Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 3. **(2008).** pp. 69 – 83

Discrete limit theorems for Estermann zeta-functions. II

Antanas Laurinčikas, Renata Macaitienė

Communicated by V. V. Kirichenko

Abstract. A discrete limit theorem in the sense of weak convergence of probability measures in the space of meromorphic functions for the Estermann zeta-function with explicitly given the limit measure is proved.

1. Introduction

Let $s = \sigma + it$ be a complex variable, k and l be coprime integers, and, for $\alpha \in \mathbb{C}$,

$$\sigma_\alpha(m) = \sum_{d/m} d^\alpha.$$

For $\sigma > \max(1, 1 + \Re\alpha)$, the Estermann zeta-function $E(s; \frac{k}{l}, \alpha)$ with parameters $\frac{k}{l}$ and α is defined by

$$E\left(s; \frac{k}{l}, \alpha\right) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)}{m^s} \exp\left\{2\pi i m \frac{k}{l}\right\}.$$

The function $E(s; \frac{k}{l}, \alpha)$ has analytic continuation to the whole complex plane, except for two simple poles at $s = 1$ and $s = 1 + \alpha$ if $\alpha \neq 0$, and a double pole at $s = 1$ if $\alpha = 0$. In view of the equation

$$E\left(s; \frac{k}{l}, \alpha\right) = E\left(s - \alpha; \frac{k}{l}, -\alpha\right),$$

Partially supported by Lithuanian Foundation of Studies and Science.
2000 Mathematics Subject Classification: *11M41.*
Key words and phrases: *Estermann zeta-function, Haar measure, limit theorem, probability measure, weak convergence.*

we may suppose that $\Re\alpha \le 0$.

The present paper is a continuation of [6], where a discrete limit theorem on the complex plane for $E(s; \frac{k}{l}, \alpha)$ has been proved. To state the latter theorem, we need some definitions and notation. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. Moreover, let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\} \stackrel{\text{def}}{=} \gamma$ for each prime p. The torus Ω is a compact topological Abelian group, therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined. This gives a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathcal{P}$ ($\mathcal{P}$ denotes the set of all prime numbers), and put, for $m \in \mathbb{N}$,

$$\omega(m) = \sum_{p^\alpha \| m} \omega^\alpha(p),$$

where $p^\alpha \parallel m$ means that $p^\alpha \mid m$ but $p^{\alpha+1} \nmid m$. Now suppose that $\Re\alpha \le 0$ and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the complex-valued random element $E(\sigma; \frac{k}{l}, \alpha; \omega)$, for $\sigma > \frac{1}{2}$, by

$$E\left(\sigma; \frac{k}{l}, \alpha; \omega\right) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)\omega(m)}{m^\sigma} \exp\left\{2\pi i m \frac{k}{l}\right\}.$$

Let $P^{\mathbb{C}}_{E,\sigma}$ be the distribution of $E(\sigma; \frac{k}{l}, \alpha; \omega)$, i.e.,

$$P^{\mathbb{C}}_{E,\sigma}(A) = m_H\left(\omega \in \Omega : E\left(\sigma; \frac{k}{l}, \alpha; \omega\right) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}).$$

In the sequel, for $N \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, we will use the notation

$$\mu_N(\ldots) = \frac{1}{N+1} \sum_{\substack{0 \le m \le N \\ \ldots}} 1,$$

where in place of dots a condition satisfied by m is to written. In [6], the following statement has been proved.

Theorem 1. *Suppose that $\Re\alpha \le 0$ and $\sigma > \frac{1}{2}$, and that $h > 0$ is a fixed number such that $\exp\left\{\frac{2\pi r}{h}\right\}$ is irrational for all $r \in \mathbb{Z} \setminus \{0\}$. Then the probability measure*

$$\mu_N\left(E\left(\sigma + imh; \frac{k}{l}, \alpha\right) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P^{\mathbb{C}}_{E,\sigma}$ *as* $N \to \infty$.

The function $E(s; \frac{k}{l}, \alpha)$ is meromorphic one. Therefore, its asymptotic behavior is better reflected by a limit theorem in the space of meromorphic functions.

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere with the metric d defined by

$$d(s_1, s_2) = \frac{2|s_1 - s_2|}{\sqrt{1+|s_1|^2}\sqrt{1+|s_2|^2}}, \quad d(s, \infty) = \frac{2}{\sqrt{1+|s|^2}}, \quad d(\infty, \infty) = 0,$$

$s, s_1, s_2 \in \mathbb{C}$. Let G be a region on the complex plane. Denote by $M(G)$ the space of meromorphic on G functions $f\colon G \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. In this topology, a sequence $\{f_n\} \subset M(G)$ converges to $f \in M(G)$ if, for every compact subset $K \subset G$,

$$\lim_{n\to\infty} \sup_{s\in K} d(f_n(s), f(s)) = 0.$$

All analytic functions on G form a subspace $H(G)$ of $M(G)$.

Let $D = \left\{s \in \mathbb{C} : \sigma > \frac{1}{2}\right\}$. Then, in the case $\Re\alpha \le 0$,

$$E\left(s; \frac{k}{l}, \alpha; \omega\right) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)\omega(m)}{m^s} \exp\left\{2\pi i m \frac{k}{l}\right\},$$

is an $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by P^H_E its distribution given, for $A \in \mathcal{B}(H(D))$, by

$$P^H_E(A) = m_H\left(\omega \in \Omega : E\left(s; \frac{k}{l}, \alpha; \omega\right) \in A\right),$$

and define the probability measure

$$P_N(A) = \mu_N\left(E\left(s + imh; \frac{k}{l}, \alpha\right) \in A\right), \quad A \in \mathcal{B}(M(D)).$$

The aim of this paper is to prove a limit theorem for the measure P_N.

Theorem 2. *Suppose that* $\Re\alpha \le 0$ *and that* $h > 0$ *is a fixed number such that* $\exp\left\{\frac{2\pi r}{h}\right\}$ *is irrational for all* $r \in \mathbb{Z} \setminus \{0\}$. *Then the probability measure* P_N *converges weakly to* P^H_E *as* $N \to \infty$.

We suppose in the sequel that $\Re\alpha \le 0$, and that $\exp\left\{\frac{2\pi r}{h}\right\}$ is irrational for all $r \in \mathbb{Z} \setminus \{0\}$.

2. Case of absolute convergence

In this section, we will prove a discrete limit theorem in the space of analytic functions for a function given by absolutely convergent Dirichlet series and related to the function $E\left(s; \frac{k}{l}, \alpha\right)$.

Let, for brevity, $s_1 = 1$, $s_2 = \begin{cases} 1+\alpha & \text{if } \alpha \neq 0, \\ 1 & \text{if } \alpha = 0, \end{cases}$
and

$$f(s) = \prod_{j=1}^{2} \left(1 - 2^{s_j - s}\right).$$

Then $f(s_j) = 0$, $j = 1, 2$, and the point $s = 1$ is a double zero of $f(s)$ if $\alpha = 0$. Define

$$\widehat{E}\left(s; \frac{k}{l}, \alpha\right) = f(s) E\left(s; \frac{k}{l}, \alpha\right)$$

Then, clearly, $\widehat{E}\left(s; \frac{k}{l}, \alpha\right)$ is an analytic function on the half-plane D. Moreover, denoting by $|\mathcal{A}|$ the number of elements of a set $\mathcal{A}$, we have that, for $\sigma > 1$,

$$\begin{aligned} \widehat{E}\left(s; \frac{k}{l}, \alpha\right) &= \prod_{j=1}^{2} \left(1 - \frac{2^{s_j}}{2^s}\right) \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)}{m^s} \exp\left\{2\pi i m \frac{k}{l}\right\} \\ &= \sum_{\mathcal{A} \subseteq \{1,2\}} \sum_{m=1}^{\infty} \sigma_\alpha(m) \exp\left\{2\pi i m \frac{k}{l}\right\} 2^{\sum_{j \in \mathcal{A}} s_j} (-1)^{|\mathcal{A}|} 2^{-|\mathcal{A}| s} m^{-s} \\ &= \sum_{j=0}^{2} \sum_{m=1}^{\infty} a_{m,j}\left(\frac{k}{l}, \alpha\right) \frac{1}{2^{js} m^s}. \end{aligned}$$

It is easily seen that, for all $m \in \mathbb{N}$ and $j = 0, 1, 2$,

$$a_{m,j}\left(\frac{k}{l}, \alpha\right) \ll |\sigma_\alpha(m)|.$$

Let $\sigma_1 > \frac{1}{2}$ be a fixed number, and, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}.$$

Define

$$\widehat{E}_n\left(s; \frac{k}{l}, \alpha\right) = \sum_{j=0}^{2} \sum_{m=1}^{\infty} \frac{a_{m,j}\left(\frac{k}{l}, \alpha\right) v_n(m)}{2^{js} m^s},$$

and, for $\widehat{\omega} \in \Omega$,

$$\widehat{E}_n\left(s; \frac{k}{l}, \alpha; \widehat{\omega}\right) = \sum_{j=0}^{2} \sum_{m=1}^{\infty} \frac{a_{m,j}\left(\frac{k}{l}, \alpha\right) \widehat{\omega}^j(2) \widehat{\omega}(m) v_n(m)}{2^{js} m^s}.$$

It was observed in [5] that the above series both converge absolutely for $\sigma > \frac{1}{2}$. This section is devoted to the weak convergence of probability measures

$$P_{N,n} = \mu_N\left(\widehat{E}_n\left(s+imh; \frac{k}{l}, \alpha\right) \in A\right), \quad A \in \mathcal{B}(H(D)),$$

and

$$\widehat{P}_{N,n} = \mu_N\left(\widehat{E}_n\left(s+imh; \frac{k}{l}, \alpha; \widehat{\omega}\right) \in A\right), \quad A \in \mathcal{B}(H(D)).$$

Theorem 3. *There exists a probability measure P_n on $(H(D), \mathcal{B}(H(D)))$ such that both the measures $P_{N,n}$ and $\widehat{P}_{N,n}$ converge weakly to P_n as $N \to \infty$.*

The proof of Theorem 3 is based on a discrete limit theorem on the torus Ω. Define

$$Q_N(A) = \mu_N\left((p^{-imh} : p \in \mathcal{P}) \in A\right), \quad A \in \mathcal{B}(\Omega).$$

Lemma 4. *The probability measure Q_N converges weakly to the Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ as $N \to \infty$.*

Proof of the lemma is given in [6], Lemma 5.

Proof of Theorem 3. Define the function $u_n : \Omega \to H(D)$ by the formula

$$u_n(\omega) = \sum_{j=0}^{2}\sum_{m=1}^{\infty} \frac{a_{m,j}\left(\frac{k}{l}, \alpha\right) v_n(m)\omega^j(2)\omega(m)}{2^{js}m^s}.$$

From the absolute convergence for $\sigma > \frac{1}{2}$ of the series $\widehat{E}(s; \frac{k}{l}, \alpha)$, we have that the function u_n is continuous. Moreover, the equality

$$u_n\left((p^{-imh} : p \in \mathcal{P})\right) = \widehat{E}_n\left(\sigma + imh; \frac{k}{l}, \alpha\right)$$

holds. Thus, $P_{N,n,} = Q_N u_n^{-1}$. This, the continuity of u_n, Lemma 4 and Theorem 5.1 of [1] show that the measure $P_{N,n}$ converges weakly to $m_H u_n^{-1}$ as $N \to \infty$.

Similarly, in the case of the measure $\widehat{P}_{N,n}$, we define the function $\widehat{u}_n : \Omega \to H(D)$ by the formula

$$\widehat{u}_n(\omega) = \sum_{j=0}^{2}\sum_{m=1}^{\infty} \frac{a_{m,j}\left(\frac{k}{l}, \alpha\right) \widehat{\omega}^j(2)\widehat{\omega}(m)\omega^j(2)\omega(m)v_n(m)}{2^{js}m^s}.$$

Then in the above way we obtain that the measure $\widehat{P}_{N,n}$ converges weakly to $m_H\widehat{u}_n^{-1}$ as $N\to\infty$. So, it remains to prove that the measures $m_Hu_n^{-1}$ and $m_H\widehat{u}_n^{-1}$ coincide. Let, for $\omega\in\Omega$, $u(\omega)=\omega\widehat{\omega}$. Then

$$\widehat{u}_n(\omega)=u_n(\omega\widehat{\omega})=u_n(u(\omega)).$$

Therefore, using the invariance of the Haar measure m_H, we find that

$$m_H\widehat{u}_n^{-1}=m_H(u_n(u))^{-1}=(m_Hu^{-1})u_n^{-1}=m_Hu_n^{-1},$$

and the theorem is proved.

We note that the requirement on the irrationality of $\exp\left\{\frac{2\pi r}{h}\right\}$, $r\in\mathbb{Z}\setminus\{0\}$, is used in the proof of Lemma 4, hence also for the proof of Theorem 3.

3. Approximation results

Let, for $\omega\in\Omega$ and $s\in D$,

$$\begin{aligned}\widehat{E}\left(s;\frac{k}{l},\alpha;\omega\right) &= \sum_{j=0}^{2}\sum_{m=1}^{\infty}a_{m,j}\left(\frac{k}{l},\alpha\right)\frac{\omega^j(2)\omega(m)}{2^{js}m^s}\\ &= \prod_{j=1}^{2}\left(1-\frac{2^{s_j}\omega(2)}{2^s}\right)\sum_{m=1}^{\infty}\frac{\sigma_\alpha(m)\omega(m)}{m^s}\exp\left\{2\pi im\frac{k}{l}\right\}.\end{aligned}$$

Then $\widehat{E}\left(s;\frac{k}{l},\alpha;\omega\right)$ is an $H(D)$-valued random element defined on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. Denote by $P_{\widehat{E}}$ the distribution of $\widehat{E}\left(s;\frac{k}{l},\alpha;\omega\right)$. In this section, we approximate in the mean the functions $\widehat{E}\left(s;\frac{k}{l},\alpha\right)$ and $\widehat{E}\left(s;\frac{k}{l},\alpha;\omega\right)$ by $\widehat{E}_n\left(s;\frac{k}{l},\alpha\right)$ and $\widehat{E}_n\left(s;\frac{k}{l},\alpha;\omega\right)$, respectively.

Theorem 5. *Let K be a compact subset of D. Then*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}\left|\widehat{E}\left(s+imh;\frac{k}{l},\alpha\right)-\widehat{E}_n\left(s+imh;\frac{k}{l},\alpha\right)\right|=0.$$

Proof. For $n\in\mathbb{N}$, define

$$l_n(s)=\frac{s}{\sigma_1}\Gamma\left(\frac{s}{\sigma_1}\right)n^s,$$

where $\Gamma(s)$ is the Euler gamma function and σ_1 is defined in Section 2. Then, see, [5], for $\sigma>\frac{1}{2}$,

$$\widehat{E}_n\left(s;\frac{k}{l},\alpha\right)=\frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\widehat{E}\left(s+z;\frac{k}{l},\alpha\right)l_n(z)\frac{\mathrm{d}z}{z}. \qquad (1)$$

Suppose that $\min\{\sigma : s \in K\} = \frac{1}{2} + \eta$, $\eta > 0$. Now we take $\sigma_2 = \frac{1}{2} + \frac{\eta}{2}$ and using (1) obtain by the residue theorem that, for $\sigma > \sigma_2$,

$$\widehat{E}_n\left(s; \frac{k}{l}, \alpha\right) = \frac{1}{2\pi i} \int\limits_{\sigma_2-\sigma-i\infty}^{\sigma_2-\sigma+i\infty} \widehat{E}\left(s+z; \frac{k}{l}, \alpha\right) l_n(z)\frac{\mathrm{d}z}{z} + \widehat{E}\left(s; \frac{k}{l}, \alpha\right). \tag{2}$$

Let L be a simple closed contour lying in D and enclosing the set K, and let δ be the distance of L from K. The an application of the Cauchy integral formula yields the estimate

$$\begin{aligned} &\sup_{s\in K}\left|\widehat{E}\left(s+imh; \frac{k}{l}, \alpha\right) - \widehat{E}_n\left(s+imh; \frac{k}{l}, \alpha\right)\right| \\ \leq\ &\frac{1}{2\pi\delta}\int\limits_L \left|\widehat{E}\left(z+imh; \frac{k}{l}, \alpha\right) - \widehat{E}_n\left(z+imh; \frac{k}{l}, \alpha\right)\right| |\mathrm{d}z|. \end{aligned}$$

Therefore, taking into account (2), we find that

$$\begin{aligned} &\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}\left|\widehat{E}\left(s+imh; \frac{k}{l}, \alpha\right) - \widehat{E}_n\left(s+imh; \frac{k}{l}, \alpha\right)\right| \\ &\ll \frac{|L|}{N\delta}\sup_{\sigma+iu\in L}\sum_{m=0}^{N}\left|\widehat{E}\left(\sigma+imh+iu; \frac{k}{l}, \alpha\right) - \widehat{E}_n\left(\sigma+imh+iu; \frac{k}{l}, \alpha\right)\right| \\ &\ll \sup_{\sigma+iu\in L}\int\limits_{-\infty}^{\infty}\frac{|l_n(\sigma_2-\sigma+i\tau)|}{|\sigma_2-\sigma+i\tau|}\left(\frac{1}{N}\sum_{m=0}^{N}\left|\widehat{E}\left(\sigma_2+iu+i\tau+imh; \frac{k}{l}, \alpha\right)\right|\right)\mathrm{d}\tau \\ &\ll \sup_{\sigma+iu\in L}\int\limits_{-\infty}^{\infty}\frac{|l_n(\sigma_2-\sigma+i\tau)|}{|\sigma_2-\sigma+i\tau|}\left(\frac{1}{N}\sum_{m=0}^{N}\left|\widehat{E}\left(\sigma_2+iu+i\tau+imh; \frac{k}{l}, \alpha\right)\right|^2\right)^{\frac{1}{2}}\mathrm{d}\tau. \end{aligned} \tag{3}$$

Since $\sigma_2 > \frac{1}{2}$ and $\Re\alpha \leq 0$, we have by [9] that

$$\int\limits_0^T \left|E\left(\sigma_2+it; \frac{k}{l}, \alpha\right)\right|^2 \mathrm{d}t \ll T.$$

Hence, it follows that also

$$\int\limits_0^T \left|\widehat{E}\left(\sigma_2+it; \frac{k}{l}, \alpha\right)\right|^2 \mathrm{d}t \ll T, \tag{4}$$

and

$$\int_0^T \left|\widehat{E}'\left(\sigma_2 + it; \frac{k}{l}, \alpha\right)\right|^2 \mathrm{d}t \ll T. \tag{5}$$

We choose the contour L to satisfy $\delta = \frac{\eta}{4}$. Then u is bounded, and the Gallagher lemma, see [8], Lemma 1.4, together with estimates (4) and (5) shows that

$$\begin{aligned}
&\frac{1}{N}\sum_{m=0}^{N}\left|\widehat{E}\Big(\sigma_2 + iu + i\tau + imh; \frac{k}{l}, \alpha\Big)\right|^2 \\
&\qquad \ll \frac{1}{Nh}\int_0^{Nh}\left|\widehat{E}\Big(\sigma_2 + iu + i\tau + it; \frac{k}{l}, \alpha\Big)\right|^2 \mathrm{d}t \\
&\qquad + \frac{1}{N}\left(\int_0^{Nh}\left|\widehat{E}'\Big(\sigma_2 + iu + i\tau + it; \frac{k}{l}, \alpha\Big)\right|^2 \mathrm{d}t\right. \\
&\qquad \left.\cdot \int_0^{Nh}\left|\widehat{E}\Big(\sigma_2 + iu + i\tau + it; \frac{k}{l}, \alpha\Big)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad \ll \frac{1}{N}(N + |\tau|) \ll 1 + |\tau|. \qquad (6)
\end{aligned}$$

This and (3) lead to the estimate

$$\begin{aligned}
&\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}\left|\widehat{E}\left(s + imh; \frac{k}{l}, \alpha\right) - \widehat{E}_n\left(s + imh; \frac{k}{l}, \alpha\right)\right| \\
\ll\ & \sup_{\sigma+iu\in L}\int_{-\infty}^{\infty}|l_n(\sigma_2 - \sigma + i\tau)|(1 + |\tau|)\mathrm{d}\tau. \qquad (7)
\end{aligned}$$

By the definition of σ_2 and the contour L, we have that $\sigma_2 - \sigma \le -\frac{\eta}{4}$ for $\sigma + iu \in L$. Moreover, the definition of the function $l_n(s)$ shows that, for $\sigma < 0$,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|l_n(\sigma + i\tau)|\,(1 + |\tau|)\mathrm{d}t = 0.$$

Therefore, this and (7) imply the assertion of the lemma.

Theorem 6. *Let K be a compact subset of D. Then, for almost all $\omega \in \Omega$,*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}\left|\widehat{E}\Big(s + imh; \frac{k}{l}, \alpha; \omega\Big) - \widehat{E}_n\Big(s + imh; \frac{k}{l}, \alpha; \omega\Big)\right| = 0.$$

Proof. In [5] it was observed that, for $\sigma > \frac{1}{2}$, the estimate

$$\int_0^T \left|\widehat{E}\left(\sigma + it; \frac{k}{l}, \alpha; \omega\right)\right|^2 \mathrm{d}t \ll T$$

holds for almost all $\omega \in \Omega$. Therefore, the proof repeats the arguments used in the proof of Theorem 5.

4. Limit theorems for $\widehat{E}\left(s; \frac{k}{l}, \alpha\right)$

On $(H(D), \mathcal{B}(H(D)))$, define two probability measures

$$Q_N(A) = \mu_N\left(\widehat{E}\left(s + imh; \frac{k}{l}, \alpha\right) \in A\right),$$

and, for $\omega \in \Omega$,

$$\widehat{Q}_N(A) = \mu_N\left(\widehat{E}\left(s + imh; \frac{k}{l}, \alpha; \omega\right) \in A\right).$$

Theorem 7. *There exists a probability measure Q on $(H(D), \mathcal{B}(H(D)))$ such that both the measures Q_N and $\widehat{Q}_N$ converge weakly to Q as $N \to \infty$.*

Proof. By Theorem 3, the probability measures $P_{N,n}$ and $\widehat{P}_{N,n}$ both converge weakly to the measure P_n. Let θ_N be a random variable defined on a certain probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mathbb{P})$ with the distribution

$$\mathbb{P}(\theta_N = mh) = \frac{1}{N+1}, \quad m = 0, 1, ..., N.$$

Define

$$X_{N,n} = X_{N,n}(s) = \widehat{E}_n\left(s + i\theta_N; \frac{k}{l}, \alpha\right),$$

and denote by $X_n = X_n(s)$ the $H(D)$-valued random element with the distribution P_n. Then Theorem 3 implies the relation

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} X_n, \tag{8}$$

where, as usual, $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution.

The further proof requires a metric on $H(D)$ which induces its topology of uniform convergence on compacta. It is known, see, for example,

[2], that there exists a sequence $\{K_n : n \in \mathbb{N}\}$ of compact subsets of D such that $D = \bigcup_{n=1}^{\infty} K_n$, $K_n \subset K_{n+1}$, and if K is a compact of the region D, then $K \subseteq K_n$ for some n. Then it is easily seen that

$$\rho(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\sup_{s \in K_n} |f(s) - g(s)|}{1 + \sup_{s \in K_n} |f(s) - g(s)|}$$

is the mentioned metric.

For every $M_r > 0$, the Chebyshev inequality yields

$$\begin{aligned} \mathbb{P}\left(\sup_{s \in K_r} |X_{N,n}(s)| > M_r\right) &= \mu_N\left(\sup_{s \in K_r} \left|\widehat{E}_n\left(s + imh; \frac{k}{l}, \alpha\right)\right| > M_r\right) \\ &\le \frac{1}{M_r(N+1)} \sum_{m=0}^{N} \sup_{s \in K_r} \left|\widehat{E}_n\left(s + imh; \frac{k}{l}, \alpha\right)\right|. \end{aligned} \tag{9}$$

Let L_r be a simple closed contour in D enclosing the set K_r, and let δ_r be the distance of L_r from K_r. Then by the Cauchy integral formula

$$\sup_{s \in K_r} \left|\widehat{E}\left(s + imh; \frac{k}{l}, \alpha\right)\right| \ll \frac{1}{\delta_r} \int_{L_r} \left|\widehat{E}\left(z + imh; \frac{k}{l}, \alpha\right)\right| |\mathrm{d}z|.$$

Therefore, in view of Theorem 5 and (6),

$$\begin{aligned} &\limsup_{N \to \infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s \in K_r} \left|\widehat{E}_n\left(s + imh; \frac{k}{l}, \alpha\right)\right| \\ \le\ &\limsup_{N \to \infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s \in K_r} \left|\widehat{E}\left(s + imh; \frac{k}{l}, \alpha\right) - \widehat{E}_n\left(s + imh; \frac{k}{l}, \alpha\right)\right| \\ &+ \limsup_{N \to \infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s \in K_r} \left|\widehat{E}\left(s + imh; \frac{k}{l}, \alpha\right)\right| \\ \le\ &C_{1r} + \limsup_{N \to \infty} \frac{|L_r|}{\delta_r(N+1)} \sup_{\sigma + iu \in L_r} \sum_{m=0}^{N} \left|\widehat{E}\left(s + iu + imh; \frac{k}{l}, \alpha\right)\right| \\ \le\ &C_{1r} + C_{2r} \stackrel{\text{def}}{=} C_r < \infty. \end{aligned} \tag{10}$$

Now let $\epsilon > 0$ be an arbitrary number. We take $M_r = M_{r,\epsilon} = C_r \frac{2^r}{\epsilon}$. Then we deduce from (9) and (10) that

$$\limsup_{N \to \infty} \mathbb{P}\left(\sup_{s \in K_r} |X_{N,n}(s)| > M_{r,\epsilon}\right) < \frac{\epsilon}{2^r}$$

for all $n, r \in \mathbb{N}$. Since (8) implies the relation

$$\sup_{s\in K_r} |X_{N,n}(s)| \xrightarrow[N\to\infty]{\mathcal{D}} \sup_{s\in K_r} |X_n(s)|,$$

hence we find that

$$\mathbb{P}\left(\sup_{s\in K_r} |X_n(s)| > M_{r,\epsilon}\right) < \frac{\epsilon}{2^r} \tag{11}$$

for all $n, r \in \mathbb{N}$. Define

$$H_\epsilon = \{f \in \ H(D) : \sup_{s\in K_r} |f(s)| \le M_{r,\epsilon}\, r \ge 1\}.$$

Then the set H_ϵ is compact on $H(D)$, and, by (11),

$$\mathbb{P}(X_n(s) \in H_\epsilon) \ge 1 - \epsilon$$

for all $n \in \mathbb{N}$. This means that the family of probability measures $\{P_n : n \in \mathbb{N}\}$ is tight. Therefore, by the Prokhorov theorem, see, for example, [1], it is relatively compact. Thus, there exists a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to some probability measure Q on $(H(D), \mathcal{B}(H(D)))$ as $k \to \infty$. Then also the relation

$$X_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} Q \tag{12}$$

holds.

Now let

$$X_N = X_N(s) = \widehat{E}\left(s + i\theta_N; \frac{k}{l}, \alpha\right).$$

Then, by Theorem 5, for every $\epsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \mathbb{P}\big(\rho\,(X_N(s), X_{N,n}(s)) \ge \epsilon\big) \\
= \quad &\lim_{n\to\infty} \limsup_{N\to\infty} \mu_N \left(\rho\left(\widehat{E}\left(s + imh; \frac{k}{l}, \alpha\right), \widehat{E}_n\left(s + imh; \frac{k}{l}, \alpha\right)\right) \ge \epsilon\right) \\
\le \quad &\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N+1)\epsilon} \sum_{m=0}^{N} \rho\left(\widehat{E}\left(s + imh; \frac{k}{l}, \alpha\right), \widehat{E}_n\left(s + imh; \frac{k}{l}, \alpha\right)\right) = 0.
\end{aligned}$$

Since the space $H(D)$ is separable, this, (8), (12) together with Theorem 4.2 of [1] show that

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} Q. \tag{13}$$

This means that the measure Q_N converges weakly to Q as $N \to \infty$. Moreover, (13) shows that the measure P is independent of the subsequence $\{P_{n_k}\}$. Since $\{P_n\}$ is relatively compact, hence we deduce that

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} Q. \tag{14}$$

Now let

$$\widehat{X}_{N,n}(s) = \widehat{E}_n\left(s + i\theta_N; \frac{k}{l}, \alpha; \omega\right),$$

and

$$\widehat{X}_N(s) = \widehat{E}_n\left(s + i\theta_N; \frac{k}{l}, \alpha; \omega\right).$$

Then, repeating the above arguments for $\widehat{X}_{N,n}(s)$ and $\widehat{X}_N(s)$, applying Theorems 3 and 6, as well as taking into account (14), we obtain that the measure $\widehat{Q}_N$ also converges weakly to Q as $N \to \infty$. The theorem is proved.

Theorem 8. *The probability measure Q_N converges weakly to $P_{\widehat{E}}$ as $N \to \infty$.*

Proof. We start with elements of the ergodic theory. Let $a_h = \{p^{-ih} : p \in \mathcal{P}\}$, and $f_h(\omega) = a_h\omega$, $\omega \in \Omega$. Then f_h is a measurable measure preserving transformation on $(\Omega, \mathcal{B}(\Omega), m_H)$. It was obtained in [3] that this transformation is ergodic.

Let $A \in \mathcal{B}(H(D))$ be an arbitrary continuity set of the limit measure Q in Theorem 7. Then, by the latter theorem,

$$\lim_{N\to\infty} \mu_N\left(\widehat{E}\left(s + imh; \frac{k}{l}, \alpha\right) \in A\right) = Q(A). \tag{15}$$

On the space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the random variable θ by the formula

$$\theta = \theta(\omega) = \begin{cases} 1 & \text{if } \widehat{E}\left(\sigma; \frac{k}{l}, \alpha; \omega\right) \in A, \\ 0 & \text{if } \widehat{E}\left(\sigma; \frac{k}{l}, \alpha; \omega\right) \notin A. \end{cases}$$

Then, denoting by $\mathbb{E}\theta$ the expectation of θ, we have that

$$\mathbb{E}\theta = \int_\Omega \theta \mathrm{d}m_H = m_H\left(\omega \in \Omega : \widehat{E}\left(s; \frac{k}{l}, \alpha; \omega\right) \in A\right) = P_{\widehat{E}}(A). \tag{16}$$

Since the transformation f_h is ergodic, the classical Birkhoff–Khinchine theorem, see, for example, [4], shows that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \theta\big(f_h^m(\omega)\big) = \mathbb{E}\theta \tag{17}$$

for almost all $\omega \in \Omega$. On the other hand, from the definitions of f_h and θ, we deduce that

$$\frac{1}{N+1} \sum_{m=0}^{N} \theta\big(f_h^m(\omega)\big) = \mu_N\left(\widehat{E}\left(s + imh; \frac{k}{l}, \alpha; \omega\right) \in A\right).$$

This, (16) and (17) give the equality

$$\lim_{N\to\infty} \mu_N \left(\widehat{E}\left(s+imh; \frac{k}{l}, \alpha; \omega\right) \in A\right) = P_{\widehat{E}}(A).$$

Therefore, in view of (15),

$$Q(A) = P_{\widehat{E}}(A) \tag{18}$$

for all continuity sets of the measure Q. Since all continuity sets constitute the determining class, (18) holds for all $A \in \mathcal{B}(H(D))$, and the theorem is proved.

5. Two-dimensional theorem

Let $H^2(D) = H(D) \times H(D)$, and

$$f(s,\omega) = \prod_{j=1}^{2} \left(1 - \frac{2^{s_j}\omega(2)}{2^s}\right).$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define an $H^2(D)$-valued random element $F(s,\omega)$ by

$$F(s,\omega) = \left(f(s,\omega), \widehat{E}\left(s; \frac{k}{l}, \alpha; \omega\right)\right).$$

In this section, we consider the weak convergence of the probability measure

$$R_N(A) \stackrel{\text{def}}{=} \mu_N \left(\left(f(s+imh), \widehat{E}\left(s+imh; \frac{k}{l}, \alpha\right)\right) \in A\right), \quad A \in \mathcal{B}(H^2(D)).$$

Theorem 9. *The probability measure R_N converges weakly to the distribution P_F of the random element $F(s,\omega)$ as $N \to \infty$.*

Proof. The function $f(s)$ is a Dirichlet polynomial. Therefore, the probability measure

$$\mu_N \left(f(s+imh) \in A\right), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the distribution of the random element $f(s,\omega)$ as $N \to \infty$. Now this, Theorem 8 and an application of the modified Cramér–Wald criterion, an example of its application is given in [7], leads to the statement of the theorem.

6. Proof of the main theorem

Theorem 2 is a consequence of Theorem 9.

Proof of Theorem 2. It is not difficult to see that, for the metric d defined in Section 1, the equality

$$d(g_1, g_2) = d\left(\frac{1}{g_1}, \frac{1}{g_2}\right), \quad g_1, g_2 \in H(D),$$

holds. Therefore, the function $u : H^2(D) \to M(D)$ defined by the formula

$$u(g_1, g_2) = \frac{g_2}{g_1}, \quad g_1, g_2 \in H(D),$$

is continuous, and $P_N = R_N u^{-1}$. Hence, by Theorem 5.1 of [1] and Theorem 9, the measure P_N converges weakly to the measure $P_F u^{-1}$, i.e., to

$$m_H\left(\omega \in \Omega : \frac{\widehat{E}\left(s; \frac{k}{l}, \alpha; \omega\right)}{f(s,\omega)} \in A\right), \quad A \in \mathcal{B}(M(D)). \tag{19}$$

However, by the definition of the random element $\widehat{E}\left(s; \frac{k}{l}, \alpha; \omega\right)$, we have that

$$\frac{\widehat{E}\left(s; \frac{k}{l}, \alpha; \omega\right)}{f(s,\omega)} = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)\omega(m)}{m^s} \exp\left\{2\pi i m \frac{k}{l}\right\} = E\left(s; \frac{k}{l}, \alpha; \omega\right).$$

Therefore, (19) coincides with

$$m_H\left(\omega \in \Omega : E\left(s; \frac{k}{l}, \alpha; \omega\right) \in A\right), \quad A \in \mathcal{B}(H(D)).$$

The theorem is proved.

References

[1] P. Billingsley, *Convergence of Probability Measures.* Wiley, New York, 1968.

[2] J. B. Conway, *Functions of One Complex Variable.* Springer – Verlag, New York, 1973.

[3] R. Kačinskaitė, A discrete limit theorem for the Matsumoto zeta-function on the complex plane. *Lith. Math. J.*, **40**(4) (2000), 364–378.

[4] U. Krengel, *Ergodic Theorems.* Walter de Gruyter, Berlin, 1985.

[5] A. Laurinčikas, Limit theorems for the Estermann zeta-function. II. *Cent. Eur. J. Math.*, **3**(4) (2005), 580–590.

[6] A. Laurinčikas, R. Macaitienė, Discrete limit theorems for the Estermann zeta-functions. I. *Algebra and Discrete Math.* Number 4 (2007), 84-101.

[7] R. Macaitienė, A joint discrete limit theorem in the space of meromorphic functions for general Dirichlet series. *Acta Appl. Math.*, **97** (2007), 99–112.

[8] H. L. Montgomery, *Topics in Multiplicative Number Theory*. Springer-Verlag, Berlin, 1971.

[9] R. Šleževičienė, On some aspects in the theory of the Estermann zeta-function. *Fiz. Mat. Fak. Moksl. Semin. Darb.*, **5** (2002), 115–130.

Contact information

Antanas Laurinčikas
Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania
E-Mail: antanas.laurincikas@maf.vu.lt

Renata Macaitienė
Department of Mathematics and Informatics, Šiauliai University, P. Visinskio 19, LT-77156 Siauliai, Lithuania
E-Mail: renata.macaitiene@mi.su.lt

Received by the editors: 26.02.2008
and in final form 14.10.2008.