm{S}(x^a s^b)$, the matrix of $x^a s^b$ on the basis x^a. The action of $x^a s^b$ on x^c is given by $x^a s^b x^c = \sum_{e \subseteq b \cap c} x^{a \cup c \setminus e}$. Therefore, the matrix $\mathrm{S}(x^a s^b)$ is given for $c, d \in \mathrm{P}[n]$ by the rule

$$\mathrm{S}(x^a s^b)_{c,d} = \begin{cases} 1 & \text{if } O\{e \subseteq b \cap d \mid c = a \cup d \setminus e\} \\ 0 & \text{otherwise} \end{cases}$$

Example 24. The graph of the matrix $\mathrm{S}(x^{\{1,2\}}s^{\{1,3\}})$ is shown in Figure 4.

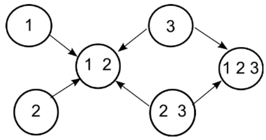

FIGURE 4. Graph of the matrix $\mathrm{R}(m^{\{1,2\}}\partial^{\{2,3\}})$.

Definition 25. The shifted Boole-Weyl algebra SBA_n is the quotient of $\mathbb{Z}_2\langle x_1, \ldots, x_n, s_1, \ldots, s_n\rangle$, the free associative $\mathbb{Z}_2$-algebra generated by $x_1, \ldots, x_n, s_1, \ldots, s_n$, by the ideal

$$\langle x_i^2 + x_i, x_ix_j + x_jx_i, s_is_j + s_js_i, s_i^2 + 1, s_ix_j + x_js_i, s_ix_i + x_is_i + s_i\rangle.$$

Theorem 26. *The map $\mathbb{Z}_2\langle x_1, \ldots, x_n, s_1, \ldots, s_n\rangle \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ sending x_i to the operator of multiplication by x_i, and s_i to the shift operator in the i-direction, descends to an isomorphism $\mathrm{SBA}_n \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ of $\mathbb{Z}_2$-algebras.*

Proof. One can check that the map

$$\mathbb{Z}_2\langle x_1, \ldots, x_n, s_1, \ldots, s_n\rangle \to \mathbb{Z}_2\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$$

sending x_i to x_i and s_i to $y_i + 1$ descends to an algebra isomorphisms $\mathrm{SBA}_n \to \mathrm{BA}_n$. The result then follows from Theorem 11. □

Theorem 27. *The map $\mathbb{Z}_2\langle x_1, \ldots, x_n, s_1, \ldots, s_n\rangle \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ sending x_i to the operator of multiplication by $w_i = x_i + 1$, and s_i to the shift operator in the i-direction, descends to an isomorphism $\mathrm{SBA}_n \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ of $\mathbb{Z}_2$-algebras.*

Proof. Follows from the fact that w_i and s_j satisfy exactly the same relation as x_i and s_j. □

Corollary 28. *Any identity in SBA_n involving x_i and s_j has an associated identity involving w_i and s_j obtained by replacing x_i by w_i.*

Lemma 29. *For $a, b, c, d \in \mathrm{P}[n]$ the following identities hold in SBA_n:*

1. $s^bm^c = m^{b+c}s^b$.
2. $m^as^bm^cs^d = \delta_{a,b+c}m^as^{b+d}$.
3. $s^bx^c = \displaystyle\sum_{k\subseteq b\cap c} x^{c\setminus k}s^b$.
4. $x^as^bx^cs^d = \displaystyle\sum_{e\subseteq b\cap c} x^{a\cup c\setminus e}s^{b+d}$.

Proof. 1. For any $f \in \mathbb{Z}_2[\mathbb{A}^n]$ we have

$$(s^b m^c f)(x) = m^c(x+b)f(x+b) = m^{b+c}(x)f(x+b) = m^{b+c}s^b f(x),$$

thus $s^b m^c = m^{b+c}s^b$.

2. $m^a s^b m^c s^d = m^a m^{b+c} s^b s^d = \delta_{a,b+c} m^{b+c} s^{b+d}$.

3. From the identity $s_i x_i = x_i s_i + s_i$ we see that as s_i pass to the right of x_i, it may or may not absorb x_i. The set $k \subseteq b \cap c$ is the set of indices for which x_i is absorbed by s_i.

4. $x^a s^b x^c s^d = \sum_{e \subseteq b\cap c} x^a x^{c\setminus e} s^b s^d = \sum_{e \subseteq b \cap c} x^{a\cup c \setminus e} s^{b+d}$. □

Example 30.

1. $s^{[n]} m^c = m^{\overline{c}} s^{[n]}, \overline{c} = [n] \setminus c.$
2. $m^{\overline{c}} s^{[n]} m^c s^d = m^{\overline{c}} s^{\overline{d}}.$
3. $s^{[n]} x^c = \displaystyle\sum_{k\subseteq c} x^k s^{[n]}.$
4. $x^a s^{[n]} x^c s^d = \displaystyle\sum_{k \subseteq c} x^{a\cup k} s^{\overline{d}}.$

From Lemma 2 we see that there are several natural basis for BA_n, namely:

$$\{m^a s^b | a, b \in \mathrm{P}[n]\}, \{x^a s^b \mid a, b \in \mathrm{P}[n]\}, \{w^a s^b \mid a, b \in \mathrm{P}[n]\}.$$

We write the coordinates of $f \in \mathrm{SBA}_n$ in these bases as:

$$f = \sum_{a,b\in \mathrm{P}[n]} f_{m,s}(a,b) m^a s^b = \sum_{a,b\in \mathrm{P}[n]} f_{x,s}(a,b) x^a s^b = \sum_{a,b\in \mathrm{P}[n]} f_{w,s}(a,b) w^a s^b.$$

These coordinates systems are connected by the relations:

$$f_{x,s}(b,c) = \sum_{a\subseteq b} f_{m,s}(a,c),\ f_{m,s}(b,c) = \sum_{a\subseteq b} f_{x,s}(a,c),\ f_{w,s}(b,c) = \sum_{a\subseteq b} f_{m,s}(\overline{a},c),$$
$$f_{m,s}(b,c) = \sum_{a\subseteq \overline{b}} f_{w,s}(a,c),\ f_{x,s}(a,c) = \sum_{a\subseteq b} f_{w,s}(b,c),\ f_{w,s}(a) = \sum_{a\subseteq b} f_{x,s}(b,c).$$

Theorem 31. *For $f, g \in \mathrm{SBA}_n$ the following identities hold for $a, b, e, h \in \mathrm{P}[n]$:*

1. $(fg)_{m,s}(a,b) = \sum_{c\in \mathrm{P}[n]} f_{m,s}(a,c) g_{m,s}(a+c, b+c).$
2. $(fg)_{x,s}(e,h) = \sum_{a\subseteq e, b, c\in \mathrm{P}[n]} O\{k \subseteq b\cap c \mid a\cup c \setminus k = e\} f_{x,s}(a,b) g_{x,s}(c, b+h).$

Proof. 1. Let $f = \sum_{a,b\in \mathrm{P}[n]} f_{m,s}(a,b)m^a s^b$, $g = \sum_{c,d\in \mathrm{P}[n]} g_{m,s}(c,d)m^c s^d$, then

$$\begin{aligned} fg &= \sum_{a,b,c,d\in \mathrm{P}[n]} f_{m,s}(a,b)g_{m,s}(c,d)m^a s^b m^c s^d \\ &= \sum_{b,c,d\in \mathrm{P}[n]} f_{m,s}(b+c,b)g_{m,s}(c,d)m^{b+c}s^{b+d} \\ &= \sum_{e,f\in \mathrm{P}[n]} \Big(\sum_{\substack{b+c=e\\ b+d=f}} f_{m,s}(b+c,b)g_{m,s}(c,d)\Big)m^e s^f \\ &= \sum_{e,f\in \mathrm{P}[n]} \Big(\sum_{b\in \mathrm{P}[n]} f_{m,s}(e,b)g_{m,s}(e+b,f+b)\Big)m^e s^f. \end{aligned}$$

2. Let $f = \sum_{a,b\in \mathrm{P}[n]} f_{x,s}(a,b)x^a s^b$, $g = \sum_{c,d\in \mathrm{P}[n]} g_{x,s}(c,d)x^c s^d$, then

$$\begin{aligned} fg &= \sum_{a,b,c,d\in \mathrm{P}[n]} f_{x,s}(a,b)g_{x,s}(c,d)x^a s^b x^c s^d \\ &= \sum_{a,b,c,d\in \mathrm{P}[n],k\subseteq b\cap c} f_{x,s}(a,b)g_{x,s}(c,d)x^{a\cup c\setminus k}s^{b+d} \\ &= \sum_{e,h\in \mathrm{P}[n]} \Big(\sum_{\substack{a,b,c,d\in \mathrm{P}[n]\\ k\subseteq b\cap c\\ a\cup c\setminus k=e,b+d=h}} f_{x,s}(a,b)g_{x,s}(c,d)\Big)x^e s^h \\ &= \sum_{e,h\in \mathrm{P}[n]} \Big(\sum_{\substack{a\subseteq e,b,c\in \mathrm{P}[n]\\ k\subseteq b\cap c\\ a\cup c\setminus k=e}} f_{x,s}(a,b)g_{x,s}(c,b+h)\Big)x^e s^h \\ &= \sum_{e,h\in \mathrm{P}[n]} \Big(\sum_{a\subseteq e,b,c\in \mathrm{P}[n]} O\{k\subseteq b\cap c \,|\, a\cup c\setminus k=e\} f_{x,s}(a,b)g_{x,s}(c,b+h)\Big)x^e s^h. \square \end{aligned}$$

Example 32. Suppose that $f = \sum_{a,b\in \mathrm{P}[n]} f_{m,s}(a,b)m^a s^b$ and $g = \sum_{c,d\in \mathrm{P}[n]} g_{m,s}(c,d)m^c s^d$ are actually regular functions on $\mathbb{Z}_2^n$, i.e. $f_{m,s}(a,b) = 0$ if $b \neq \varnothing$, and $g_{m,s}(c,d) = 0$ if $d \neq \varnothing$. A non-vanishing term in the expression

$$(fg)_{m,s}(a,b) = \sum_{c\in \mathrm{P}[n]} f_{m,s}(a,c)g_{m,s}(a+c,b+c)$$

must have $c = \varnothing$, and then we must also have that $c = \varnothing + c = \varnothing$, and $a + c = a + \varnothing = a$. Thus in this case the product fg is, as expected, just the pointwise product of functions on $\mathbb{Z}_2^n$.

Example 33. Let $f = \sum_{a,b \in \mathrm{P}[n]} f_{m,s}(a,b) m^a s^b$, and suppose that $g = \sum_{c,d \in \mathrm{P}[n]} g_{m,s}(c,d) m^c s^d$ is such that $g_{m,s}(c,d) = 0$ if $c \neq [n]$. Then a non-vanishing summand in the formula

$$(fg)_{m,s}(a,b) = \sum_{c \in \mathrm{P}[n]} f_{m,s}(a,c) g_{m,s}(a+c, b+c)$$

can only arise for $c = \overline{a}$. Therefore $(fg)_{m,s}(a,b) = f_{m,s}(a,\overline{a}) g_{m,s}([n], b+\overline{a})$. For example, we have

$$\Big(\sum_{a \in \mathrm{P}[n]} m^a s^{\overline{a}}\Big)\Big(\sum_{d \in \mathrm{P}[n]} m^{[n]} s^d\Big) = \sum_{a,b \in \mathrm{P}[n]} m^a s^b.$$

As another example consider $f = \sum_{a,b \in \mathrm{P}[n]} m^a s^b$ and $g = m^{[n]} s^{[n]}$. In this case we get that:

$$\Big(\sum_{a,b \in \mathrm{P}[n]} m^a s^b\Big)\Big(m^{[n]} s^{[n]}\Big) = \sum_{a \in \mathrm{P}[n]} m^a s^a.$$

6. Quantum operational logic

In this section we study quantum Boolean algebras from a logical viewpoint. Propositional logic may be approached from a myriad of viewpoints. Here we take a revisionist approach bias towards the theory of operads and props. We believe this approach may be of interest in itself, and is certainly pretty convenient for our current purposes as it would readily generalize to cover quantum operational logic. We assume the reader to be familiar with the language of operads and props [4, 12, 13, 15, 16]. First we review the basic principles of classical propositional logic [6] which may be summarized as:

• On the syntactic side, propositions are words in a certain language. Propositions are either simple or composite. Let x be the finite set of simple propositions, and $\mathbb{P}(x)$ be the set of all propositions. Composite propositions are obtained from the simple propositions using the logical connectives. There are several options for the choice of connectives, the most common ones being $\vee, \wedge, \rightarrow, \neg$.

• On the semantics side, a truth function $\widehat{p} \colon \mathbb{Z}_2^x \rightarrow \mathbb{Z}_2$ is associated to each proposition $p \in \mathbb{P}(x)$, where $\mathbb{Z}_2^x$ is the set of maps from x to $\mathbb{Z}_2$. The map $\mathbb{P}(x) \rightarrow \mathbb{Z}_2[\mathbb{A}^x]$ sending a proposition p to its truth function $\widehat{p} \in \mathbb{Z}_2[\mathbb{A}^x] = \mathrm{M}(\mathbb{Z}_2^x, \mathbb{Z}_2)$ is such that:

– $\widehat{a}$ is evaluation at a, i.e. $\widehat{a}f = f(a)$ for $a \in x$, and $f \in \mathbb{Z}_2^x$.
– $\widehat{p \vee q} = \widehat{p} \vee \widehat{p}, \widehat{p \wedge q} = \widehat{p} \wedge \widehat{p}, \widehat{p \to q} = \widehat{p} \to \widehat{p}, \widehat{\neg p} = \neg\widehat{p}$, where the action of the connectives on truth functions comes from the corresponding operations on $\mathbb{Z}_2$.

The map $\mathbb{P}(x) \to \mathbb{Z}_2[\mathbb{A}^x]$ is surjective, and there is a systematic procedure to tell when two propositions have the same associated truth function. For our purposes, it is convenient to describe $\mathbb{P}(x)$ using the binary connectives product . and sum $+$, and the constants $0, 1$. In logical terms the product . is the logical conjunction, $+$ is the exclusive or, and 0 and 1 represent falsity and truth, respectively. $\mathbb{P}(x)$ is defined recursively as the set of words in the symbols $a \in x, ., +, 0, 1, (,)$ such that:

- $x \subseteq \mathbb{P}(x)$, $0 \in \mathbb{P}(x)$, and $1 \in \mathbb{P}(x)$.
- If $p, q \in \mathbb{P}(x)$, then (pq) and $(p+q)$ are also in $\mathbb{P}(x)$.

We defined recursively the notion of sub-words in $\mathbb{P}(x)$. For all $p, q, r \in \mathbb{P}(x)$ set:

- p is a sub-word of p;
- p is a sub-word of (pq) and $(p+q)$;
- if p is a sub-word of q and q is a sub-word r, then p is a sub-word of r.

Next we define an equivalence relation $\mathbb{R}(x)$, also denoted by $\sim$, on $\mathbb{P}(x)$. Given $p, q \in \mathbb{P}(x)$ we set:

$$p\mathbb{R}(x)q \quad \text{if and only if} \quad \widehat{p} = \widehat{q}.$$

The relation $\mathbb{R}(x)$ can be defined in syntactic terms as follows. Propositions p and q are related if and only if either $p = q$ or there exists a sequence $p_1, \ldots, p_k$, for some $k \geqslant 1$, such that $p_1 = p$, and $p_k = q$, and p_{i+1} is obtained from p_i by replacing a sub-word of p_i by an equivalent word according to the following relations valid for all propositions $p, q, r \in \mathbb{P}(x)$:

- Associativity and commutativity for . and $+$:

$$p(qr) \sim (pq)r, pq \sim qp, (p+q)+r \sim p+(q+r), p+q \sim q+p.$$

- Distributivity: $p(q+r) \sim pq + pr$.
- Additive and multiplicative units: $0 + p \sim p$ and $1p \sim p$.
- Additive nilpotency: $p + p \sim 0$.
- Multiplicative idempotency: $pp \sim p$.

Let Set be the category of sets, and set be the full subcategory of finite sets. Let

$$\mathbb{Z}_2^{(\,\cdot\,)} : \mathrm{set}^\circ \to \mathrm{set}$$

be the functor sending a finite set x to the free Boolean algebra generated by x, i.e. $\mathbb{Z}_2^x \simeq \mathrm{P}(x)$; and sending a map $f\colon x \to y$ to the map $\mathbb{Z}_2^y \to \mathbb{Z}_2^x$ sending $g \in \mathbb{Z}_2^y$ to $g \circ f$. Let

$$\mathbb{Z}_2[\mathbb{A}^{(\,\cdot\,)}]\colon \text{set} \to \text{set}$$

be the functor given by

$$\mathbb{Z}_2[\mathbb{A}^{(\,\cdot\,)}] = \mathbb{Z}_2^{(\,\cdot\,)} \circ \mathbb{Z}_2^{(\,\cdot\,)} = \mathbb{Z}_2^{\mathbb{Z}_2^{(\,\cdot\,)}},$$

i.e. $\mathbb{Z}_2[\mathbb{A}^x]$ is the algebra of regular Boolean functions on the affine space $\mathbb{Z}_2^x$. Recall that an operad O in Set is given by a sequence of sets $\{O(n)\}_{n\in\mathbb{N}}$, together with right actions of the permutations groups $O(n) \times S_n \to O(n)$ and composition maps $c_k\colon O(k) \times O(n_1) \times \cdots \times O(n_k) \to O(n_1 + \cdots n_k)$ which satisfy the equivariance, associativity, and unity axioms [16]. Any set X determines the endomorphism operad End_X defined by the sequence

$$\{\mathrm{End}_X(n)\}_{n\in\mathbb{N}} = \{\mathrm{M}(X^n, X)\}_{n\in\mathbb{N}}.$$

A permutation $\alpha \in S_n$ acts on a map $f\colon X^n \to X$ by

$$f\alpha(x_1, \ldots, x_n) = f(x_{\alpha^{-1}1}, \ldots, x_{\alpha^{-1}n}).$$

The composition maps arise as follows:

$$\begin{aligned} &\mathrm{M}(X^k, X) \times \mathrm{M}(X^{n_1}, X) \times \cdots \times \mathrm{M}(X^{n_1}, X) \\ &\quad \simeq \mathrm{M}(X^k, X) \times \mathrm{M}(X^{n_1+\cdots+n_k}, X^k) \to \mathrm{M}(X^{n_1+\cdots+n_k}, X^k), \end{aligned}$$

where $\simeq$ stands for the natural isomorphism, and the last arrow is composition of maps.

Theorem 34. *The sequence $\{\mathbb{Z}_2[\mathbb{A}^n]\}_{n\in\mathbb{N}}$ defines an operad equivalent to the endomorphism operad of $\mathbb{Z}_2$ in* set.

Proof. The result follows from the identifications

$$\mathbb{Z}_2[\mathbb{A}^n] = \mathbb{Z}_2^{\mathbb{Z}_2^n} = \mathrm{M}(\mathbb{Z}_2^n, \mathbb{Z}_2) = \mathrm{End}_{\mathbb{Z}_2}(n). \qquad \square$$

A S-collection is a sequence of sets $X = \{X_n\}_{n\in\mathbb{N}}$ such that X_n comes with a right S_n-action. A sequence of sets $A = \{A_n\}_{n\in\mathbb{N}}$ generates a S-collection with free S_n actions, namely the sequence

$$A \times S = \{A_n \times S_n\}_{n\in\mathbb{N}}.$$

A S-collection X generates the free operad $FX = \{FX_n\}_{n\in\mathbb{N}}$ described in [16]. For an S-collection of the form $A \times S$, with A any sequence of sets, the free operad

$$FA := F(A \times S)$$

generated by $A \times S$ admits the following description. FA_n is the set of all pairs (t, α) such that:

- t is a A-decorated planar rooted tree with n marked incoming leaves and one outgoing vertex, namely the root. A-decorated means that a choice of an element in A_k is made for each vertex in the tree with incoming valence k (other than the marked incoming leaves.)
- A numbering of the marked incoming leaves, i.e. a bijection from the marked incoming leaves to $[n]$.
- The action of S_n on FA_n permutes the numberings of the leaves.
- The operadic compositions is given by the grafting of trees.

Let $\mathbb{P}$ be the free operad in Set generated by the following sequence:

$$A_0 = \{0, 1\}, A_2 = \{+, .\} \quad \text{and} \quad A_k = \varnothing \text{ for } k \neq 0, 2.$$

Proposition 35. *For $x \in \mathrm{set}$, the set of all propositions $\mathbb{P}(x)$ is equal to the free $\mathbb{P}$-algebra generated by x.*

Proof. By definition the free $\mathbb{P}$-algebra generated by x is given by

$$\mathbb{P}(x) = \coprod_{n=0}^{\infty} \mathbb{P}(n) \times_{S_n} x^n,$$

where $\mathbb{P}(n) \times_{S_n} x^n$ is the quotient of $\mathbb{P}(n) \times x^n$ by the relations:

$$(f\alpha, a_1, \dots, a_n) \sim (f, a_{\alpha^{-1}1}, \dots, a_{\alpha^{-1}n})$$

for $f \in \mathbb{P}(n), \alpha \in S_n, \mathrm{and}(a_1, \dots, a_n) \in x^n$. Since $\mathbb{P}$ is the free operad generated by the sequences of sets A, the free algebra can equivalently be described as the set of pairs (t, f) where:

- t is a A-decorated planar rooted tree with n marked incoming leaves. Thus actually t is a binary tree with branches corresponding to $+$ and $.$, the sum a product symbols.
- f is a map from the n marked incoming leaves to x.

It is clear that such pairs (t, f) are in bijective correspondence with proposition in $\mathbb{P}(x)$. □

We also denote by $\mathbb{P}$ the functor $\mathbb{P}\colon \mathrm{set} \to \mathrm{Set}$ sending x to the set of propositions $\mathbb{P}(x)$, and $f\colon x \to y$ to its unique extension $\mathbb{P}(f)\colon \mathbb{P}(x) \to \mathbb{P}(y)$ sending x to y via f, and respecting the logical connectives. Let Req be the category of equivalence relations. Objects in Req are pairs (X, R) where X is a set and R is an equivalence relation on X. A morphism $f\colon (X, R) \to (Y, S)$ in Req is a map $f\colon X \to Y$ such that $fR \subseteq S$. We have a functor $\mathrm{Req} \to \mathrm{Set}$ sending (X, R) to the quotient set X/R.

Proposition 36. *The pair* $(\mathbb{P}, \mathbb{R})$ *constructed above defines a functor* $(\mathbb{P}, \mathbb{R})\colon \mathrm{set} \to \mathrm{Req}$ *sending* x *to* $(\mathbb{P}(x), \mathbb{R}(x))$. *Thus we obtain the quotient functor* $\mathbb{P}/\mathbb{R}\colon \mathrm{set} \to \mathrm{set}$.

Proposition 37. *The functors* $\mathbb{P}/\mathbb{R}$ *and* $\mathbb{Z}_2[\mathbb{A}^{(\,\cdot\,)}]$ *are naturally isomorphic, both as set valued functors and as Boolean algebras valued functors.*

Proof. It follows from Lemma 1 that $\mathbb{P}/\mathbb{R}(x)$ and $\mathbb{Z}_2[\mathbb{A}^x]$ are naturally isomorphic Boolean algebras. $\square$

Proposition 38. *The operads* $\{\mathbb{P}/\mathbb{R}[n]\}_{n\in\mathbb{N}}$ *and* $\{\mathbb{Z}_2[\mathbb{A}^n]\}_{n\in\mathbb{N}}$ *are isomorphic.*

Proof. We know from Proposition 37 that $\mathbb{P}/\mathbb{R}[n]$ and $\mathbb{Z}_2[\mathbb{A}^n]$ are isomorphic Boolean algebras. The operadic operations on $\mathbb{P}/\mathbb{R}[n]$ arise from the operations of substitution and renaming of variables on propositional formulae. It is well-known that the formation of truth functions behaves well with respect to the operation of substitution and renaming of variables on propositional formulae. Thus $\{\mathbb{P}/\mathbb{R}[n]\}_{n\in\mathbb{N}}$ and $\{\mathbb{Z}_2[\mathbb{A}^n]\}_{n\in\mathbb{N}}$ agree also as operads. $\square$

Recall that $\mathbb{Z}_2[\mathbb{A}^x]$ is a poset with $f \leqslant g$ if and only if $f(a) \leqslant g(a)$ for all $a \in \mathbb{Z}_2^x$. Classical propositional logic main concern is the pre-order $\vdash$ of entailment on $\mathbb{P}(x)$. The entailment relation $\vdash$ can be defined semantically as follows: for $p, q \in \mathbb{P}(x)$ set

$$p \vdash q \quad \text{if and only if} \quad \widehat{p} \leqslant \widehat{q},$$

or equivalently, $p \vdash q$ if and only if there is $r \in \mathbb{P}(x)$ such that $\widehat{p} = \widehat{qr}$. The entailment relation $\vdash$ can be defined syntactically as follows:

$$p \vdash q \quad \text{if and only if} \quad \text{there exists } r \in \mathbb{P}(x) \text{ such that } p \sim qr.$$

Next we move from the propositional settings to the operational settings, always within a Boolean context. Recall from the introduction that quantum observables are operators instead of propositions. In analogy with the classical case we identify operators with words in a certain language. On the semantic side truth functions are replaced by Boolean differential operators. We think of quantum operational logic as arising from the following principles:

1) Simple-composite operators. On the syntactic side operators are words in a certain language. For a set x we let $\widetilde{x} = \{\widetilde{a} \mid a \in x\}$ be a set disjoint from x whose elements are of the form $\widetilde{a}$ for $a \in x$. Given x, the set $\mathbb{O}(x)$ of operators is obtained from the set of simple operators $x \sqcup \widetilde{x} \subseteq \mathbb{O}(x)$ using the binary connectives product . and sum $+$, and the constants $0, 1$. Explicitly, $\mathbb{O}(x)$ is defined recursively as the set of words in the symbols $a \in x \sqcup \widetilde{x}, ., +, 0, 1, (,)$ such that:

- $x \sqcup \widetilde{x} \subseteq \mathbb{O}(x)$, $0 \in \mathbb{O}(x)$, and $1 \in \mathbb{O}(x)$.
- If $p, q \in \mathbb{O}(x)$, then (pq) and $(p+q)$ are also in $\mathbb{O}(x)$.

2) The logical interpretation of the connectives . and $+$. The product pq generalizes the classical connective AND, but there is also an ordering behind it: the operator pq may be interpreted as "act with the operator q, ANDTHEN act with the operator p". The connective $+$ corresponds to the exclusive or XOR. The constants 0 and 1 stand for the null operator and the identity operator, respectively. The logical interpretation is RESETTO0 and LEAVEASIS, respectively.

3) The algebra $\mathrm{BDO}_x = \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^x])$ of Boolean differential operators on $\mathbb{Z}_2^x$. On the semantic side BDO_x is thought as the quantum analogue for the Boolean algebra $\mathbb{Z}_2[\mathbb{A}^x]$ of truth functions. Just as we have a map from propositions to truth functions, we have a map

$$\widehat{(\,\cdot\,)}\colon \mathbb{O}(x) \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^x])$$

from operators to Boolean differential operators given by:

- $\widehat{a}$ is the operator of multiplication by a, for $a \in x$.
- $\widehat{\widetilde{a}}$ is the Boolean partial derivative ∂_a along a, for $a \in x$.
- $\widehat{p+q} = \widehat{p} + \widehat{q}$ and $\widehat{pq} = \widehat{p}\widehat{q}$ for $p, q \in \mathbb{O}(x)$.
- $\widehat{0}$ is the operator identically equal to 0, and $\widehat{1}$ is the identity operator.

We think of the composition $\circ$ of operators in $\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^x])$ as the quantum analogue of the meet $\wedge$, or equivalently the product, on $\mathbb{Z}_2[\mathbb{A}^x] = \mathrm{M}(\mathbb{Z}_2^x, \mathbb{Z}_2)$. Indeed $\circ$ is an extension of the classical meet. Consider the inclusion map $\mathbb{Z}_2[\mathbb{A}^x] \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^x])$ sending $f \in \mathbb{Z}_2[\mathbb{A}^x]$ to the operator of multiplication by f. This map is additive and multiplicative, thus showing that the quantum structures are, as they should, an extension of the classical ones. The map $\mathbb{O}(x) \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^x])$ turns out to be surjective, and there is a well-defined procedure to tell when two operators are assigned the same Boolean differential operator, which we proceed to introduce. Sub-words are defined in $\mathbb{O}(x)$ just as in propositional logic. We define an equivalence relation $\mathbb{R}(x)$, also denoted by $\sim$, on $\mathbb{O}(x)$. For $p, q \in \mathbb{O}(x)$ we set:

$$p\mathbb{R}(x)q \quad \text{if and only if} \quad \widehat{p} = \widehat{q}.$$

$\mathbb{R}(x)$ is defined syntactically as follows: p and q are related if and only if either $p = q$ or there exists a sequence $p_1, \ldots, p_k$, for some $k \geqslant 1$, such that $p_1 = p$, and $p_k = q$, and p_{i+1} is obtained from p_i by replacing a sub-word of p_i by an equivalent word according to the following relations:

- Associativity for the product: $p(qr) \sim (pq)r$.
- Associativity and commutativity for $+$:

$$(p+q)+r \sim p+(q+r), p+q \sim q+p.$$

- Distributivity: $p(q+r) \sim pq+pr$.
- Additive and multiplicative units: $0+p \sim p$ and $1p \sim p$
- Additive nilpotency: $p+p \sim 0$.
- Multiplicative idempotency and nilpotency: $aa \sim a$ and $\widetilde{a}\widetilde{a} \sim 0$, for $a \in x$.
- Commutation relations:

$$ba \sim ab \text{ and } \widetilde{a}\widetilde{b} \sim \widetilde{b}\widetilde{a} \quad \text{for } a, b \in x,$$
$$\widetilde{b}a \sim a\widetilde{b} \quad \text{for } a \neq b \in x, \quad \text{and} \quad \widetilde{a}a \sim a\widetilde{a} + \widetilde{a} + 1 \quad \text{for } a \in x.$$

Let $\mathbb{B}$ be the category of finite sets and bijections. We recall that an S-collection as defined above is the same as a functor from $\mathbb{B}$ to Set, also known as a combinatorial species following the Montreal school initiated by Joyal [2,11]. Note that we may indeed regard $\mathbb{O}, \mathbb{R}$, and $\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^{(\,\cdot\,)}])$ as functors $\mathbb{B} \to \mathrm{Set}$. The pair $(\mathbb{O}, \mathbb{R})$ defines a functor $(\mathbb{O}, \mathbb{R})\colon \mathbb{B} \to \mathrm{Req}$.

Proposition 39. *The functors* $\mathbb{O}/\mathbb{R}$ *and* $\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^{(\,\cdot\,)}])$ *are naturally isomorphic as Boole-Weyl algebra valued functors.*

Proof. It follows from Theorem 11 that for any finite set x we have natural isomorphisms

$$\mathbb{O}/\mathbb{R}(x) \simeq \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^x])$$

respecting the structures of Boolean algebras on both sides. $\square$

Next we define the entailment pre-order $\vdash$ on $\mathbb{O}(x)$. First we define a pre-order on $\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^x])$. Given $S, T \in \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^x])$ we set

$$S \leqslant T \quad \text{if and only if} \quad \text{there exists } R \text{ such that } S = TR.$$

For example, if $S = \pi_A$ and $T = \pi_B$ are projections onto the subspaces A and B of $\mathbb{Z}_2[\mathbb{A}^x]$, respectively, then $\pi_A \leqslant \pi_B$ if and only if $A \subseteq B$, since $\pi_A = \pi_B R$ implies that $\pi_B \pi_A = \pi_B \pi_B R = \pi_B R = \pi_A$, and thus $A \subseteq B$.

Entailment is defined semantically as follows, for $p, q \in \mathbb{O}(x)$ we set

$$p \vdash q \quad \text{if and only if} \quad \text{there is } r \in \mathbb{O}(x) \text{ such that } \widehat{p} = \widehat{qr}.$$

Equivalently, the entailment relation $\vdash$ can be defined syntactically as:

$$p \vdash q \quad \text{if and only if} \quad \text{there is } r \in \mathbb{O}(x) \text{ such that } p \sim qr.$$

As expected, entailment on operators is an extension of entailment on propositions.

Proposition 40. *The* $(\mathbb{P}(\,\cdot\,), \vdash)$, $(\mathbb{O}(\,\cdot\,), \vdash)$, $(\mathbb{Z}_2[\mathbb{A}^{(\,\cdot\,)}], \leqslant)$ *and* $(\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^{(\,\cdot\,)}]), \leqslant)$ *may be regarded as functors from* $\mathbb{B}$ *to the category of pre-ordered sets. Moreover, these functors fit into the following commutative diagram of natural transformations:*

$$\begin{tikzcd}
(\mathbb{P}(\,\cdot\,), \vdash) \arrow[r] \arrow[d] & (\mathbb{O}(\,\cdot\,), \vdash) \arrow[d] \\
(\mathbb{Z}_2[\mathbb{A}^{(\,\cdot\,)}], \leqslant) \arrow[r] & (\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^{(\,\cdot\,)}]), \leqslant)
\end{tikzcd}$$

where the top horizontal arrow is the natural inclusion map, the bottom horizontal arrow sends f to the operator of multiplication by f, and the vertical arrows are the valuation maps from propositions and operators to truth functions and Boolean differential operators, respectively.

Let $\mathbb{Z}_2$-Vect be the category of vector spaces and linear transformation over $\mathbb{Z}_2$. We recall from [15] that a prop P in $\mathbb{Z}_2$-Vect is a strict symmetric monoidal category such that:

- The objects are the natural numbers $\mathbb{N}$.
- The monoidal structure is given on objects by $n \otimes m = m + n$.
- The symmetric monoidal structure is enriched over $\mathbb{Z}_2$-Vect. In particular, the set of morphisms $P(n, m)$ are $\mathbb{Z}_2$-vector spaces, and the product $f \otimes g$ on morphisms is bilinear.

Each vector space V over $\mathbb{Z}_2$ determines the prop End_V of endomorphisms, given by

$$\mathrm{End}_V(n,m) = \mathrm{Hom}_{\mathbb{Z}_2}(V^{\otimes n}, V^{\otimes m}).$$

As we saw in Proposition 39, propositional logic give us a syntactical presentation of the endomorphism operad of $\mathbb{Z}_2$ in the category of sets. Next we extend this result to quantum operational logic which give us a syntactical presentation of the diagonal part of the endomorphism prop of $\mathbb{Z}_2[\mathbb{A}^1]$ in the category $\mathbb{Z}_2$-vect. It is easy to check that if P is a prop, then its diagonal part $P(n,n)$ is naturally an S-collection, with the right action $P(n,n)\times S_n \to P(n,n)$ given by

$$f\alpha = \alpha\circ f\circ \alpha^{-1}.$$

Theorem 41. *The functor $\mathbb{O}/\mathbb{R}$ is isomorphic to the diagonal part of the prop $\mathrm{End}_{\mathbb{Z}_2[\mathbb{A}^1]}$ as Boole-Weyl algebra valued functors.*

Proof. We have the following chain of natural isomorphism

$$\begin{aligned}\mathbb{O}/\mathbb{R}[n] &\simeq \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n]) \simeq \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^1]^{\otimes n})\\ &= \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^1]^{\otimes n}, \mathbb{Z}_2[\mathbb{A}^1]^{\otimes n}) = \mathrm{End}_{\mathbb{Z}_2[\mathbb{A}^1]}(n,n)\end{aligned}$$

respecting the structures of Boole-Weyl algebras, where the first isomorphism comes from Theorem 11, and the second isomorphism comes from the fact that fact that $\mathbb{Z}_2[\mathbb{A}^n]\simeq\mathbb{Z}_2[\mathbb{A}^1]^{\otimes n}$. These isomorphisms are S_n-equivariant since one can check for $\alpha\in S_n$ that

$$\alpha\circ x_i\circ\alpha^{-1} = x_{\alpha^{-1}i} \quad\text{and}\quad \alpha\circ y_i\circ\alpha^{-1} = y_{\alpha^{-1}i}. \qquad \square$$

7. Set theoretical viewpoint

The link between classical propositional logic and the algebra of sets arises as follows. Recall that there is a map $\mathbb{P}(x)\to \mathrm{M}(\mathbb{Z}_2^x,\mathbb{Z}_2)$ sending each proposition to its truth function. Since $\mathrm{M}(\mathbb{Z}_2^x,\mathbb{Z}_2)$ can be identified with $\mathrm{PP}(x)$ we obtain a map $\mathbb{P}(x)\to\mathrm{PP}(x)$ assigning to each proposition p a set $\widehat{p}$ of subsets of x. Moreover, the logical connectives intertwine nicely with the set theoretical operations on subsets, namely:

$$\widehat{p+q} = (\widehat{p}\cup\widehat{q})\setminus(\widehat{p}\cap\widehat{q}), \quad \widehat{pq} = \widehat{p\wedge q} = \widehat{p}\cap\widehat{q}, \quad \widehat{\neg p} = \overline{\widehat{p}},$$
$$\widehat{p\vee q} = \widehat{p}\cup\widehat{q}\widehat{p\to q} = \overline{\widehat{p}}\cup\widehat{q}.$$

We stress the, often overlooked, fact that classical propositional logic describes the set theoretical operations present in $\mathrm{PP}(x)$ that are common to all sets of the form $\mathrm{P}(y)$, i.e. the extra algebraic structures present in $\mathrm{P}(y)$ when $y = \mathrm{P}(x)$ play no significative role in the logic/set theory relation outlined above. Thus whereas the axioms characterizing the algebras $\mathrm{P}(x)$ have been massively studied, the algebraic structures characterizing $\mathrm{P}^n(x)$, for $n \geqslant 2$, have seldom attracted any attention. We proceed to consider the analogue statements in the quantum operational scenario for sets of the form $[n]$. It should be clear, however, that similar constructions apply for arbitrary finite sets. As in the classical case we have a map $\mathbb{O}_n \to \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ sending operators (words in a certain language) to Boolean differential operators. As shown in Section 4 it is possible to identify $\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n])$ with the Boolean-Weyl algebra BA_n, and with the shifted Boolean-Weyl algebra SBA_n. Moreover, we described several explicit bases for these algebras. For example, each $f \in \mathrm{BA}_n$ can be written in an unique way as:

$$f = \sum_{a,b \in \mathrm{P}[n]} f(a,b) x^a y^b.$$

Thus Boolean differential operators can be identified with maps from $\mathrm{P}[n] \times \mathrm{P}[n]$ to $\mathbb{Z}_2$, and we get the identifications:

$$\begin{aligned}\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n]) &\simeq \mathrm{BDO}_n \simeq \mathrm{BA}_n \simeq \mathrm{M}(\mathrm{P}[n] \times \mathrm{P}[n], \mathbb{Z}_2) \\ &\simeq \mathrm{P}(\mathrm{P}[n] \times \mathrm{P}[n]) \simeq \mathrm{PP}([n] \sqcup [n]).\end{aligned}$$

We adopt the following conventions. We identify $[n] \sqcup [n]$ with the set

$$[n, \widetilde{n}] = \{1, 2, \ldots, n, \widetilde{1}, \widetilde{2}, \ldots, \widetilde{n}\}.$$

Given $a \subseteq [n]$ we let $\widetilde{a} = \{\widetilde{i} \mid i \in a\}$ be the corresponding subset of $[\widetilde{n}] = \{\widetilde{1}, \widetilde{2}, \ldots, \widetilde{n}\}$. An element $a \in \mathrm{P}[n, \widetilde{n}]$ will be written as $a = a_1 \sqcup \widetilde{a}_2$ with $a_1, a_2 \in \mathrm{P}[n]$. Note that we have a natural map $\pi \colon \mathrm{P}[n, \widetilde{n}] \to \mathrm{P}[n] \times \mathrm{P}[n]$ given by $\pi(a) = (\pi_1(a), \pi_2(a)) = (a_1, a_2)$. We use indices without tilde to denote monomials of regular functions, and indices with tilde to denote Boolean derivatives or shift operators. The identification $\mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2[\mathbb{A}^n]) = \mathrm{PP}[n, \widetilde{n}]$ allows us to give set theoretical interpretations to the algebraic structures on Boolean differential operators. Unlike the classical set theoretical structures, the quantum operational structures are not defined for arbitrary sets of the form $\mathrm{P}(y)$, quite to the contrary, they very much depend on the fact that $y = \mathrm{P}[n, \widetilde{n}]$.

Below we consider pairs (A, M) where A is a $\mathbb{Z}_2$-algebra and M is an A-module. A morphism $(f, g) : (A_1, M_1) \to (A_2, M_2)$ between such pairs, is given by $\mathbb{Z}_2$-linear maps $f : A_1 \to A_2$ and $g : M_1 \to M_2$ such that f is an algebra morphism, and $g(am) = f(a)g(m)$ for all $a \in A, m \in M$. The additive structure $+\colon \mathrm{PP}[n, \widetilde{n}] \times \mathrm{PP}[n, \widetilde{n}] \to \mathrm{PP}[n, \widetilde{n}]$ on $\mathrm{PP}[n, \widetilde{n}]$ is given by

$$A + B = A \cup B \setminus (A \cap B).$$

We consider several isomorphic products $\circ, \bullet, \star$, and $*$ on $\mathrm{PP}[n, \widetilde{n}]$ displaying different combinatorial properties. The products correspond with the various bases for BA_n and SBA_n. We first introduce the product $\circ$ on $\mathrm{PP}[n, \widetilde{n}]$.

Theorem 42. *There are maps*

$$\circ\colon \mathrm{PP}[n, \widetilde{n}] \times \mathrm{PP}[n, \widetilde{n}] \to \mathrm{PP}[n, \widetilde{n}] \quad \textit{and} \quad \circ\colon \mathrm{PP}[n, \widetilde{n}] \times \mathrm{PP}[n] \to \mathrm{PP}[n],$$

turning $\mathrm{PP}[n, \widetilde{n}]$ *into a* $\mathbb{Z}_2$*-algebra and* $\mathrm{PP}[n]$ *into a module over* $\mathrm{PP}[n, \widetilde{n}]$*, such that the pair* $(\mathrm{PP}[n, \widetilde{n}], \mathrm{PP}[n])$ *is isomorphic to* $\big(\mathrm{End}_{\mathbb{Z}_2}(\mathrm{M}(\mathbb{Z}_2^n, \mathbb{Z}_2)), \mathrm{M}(\mathbb{Z}_2^n, \mathbb{Z}_2)\big)$ *via the maps*

$$A \to \sum_{a \in A} m^{a_1} \partial^{a_2} \quad \textit{and} \quad F \to \sum_{a \in F} m^{a}.$$

Proof. From Theorem 17 and Proposition 6 we see that the desired products $\circ$ are constructed as follows. For $A, B \in \mathrm{PP}[n, \widetilde{n}]$, the product $AB \in \mathrm{PP}[n, \widetilde{n}]$ is given by

$$A \circ B = \Big\{a \in \mathrm{P}[n, \widetilde{n}] \Big| O\{b \in \mathrm{P}[n], c \in B \,|c_2 \subseteq a_2, a_1 \sqcup \widetilde{b} \in A, \\ a_2 \setminus c_2 \subseteq a_1 + c_1 \subseteq b\}\Big\}.$$

Let $A \in \mathrm{PP}[n, \widetilde{n}]$ and $F \in \mathrm{PP}[n]$, then $AF \in \mathrm{PP}[n]$ is given by

$$A \circ F = \Big\{a \in \mathrm{P}[n] \mid O\{b \subseteq c \in \mathrm{P}[n] \mid a \sqcup \widetilde{c} \in A, a + b \in F\}\Big\}. \qquad \square$$

We provide a few applications of Theorem 42.

Example 43. In $\mathrm{PP}[3, \widetilde{3}]$ we have

$$\{\{1, 2, \overline{2}, \overline{3}\}\} \circ \{\{1, 3, \overline{1}, \overline{2}\}\} = \{\{1, 2, \overline{1}, \overline{2}\}, \{1, 2, \overline{1}, \overline{2}, \overline{3}\}\}.$$

Indeed $a \in \{\{1, 2, \overline{2}, \overline{3}\}\} \circ \{\{1, 3, \overline{1}, \overline{2}\}\}$ if there is a odd number of suitable pairs b, c. Note that $c = \{1, 3, \overline{1}, \overline{2}\}, \{1, 2\} \subseteq a_2, a_1 \sqcup \widetilde{b} = \{1, 2, \overline{2}, \overline{3}\}$, and thus necessarily $a_1 = \{1, 2\}$ and $b = \{2, 3\}$. Moreover, we must have $a_2 \setminus \{1, 2\} \subseteq \{1, 2\} + \{1, 3\} \subseteq \{2, 3\}$, that is, $a_2 \setminus \{1, 2\} \subseteq \{2, 3\}$. Thus either $a_2 = \{1, 2\}$ or $a_2 = \{1, 2, 3\}$ yielding the desired result.

Example 44. For $A \in \mathrm{PP}[n]$ set $A' = \pi_2^{-1}(A)$. Let $F \in \mathrm{PP}[n]$, then $A' \circ F = \{a \in \mathrm{P}[n] \mid O\{b \subseteq c \in \mathrm{P}[n] \mid \widetilde{c} \in A, a+b \in F\}\}$. Note that $\sum_{a \in \mathrm{P}[n]} m^a = 1$ and thus:

$$\left(\sum_{a\in A} \partial^a\right) \circ \left(\sum_{b \in F} m^b\right) = \sum_{a \in A' \circ F} m^a.$$

Example 45. For $A \in \mathrm{PP}[n]$ let $\widehat{A} = \{a \in \mathrm{P}[n, \widetilde{n}] \mid a_1 = a_2 \in A\}$. Let $F \in \mathrm{PP}[n]$, then $\widehat{A} \circ F = \{a \in \mathrm{P}[n] \mid O\{e \subseteq a \mid a+e \in F\}\}$ and therefore

$$\left(\sum_{a\in A} m^a\partial^a\right) \circ \left(\sum_{b \in F} m^b\right) = \sum_{a \in \widehat{A} \circ F} m^a.$$

Next we introduce the product $\bullet$ on $\mathrm{PP}[n, \widetilde{n}]$.

Theorem 46. *There are maps*

$$\bullet\colon \mathrm{PP}[n,\widetilde{n}] \times \mathrm{PP}[n,\widetilde{n}] \to \mathrm{PP}[n,\widetilde{n}] \quad \textit{and} \quad \bullet\colon \mathrm{PP}[n,\widetilde{n}] \times \mathrm{PP}[n] \to \mathrm{PP}[n]$$

such that the pair $(\mathrm{PP}[n,\widetilde{n}], \mathrm{PP}[n])$ *is isomorphic to* $\big(\mathrm{End}_{\mathbb{Z}_2}(\mathrm{M}(\mathbb{Z}_2^n, \mathbb{Z}_2)), \mathrm{M}(\mathbb{Z}_2^n, \mathbb{Z}_2)\big)$ *via the maps*

$$A \to \sum_{a \in A} x^{a_1}\partial^{a_2} \quad \textit{and} \quad F \to \sum_{a \in F} x^a.$$

Proof. From Theorem 17 and Proposition 6 the desired products $\bullet$ are constructed as follows. For $A, B \in \mathrm{PP}[n,\widetilde{n}]$, the product $A \bullet B \in \mathrm{PP}[n,\widetilde{n}]$ is such that $a \in A \bullet B$ if and only if

$$\begin{aligned} O\{b \in A, c \in B, k_1 \subseteq k_2 \,\big|\, & b_1 \subseteq a_1, c_1 \subseteq a_2, k_2 \subseteq b_2 \cap c_1, \\ & b_1 \cup (c_1 \setminus k_2) = a_1, b_2 \setminus k_1 = a_2 \setminus c_2\}. \end{aligned}$$

Let $A \in \mathrm{PP}[n,\widetilde{n}]$ and $F \in \mathrm{PP}[n]$, then $A \bullet F \in \mathrm{PP}[n]$ is given by

$$A \bullet F = \Big\{a \in \mathrm{P}[n] \mid O\{b \in A, c \in F \,\big|\, b_2 \subseteq c, b_1 \cup (c \setminus b_2) = a\}\Big\}. \qquad \square$$

We provide a few applications of Theorem 46.

Example 47. In $\mathrm{PP}[3, \widetilde{3}]$ we have

$$\{\{1,3,\overline{2}\}\} \bullet \{\{2,\overline{1}\}\} = \{\{1,2,3,\overline{1},\overline{2}\}, \{1,3,\overline{1},\overline{2}\}, \{1,3,\overline{1}\}\}.$$

Indeed, we must have $b = \{1,3,\overline{2}\}$ and $c = \{2,\overline{1}\}$, and thus there are three options for $k_1 \subseteq k_2 \subseteq [2]$, namely $\varnothing \subseteq \varnothing$, $\varnothing \subseteq \{2\}$ and $\{2\} \subseteq \{2\}$, giving rise to the sets $\{1,2,3,\overline{1},\overline{2}\}$, $\{1,3,\overline{1},\overline{2}\}$, $\{1,3,\overline{1}\}$, respectively.

Example 48. For $A \in \mathrm{PP}[n]$ set $A' = \pi^{-1}(\{\varnothing\} \times A)$. Let $F \in \mathrm{PP}[n]$, then $A' \bullet F = \{a \in \mathrm{P}[n] \big| O\{b \in A, c \in F \mid b \subseteq c, c \setminus b = a\}\}$. Therefore we get that

$$\left(\sum_{a\in A} \partial^a\right) \circ \left(\sum_{b\in F} x^b\right) = \sum_{a\in A'\bullet F} x^a.$$

Example 49. For $A \in \mathrm{PP}[n]$ let $\widehat{A} = \{a \in \mathrm{P}[n,\widetilde{n}] \mid a_1 = a_2 \in A\}$. Let $F \in \mathrm{PP}[n]$, then $\widehat{A} \bullet F = \{a \in F \mid O\{b \in A \mid b \subseteq a\}\}$ and therefore

$$\left(\sum_{a\in A} x^a\partial^a\right) \circ \left(\sum_{b\in F} x^b\right) = \sum_{a\in \widehat{A}\bullet F} x^a = \sum_{a\in F} O\{b \in A \mid b \subseteq a\}x^a.$$

In particular we have $\widehat{\mathrm{P}[n]} \bullet \mathrm{P}[n] = \{\varnothing\}$, and thus

$$\left(\sum_{a\in \mathrm{P}[n]} x^a\partial^a\right) \circ \left(\sum_{b\in \mathrm{P}[n]} x^b\right) = 1.$$

Next we introduce the product $\star$ on $\mathrm{PP}[n,\widetilde{n}]$.

Theorem 50. *There are maps*

$$\star\colon \mathrm{PP}[n,\widetilde{n}] \times \mathrm{PP}[n,\widetilde{n}] \to \mathrm{PP}[n,\widetilde{n}] \quad \textit{and} \quad \star\colon \mathrm{PP}[n,\widetilde{n}] \times \mathrm{PP}[n] \to \mathrm{PP}[n],$$

turning $\mathrm{PP}[n,\widetilde{n}]$ *into a* $\mathbb{Z}_2$*-algebra and* $\mathrm{PP}[n]$ *into a module over* $\mathrm{PP}[n,\widetilde{n}]$, *such that the pair* $(\mathrm{PP}[n,\widetilde{n}], \mathrm{PP}[n])$ *is isomorphic to* $\big(\mathrm{End}_{\mathbb{Z}_2}(\mathrm{M}(\mathbb{Z}_2^n,\mathbb{Z}_2)), \mathrm{M}(\mathbb{Z}_2^n,\mathbb{Z}_2)\big)$ *via the maps*

$$A \to \sum_{a\in A} m^{a_1}s^{a_2} \quad \textit{and} \quad F \to \sum_{a\in F} m^a.$$

Proof. From Theorem 31 and Proposition 22 we see that the desired products $\star$ are constructed as follows. For $A, B \in \mathrm{PP}[n,\widetilde{n}]$, the product $A \star B \in \mathrm{PP}[n,\widetilde{n}]$ is given by

$$A \star B = \{a \in \mathrm{P}[n,\widetilde{n}] \mid O\{b \in \mathrm{P}[n] \mid a_1 \sqcup \widetilde{b} \in A, (a_1 + b) \sqcup \widetilde{(a_2 + b)} \in B\}\}.$$

Let $A \in \mathrm{PP}[n,\widetilde{n}]$ and $F \in \mathrm{PP}[n]$, then $A \star F \in \mathrm{PP}[n]$ is given by

$$A \star F = \Big\{a \in \mathrm{P}[n] \mid O\{b \in \mathrm{P}[n] \mid a \sqcup \widetilde{b} \in A, a + b \in F\}\Big\}. \qquad \square$$

We provide a few applications of Theorem 50.

Example 51. In $\mathrm{PP}[3,\widetilde{3}]$ we have $\{\{1,2,3,\overline{3}\}\}\star\{\{1,2,\overline{2},\overline{3}\}\}=\{\{1,2,3,\overline{2}\}\}$. From the equation $a_1 \sqcup \widetilde{b} \in A$ we see that $a_1 = \{1,2,3\}$ and $b = \{3\}$. Also we must have $a_1 + \{3\} = \{1,2\}$, which holds, and $a_2 + \{3\} = \{2,3\}$ which implies that $a_2 = \{2\}$.

Example 52. For $A \in \mathrm{PP}[n]$ set $A' = \pi_2^{-1}(A)$. Let $F \in \mathrm{PP}[n]$, then $A' \star F = \{a \in \mathrm{P}[n] \,\big|\, O\{b \in A \mid a+b \in F\}\}$. Therefore

$$\Big(\sum_{a\in A} s^a\Big) \circ \Big(\sum_{b\in F} m^b\Big) = \sum_{a\in A'\star F} m^a,$$

in particular, $\Big(\sum_{a\in A} s^a\Big) \circ \Big(\sum_{b\in \mathrm{P}[n]} m^b\Big) = OA\sum_{a\in \mathrm{P}[n]} m^a$.

Example 53. For $A \in \mathrm{PP}[n]$ set $\widehat{A} = \{a \in \mathrm{P}[n,\widetilde{n}] \mid a_1 = a_2 \in A\}$. Let $F \in \mathrm{PP}[n]$, then $\widehat{A}\star F = \varnothing$ if $\varnothing \notin F$ and $\widehat{A}\star F = A$ if $\varnothing \in F$. Therefore

$$\Big(\sum_{a\in A} m^a s^a\Big) \circ \Big(\sum_{b\in F} m^b\Big) = c\sum_{a\in A} m^a,$$

where $c = 1$ if $\varnothing \in F$, and $c = 0$ if $\varnothing \notin F$.

Example 54. Let $\widehat{A}$ be as Example 53, then $\widehat{A}\star\widehat{A} = \widehat{A}$ if $\varnothing \in A$, and $\widehat{A}\star\widehat{A} = \varnothing$ otherwise. Indeed, $a \in \mathrm{P}[n,\widetilde{n}]$ belongs to $\widehat{A}\star\widehat{A}$ if $a_1 \sqcup \widetilde{b} \in \widehat{A}$, i.e. $a_1 = b \in A$, and $(a_1+b)\sqcup(\widetilde{a_2+b}) \in \widehat{A}$, i.e. $\varnothing \in A$ and $a_1 = a_2 \in A$.

Example 55. For $A \in \mathrm{PP}[n]$ set $\widetilde{A} = \{a \in \mathrm{P}[n,\widetilde{n}] \mid \overline{a_1} = a_2 \in A\}$. Then $\widetilde{A}\star\widetilde{A} = \widetilde{A}$ if $[n] \in A$, and $\widehat{A}\star\widehat{A} = \varnothing$ if $[n] \notin A$. Indeed, $a \in \mathrm{P}[n,\widetilde{n}]$ belongs to $\widetilde{A}\star\widetilde{A}$ if $a_1 \sqcup \widetilde{b} \in \widetilde{A}$, i.e. $\overline{a}_1 = b \in A$, and $(a_1+b)\sqcup(\widetilde{a_2+b}) \in \widehat{A}$, i.e. $[n] \in A$ and $a_2 = \varnothing + b = b = \overline{a}_1$.

Finally we introduce the product $*$ on $\mathrm{PP}[n,\widetilde{n}]$.

Theorem 56. *There are maps*

$$*\colon \mathrm{PP}[n,\widetilde{n}] \times \mathrm{PP}[n,\widetilde{n}] \to \mathrm{PP}[n,\widetilde{n}] \quad \textit{and} \quad *\colon \mathrm{PP}[n,\widetilde{n}] \times \mathrm{PP}[n] \to \mathrm{PP}[n],$$

turning $\mathrm{PP}[n,\widetilde{n}]$ *into a* $\mathbb{Z}_2$*-algebra and* $\mathrm{PP}[n]$ *into a module over* $\mathrm{PP}[n,\widetilde{n}]$, *such that the pair* $(\mathrm{PP}[n,\widetilde{n}],\mathrm{PP}[n])$ *is isomorphic to* $\big(\mathrm{End}_{\mathbb{Z}_2}(\mathrm{M}(\mathbb{Z}_2^n,\mathbb{Z}_2)),$ $\mathrm{M}(\mathbb{Z}_2^n,\mathbb{Z}_2)\big)$ *via the maps*

$$A \to \sum_{a\in A} x^{a_1}s^{a_2} \quad \textit{and} \quad F \to \sum_{a\in F} x^a.$$

Proof. From Theorem 31 and Proposition 22 we see that the desired products $\star$ are constructed as follows. For $A, B \in \mathrm{PP}[n, \widetilde{n}]$, the product $A * B \in \mathrm{PP}[n, \widetilde{n}]$ is given by

$$\{a \in \mathrm{P}[n, \widetilde{n}] | O\{b, c, d, e \in \mathrm{P}[n] \; | e \subseteq c \cap d, b \cup d \setminus e = a_1, b \sqcup \widetilde{c} \in A, \\ d \sqcup (\widetilde{c + a_2}) \in B\}\}.$$

Let $A \in \mathrm{PP}[n, \widetilde{n}]$ and $F \in \mathrm{PP}[n]$, then $A * F \in \mathrm{PP}[n]$ is given by

$$A * F = \{a \in \mathrm{P}[n] \mid O\{b, c, d \in \mathrm{P}[n], e \in F \,|\, c \subseteq d \cap e,\ b \cup e \setminus c = a,\ b \sqcup \widetilde{d} \in A\}\}.$$

□

We provide a few applications of Theorem 56.

Example 57. In $\mathrm{PP}[3, \widetilde{3}]$ we have

$$\{\{1, \overline{2}\}\} * \{\{2, 3, \overline{1}, \overline{2}\}\} = \{\{1, 3, \overline{1}\}, \{1, 2, 3, \overline{1}\}\}.$$

Indeed we must have $b = \{1\}$, $c = \{2\}$, $d = \{2, 3\}$, and $a_2 = \{2\} + \{1, 2\} = \{1\}$. Since $e \subseteq \{2\} \cap \{2, 3\} = \{2\}$, there are two options, either $e = \varnothing$ and then $a_1 = \{1, 2, 3\}$ and $a = \{1, 2, 3, \overline{1}\}$, or $e = \{2\}$ and then $a_1 = \{1, 3\}$ and $a = \{1, 3, \overline{1}\}$.

Example 58. For $A \in \mathrm{PP}[n]$ set $A' = \pi^{-1}(\{\varnothing\} \times A)$. Let $F \in \mathrm{PP}[n]$ then $A' * F = \{a \in \mathrm{P}[n] \mid O\{c \in \mathrm{P}[n], d \in A, e \in F \mid c \subseteq d \cap e, e \setminus c = a\}\}$. Therefore

$$\left(\sum_{a \in A} s^a\right) \circ \left(\sum_{b \in F} x^b\right) = \sum_{a \in A' * F} x^a.$$

Example 59. For $A \in \mathrm{PP}[n]$ let $\widehat{A} = \{a \in \mathrm{P}[n, \widetilde{n}] \mid a_1 = a_2 \in A\}$. Let $F \in \mathrm{PP}[n]$, then $\widehat{A} * F = \Big\{a \in \mathrm{P}[n] \Big| O\{b \in A, c \in \mathrm{P}[n], e \in F \mid c \subseteq b \cap e, b \cup e \setminus c = a\}\Big\}$. Therefore

$$\left(\sum_{a \in A} x^a s^a\right) \circ \left(\sum_{b \in F} x^b\right) = \sum_{a \in \widehat{A} * F} x^a.$$

8. Final remarks

Our work leaves several open problems and questions for future research: 1) We considered the structural aspects of quantization in characteristic 2. Dynamical aspects will be considered elsewhere. 2) We studied the analogue for the Weyl algebra in characteristic 2. Recently, there have been several attempts to construct a suitable category that could play the role of the category of modules over the non-existing field with characteristic one [8, 9, 21]. It is natural to ponder whether it is possible to build analogues for the Weyl algebra in such new contexts. 3) Categorification of the Weyl algebra has been considered in [11]; categorification of the Boole-Weyl and shifted Boole-Weyl algebras remain to be addressed. 4) The symmetric powers of Weyl algebras and linear Boolean algebras in characteristic zero were studied in [10] and [12], respectively. The analogue problems for the quantum Boolean algebras and the linear quantum Boolean algebras are open. 5) Our logical interpretation of quantum Boolean algebras was based on a specific choice of connectives. It remains to study other connectives, perhaps with a more direct logical meaning.

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