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Group action on bimodule categories

Yuriy A. Drozd

To the memory of A. V. Roiter

Abstract. We consider actions of groups on categories and bimodules, the related crossed group categories and bimodules, and prove for them analogues of the result know for representations of crossed group algebras and categories.

Skew group algebras arise naturally in lots of questions. In particular, the properties of the categories of representations of skew group algebras and, more generally, skew group categories have been studied in [11, 8]. On the other hand, "*matrix problems*," especially, *bimodule categories* play now a crucial role in the theory of representations [5, 6]. The situation, when a group acts on a bimodule, thus also on the bimodule category is also rather typical. Therefore one needs to deal with *skew bimodules* and their bimodule categories. In this paper we shall study skew bimodules and bimodule categories and prove for them some analogues of the results of [11, 8].

In Section 1 we recall general notions related to bimodule categories. In Section 2 we consider actions of groups on bimodule and bimodule categories and the arising functors. The main results are those of Section 3, where we define *separable actions* and prove that in the separable case the bimodule category of the skew bimodule is equivalent to the skew category of the original one. We also consider specially the case of the abelian groups, since in this case the original category can be restored from the skew one using the group of characters. Section 4 is devoted to the decomposition of objects in skew group categories, especially, to the number of non-isomorphic direct summands in such decompositions. We

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also consider the *radical* and *almost split morphisms* in the skew group categories (under the separability condition).

1. Bimodule categories

We recall the main definitions related to bimodule categories [5, 6]. We fix a commutative ring $\mathbf{K}$. All categories that we consider are supposed to be $\mathbf{K}$-*categories*, which means that all sets of morphisms are $\mathbf{K}$-modules, while the multiplication is $\mathbf{K}$-bilinear. We denote the set of morphisms from an object X to an object Y in a category $\mathcal{A}$ by $\mathcal{A}(X,Y)$. A *module* (more precise, a *left module*) over a category $\mathcal{A}$, or a $\mathcal{A}$-*module* is, by definition, a $\mathbf{K}$-linear functor $M : \mathcal{A} \to \mathbf{K}\text{-}\mathsf{Mod}$, where $\mathbf{K}\text{-}\mathsf{Mod}$ denotes the category of $\mathbf{K}$-modules. If M is such a module, $x \in M(X)$ and $a \in \mathcal{A}(X,Y)$, we write, as usually, ax instead of $M(a)(x)$. Such modules have all usual properties of modules over rings. The category of all $\mathcal{A}$-modules is denoted by $\mathcal{A}\text{-}\mathsf{Mod}$. A *bimodule* over a category $\mathcal{A}$, or an $\mathcal{A}$-bimodule, is, by definition, a $\mathbf{K}$-bilinear functor $\mathcal{B} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{K}\text{-}\mathsf{Mod}$, where $\mathcal{A}^{\mathrm{op}}$ is the opposite category to $\mathcal{A}$. If $x \in \mathcal{B}(X,Y)$, $a : X' \to X$ (i.e. $a : X \to X'$ in $\mathcal{A}^{\mathrm{op}}$), $b : Y \to Y'$, we write bxa instead of $\mathcal{B}(a,b)(x)$ (this element belongs to $\mathcal{B}(X',Y')$). In particular, xa and bx denote, respectively, $\mathcal{B}(a,1_Y)(x)$ and $\mathcal{B}(1_X,b)(x)$. If a bimodule $\mathcal{B}$ is fixed, we often write $x : X \dashrightarrow Y$ instead of $x \in \mathcal{B}(X,Y)$.

A category $\mathcal{A}$ is called *fully additive* if it is additive (i.e. has direct sums $X \oplus Y$ of any pair of objects X, Y and a zero object 0) and every idempotent endomorphism $e \in \mathcal{A}(X,X)$ *splits*, i.e. there is an object Y and a pair of morphisms $\iota : Y \to X$ and $\pi : X \to Y$ such that $\pi\iota = 1_Y$ and $\iota\pi = e$. Choosing an object Y' and morphisms $\iota' : Y' \to X$ and $\pi' : X \to Y'$ such that $\pi'\iota' = 1_{Y'}$ and $\iota'\pi' = 1-e$, we present X as a direct sum $Y \oplus Y'$, where ι and ι' are canonical embeddings, while π and π' are canonical projections. For every $\mathbf{K}$-category $\mathcal{A}$ there is the smallest fully additive category $\operatorname{add}\mathcal{A}$ containing $\mathcal{A}$. This category is unique (up to equivalence). It can be identified either with the category of matrix idempotents over $\mathcal{A}$ or with the category of finitely generated projective $\mathcal{A}$-modules [9]. We call it the *additive hull* of $\mathcal{A}$. Each $\mathcal{A}$-module M (bimodule $\mathcal{B}$) extends uniquely (up to isomorphism) to a module (bimodule) over the category $\operatorname{add}\mathcal{A}$, which we also denote by M (respectively, by $\mathcal{B}$)

If $\mathcal{B}$ is an $\mathcal{A}$-bimodule, a *differentiation* from $\mathcal{A}$ to $\mathcal{B}$ is, by definition, a set of $\mathbf{K}$-linear maps

$$\partial = \{\, \partial(X,Y) : \mathcal{A}(X,Y) \to \mathcal{B}(X,Y) \mid X, Y \in \operatorname{Ob}\mathcal{A} \,\},$$

satisfying the *Leibniz rule*:

$$\partial(ab) = (\partial a)b + a(\partial b)$$

for any morphisms a, b such that the product ab is defined. It implies, in particular, that $\partial 1_X = 0$ for any object X. Again, such a differentiation extends to the additive hull of $\mathcal{A}$ and we denote this extension by the same letter ∂. A triple $\mathfrak{T} = (\mathcal{A}, \mathcal{B}, \partial)$, where $\mathcal{A}$ is a category, $\mathcal{B}$ is a $\mathcal{A}$-bimodule and ∂ is a differentiation from $\mathcal{A}$ to $\mathcal{B}$, is called a *bimodule triple.* If $\mathfrak{T}' = (\mathcal{A}', \mathcal{B}', \partial')$ is another bimodule triple, a *bifunctor* from $\mathfrak{T}$ to $\mathfrak{T}'$ is defined as a pair $F = (F_0, F_1)$, where $F_0 : \mathcal{A} \to \mathcal{A}'$ is a functor, $F_1 : \mathcal{B} \to \mathcal{B}'(F_0)$ is a homomorphism of $\mathcal{A}$-bimodule, where $\mathcal{B}'(F_0)$ is the $\mathcal{A}$-bimodule obtained from $\mathcal{B}'$ by the transfer along F_0 (i.e. $F_1(x) : F_0(X) \dashrightarrow F_0(Y)$ if $x : X \dashrightarrow Y$, and $F_1(bxa) = F_0(b)F_1(x)F_0(a)$), such that $F_1(\partial a) = \partial'(F_0(a))$ for all $a \in \operatorname{Mor}\mathcal{A}$. As a rule, we write $F(a)$ and $F(x)$ instead of $F_0(a)$ and $F_1(x)$.

Let $F = (F_0, F_1)$ and $G = (G_0, G_1)$ be two bifunctors from a triple $\mathfrak{T} = (\mathcal{A}, \mathcal{B}, \partial)$ to another triple $\mathfrak{T}' = (\mathcal{A}', \mathcal{B}', \partial')$. A *morphism of bifunctors* $\phi : F \to G$ is defined as a morphism of functors $\phi : F_0 \to G_0$ such that

$$\begin{aligned} &\phi(Y)F_1(x) = G_1(x)\phi(X) \quad \text{for each} \quad x \in \mathcal{B}(X, Y), \\ &\partial'\phi(X) = 0 \quad \text{for each} \quad X \in \operatorname{Ob}\mathcal{A}. \end{aligned}$$

If ϕ is an isomorphism of functors, the inverse morphism is obviously a morphism of bifunctors too. Then we call ϕ an *isomorphism of bifunctors* and write $\phi : F \xrightarrow{\sim} G$. If such an isomorphism exists, we say that the bifunctors F are G *isomorphic* and write $F \simeq G$.

We call a bifunctor $F : \mathfrak{T} \to \mathfrak{T}'$ an *equivalence of bimodule triples* if there is such a bifunctor $G : \mathfrak{T}' \to \mathfrak{T}$ that $FG \simeq \mathrm{id}_{\mathfrak{T}'}$ and $GF \simeq \mathrm{id}_{\mathfrak{T}}$, where $\mathrm{id}_{\mathfrak{T}}$ denotes the identity bifunctor $\mathfrak{T} \to \mathfrak{T}$. If such a bifunctor exists, we call the triples $\mathfrak{T}$ and $\mathfrak{T}'$ *equivalent* and write $\mathfrak{T} \simeq \mathfrak{T}'$.

Lemma 1.1. A bifunctor $F = (F_0, F_1)$ is an equivalence of bimodule triples if and only if the following conditions hold:

1. The functor F_0 is *fully faithful*, i.e. all induced maps $\mathcal{A}(X, Y) \to \mathcal{A}'(F_0X, F_0Y)$ are bijective.

2. This functor is also ∂-*dense*, i.e. for every object X' of the category $\mathcal{A}'$ there are an object $X \in \operatorname{Ob}\mathcal{A}$ and an isomorphism $\alpha : X' \to F_0X$ such that $\partial\alpha = 0$.

3. The map $F_1(X, Y) : \mathcal{B}(X, Y) \to \mathcal{B}'(F_0X, F_0Y)$ is bijective for any $X, Y \in \operatorname{Ob}\mathcal{A}$.

Moreover, if these conditions hold, there is a bifunctor $G : \mathfrak{T}' \to \mathfrak{T}$ and an isomorphism $\lambda : \mathrm{id}_{\mathfrak{T}'} \to FG$ such that $GF = \mathrm{id}_{\mathfrak{T}}$ and $\lambda(FX) = 1_{FX}$ for all $X \in \mathrm{Ob}\,\mathcal{A}$.

Proof. The necessity of these conditions is evident, so we prove their sufficiency. Suppose that these conditions hold. For each object $X' \in \mathcal{A}'$ choose an object X and an isomorphism $\alpha : X' \to F_0X$ such that $\partial a = 0$, always setting $\alpha = 1_{X'}$ for $X' = F_0X$. Set $G_0X' = X$ and $\lambda(X') = \alpha$. For each morphism $a : X' \to Y'$ set $G_0a = F_0^{-1}(X,Y)(\lambda(Y')a\lambda^{-1}(X'))$, where $X = G_0X'$, $Y = G_0Y'$ (then $\lambda(X') : X' \xrightarrow{\sim} F_0X$, $\lambda(Y') : Y' \xrightarrow{\sim} F_0Y$). Obviously, the set $\{\lambda(X')\}$ defines an isomorphism of functors $\lambda : \mathrm{id} \to F_0G_0$. We also define a homomorphism of bimodules $G_1 : \mathcal{B}' \to \mathcal{B}(G_0)$ setting $G_1(x) = F_1(X,Y)^{-1}(\lambda(Y')x\lambda^{-1}(X'))$ if $x : X' \dashrightarrow Y'$, $X = G_0X'$, $Y = G_0Y'$. Then $G = (G_0, G_1)$ is a bifunctor $\mathfrak{T}' \to \mathfrak{T}$ and λ is an isomorphism of bifunctors $\mathrm{id}_{\mathfrak{T}'} \to FG$. Moreover, by this construction, $GF = \mathrm{id}_{\mathfrak{T}}$ and $\lambda(FX) = 1_{FX}$ for all X. □

Every bimodule triple $\mathfrak{T} = (\mathcal{A}, \mathcal{B}, \partial)$ gives rise to the *bimodule category* (or the *category of representations*, or the *category of elements*) of this triple [5]. The *objects* of this category are elements $\bigcup_X \mathcal{B}(X,X)$, where X runs through objects of the category $\mathrm{add}\,\mathcal{A}$. *Morphisms* from an object $x : X \dashrightarrow X$ to an object $y : Y \dashrightarrow Y$ are such morphisms $a : X \to Y$ that $ax = ya + \partial(a)$ in $\mathcal{B}(X,Y)$. It is easy to see that these definitions really define a fully additive $\mathbf{K}$-category $\mathsf{El}(\mathfrak{T})$. The set of morphisms $x \to y$ in this category is denoted by $\mathrm{Hom}_{\mathfrak{T}}(x,y)$. If $\partial = 0$, we write $\mathsf{El}(\mathcal{A}, \mathcal{B})$ or even $\mathsf{El}(\mathcal{B})$ instead of $\mathsf{El}(\mathcal{A}, \mathcal{B}, \partial)$. Each bifunctor between bimodule triples $F : \mathfrak{T} \to \mathfrak{T}'$ gives rise to a functor $F_* : \mathsf{El}(\mathfrak{T}) \to \mathsf{El}(\mathfrak{T}')$, which maps an object x to the object $F_1(x)$ and a morphism $a : x \to y$ to the morphism $F_0(a) : F_1(x) \to F_1(y)$. As well, each morphism of bifunctors $\phi : F \to G$ induces a morphism of functors $\phi_* : F_* \to G_*$, which correlate an object $x \in \mathcal{B}(X,X)$ with the morphism $\phi(X)$ considered as a morphism $F(x) \to G(x)$. Obviously, if ϕ is an isomorphism of bifunctors, ϕ_* is an isomorphism of functors. Especially, if F is an equivalence of bimodule triples, the functor F_* is an equivalence of their bimodule categories.

If $\mathcal{B} = \mathcal{A}$ and $\partial = 0$, we say that the bimodule triple $\mathfrak{T} = (\mathcal{A}, \mathcal{A}, 0)$ is the *principle triple* for the category $\mathcal{A}$. Obviously, a bifunctor between principle triples is just a functor between the corresponding categories and a morphism of such bifunctors is just a morphism of functors. The bimodule category of the principle triple for a category $\mathcal{A}$ is denoted by $\mathsf{El}(\mathcal{A})$.

If $\mathcal{A}$ and $\mathcal{A}'$ are two categories, one can consider $\mathcal{A}$-$\mathcal{A}'$-*bimodules*, i.e. bilinear functors $\mathcal{B} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A}' \to \mathbf{K}\text{-}\mathsf{Mod}$. Actually, any such bimodule can be identified with a $\mathcal{A}\times\mathcal{A}'$-bimodule $\tilde{\mathcal{B}}$ with $\tilde{\mathcal{B}}((X,X'),(Y,Y'))=\mathcal{B}(X,Y')$ and $(a,a')x(b,b')=axb'$. Such bimodules are called *bipartite.* In particular, every $\mathcal{A}$-bimodule $\mathcal{B}$ defines a bipartite $\mathcal{A}$-$\mathcal{A}$-bimodule, which we denote by $\mathcal{B}^{(2)}$ and call the *double* of the $\mathcal{A}$-bimodule $\mathcal{B}$. Certainly, bimodules $\mathcal{B}$ and $\mathcal{B}^{(2)}$ are quite different and they define different bimodule categories. If $\mathcal{B}=\mathcal{A}$ the category $\mathsf{El}(\mathcal{A}^{(2)})$ coincides with the *category of morphisms* of the additive hull $\operatorname{add}\mathcal{A}$.

Further on we often identify the categories $\mathcal{A}$ and $\operatorname{add}\mathcal{A}$ and say "an object (morphism) of $\mathcal{A}$" instead of "an object (morphism) of $\operatorname{add}\mathcal{A}$." We hope that this petty ambiguity will not embarrass the reader.

2. Group actions

Let $\mathfrak{T}=(\mathcal{A},\mathcal{B},\partial)$ be a bimodule triple and G be a group. One says that the group G *acts on the triple* $\mathfrak{T}$ if a bifunctor $T_\sigma:\mathfrak{T}\to\mathfrak{T}$ is defined for each element $\sigma\in G$ so that $T_1=\mathrm{id}_{\mathfrak{T}}$ and $T_{\sigma\tau}\simeq T_\sigma T_\tau$ for any $\sigma,\tau\in\mathsf{G}$. It implies, in particular, that all T_σ are equivalences. Further on we write X^σ instead of $T_\sigma(X)$. We only note that according to this notation $X^{\sigma\tau}\simeq(X^\tau)^\sigma$. A *system of factors* λ for such an action is defined as a set of isomorphisms of bifunctors $\lambda_{\sigma,\tau}:T_{\sigma\tau}\xrightarrow{\sim}T_\sigma T_\tau$, which satisfy the relations:

$$\lambda^\rho_{\sigma,\tau}\lambda_{\rho,\sigma\tau}=\lambda_{\rho,\sigma}\lambda_{\rho\sigma,\tau} \tag{2.1}$$

for any triple of elements $\rho,\sigma,\tau\in\mathsf{G}$, and $\lambda_{\sigma,1}=\lambda_{1,\sigma}=1$ for any $\sigma\in\mathsf{G}$. We omit the arguments (objects of $\mathcal{A}$) in these formulae (and later on in analogous cases), since their values can easily be restored. Since $\lambda_{\sigma,\tau}$ is a morphism of bifunctors, one has $\lambda_{\sigma,\tau}:X^{\sigma\tau}\to(X^\tau)^\sigma$ and

$$\lambda_{\sigma,\tau}x^{\sigma\tau}=(x^\tau)^\sigma\lambda_{\sigma,\tau} \tag{2.2}$$

for every morphism from $\mathcal{A}$ and every element from $\mathcal{B}$, and also $\partial\lambda_{\sigma,\tau}=0$ for all σ,τ. Note also that the relations (2.1) and (2.2) imply, in particular, that

$$\lambda^\sigma_{\sigma^{-1},\sigma}=\lambda_{\sigma,\sigma^{-1}}\ \text{ and }\ \lambda_{\sigma,\sigma^{-1}}x=(x^{\sigma^{-1}})^\sigma\lambda_{\sigma,\sigma^{-1}}.$$

Given an action $T=\{\,T_\sigma\,\}$ of a group G on a bimodule triple $\mathfrak{T}=(\mathcal{A},\mathcal{B},\partial)$ and a system of factors λ for this action, we define the *crossed group triple* $\mathfrak{T}\mathsf{G}=\mathfrak{T}(\mathsf{G},T,\lambda)$. Namely, we consider the *crossed group category* $\mathcal{A}\mathsf{G}=\mathcal{A}(\mathsf{G},T,\lambda)$ [11, 8]. Its objects coincide with those of $\mathcal{A}$, but morphisms $X\to Y$ in the category $\mathcal{A}\mathsf{G}$ are defined as formal (finite) linear combinations $\sum_{\sigma\in\mathsf{G}}a_\sigma[\sigma]$, where $a_\sigma\in\mathcal{A}(X^\sigma,Y)$, and the multiplication of such morphisms is defined by bilinearity and the rule

$$a_\sigma[\sigma]b_\tau[\tau]=a_\sigma b^\sigma_\tau\lambda_{\sigma,\tau}[\sigma\tau]. \tag{2.3}$$

The condition (2.1) for a system of factors is equivalent to the associativity of this multiplication. The $\mathcal{A}\mathsf{G}$-bimodule $\mathcal{B}\mathsf{G} = \mathcal{B}(\mathsf{G}, T, \lambda)$ is constructed in an analogous way: elements of $\mathcal{B}\mathsf{G}(X, Y)$ are formal (finite) linear combinations $\sum_{\sigma\in\mathsf{G}} x_\sigma[\sigma]$, where $x_\sigma \in \mathcal{B}(X^\sigma, Y)$, and their products with morphisms from $\mathcal{A}\mathsf{G}$ are defined by the same formula (2.3), with the only difference that one of the elements a_σ, b_τ is a morphism from $\mathcal{A}$, while the second one is an element from $\mathcal{B}$. The differentiation ∂ extends to $\mathcal{A}\mathsf{G}$ if we set $\partial(\sum_\sigma a_\sigma[\sigma]) = \sum_\sigma \partial a_\sigma[\sigma]$. We identify every morphism $a \in \mathcal{A}(X, Y)$ with the morphism $a[1] \in \mathcal{A}\mathsf{G}(X, Y)$ and every element $x \in \mathcal{B}(X, Y)$ with the element $x[1] \in \mathcal{B}\mathsf{G}(X, Y)$ getting the embedding bifunctor $\mathfrak{T} \to \mathfrak{T}\mathsf{G}$.

An action T of a group G on a bimodule triple $\mathfrak{T}$ induces its action T_* on the bimodule category $\mathsf{El}(\mathfrak{T})$: an element $\sigma \in \mathsf{G}$ defines the functor $(T_\sigma)_* : x \mapsto x^\sigma$. Moreover, if λ is a system of factors for the action T, it induces the system of factors λ_* for the action T_*: one has to set $(\lambda_*)_{\sigma,\tau}(x) = \lambda_{\sigma,\tau}(X)$ if $x \in \mathcal{B}(X, X)$. Thus the crossed group category $\mathsf{El}(\mathfrak{T})\mathsf{G} = \mathsf{El}(\mathfrak{T})(\mathsf{G}, T_*, \lambda_*)$ is defined, as well as the embedding $\mathsf{El}(\mathfrak{T}) \to \mathsf{El}(\mathfrak{T})\mathsf{G}$. One can also define the natural functor $\Phi : \mathsf{El}(\mathfrak{T})\mathsf{G} \to \mathsf{El}(\mathfrak{T}\mathsf{G})$ as follows. For an object $x \in \mathcal{B}(X, X)$, set $\Phi(x) = x[1] \in \mathcal{B}\mathsf{G}(X, X)$. Let $\alpha = \sum_\sigma a_\sigma[\sigma]$ be a morphism from x to $y \in \mathcal{B}(Y, Y)$ in the category $\mathsf{El}(\mathfrak{T})\mathsf{G}$. It means that $a_\sigma : x^\sigma \to y$ in the category $\mathsf{El}(\mathfrak{T})$, i.e. $a_\sigma \in \mathcal{A}(X^\sigma, Y)$ and $a_\sigma x^\sigma = y a_\sigma + \partial a_\sigma$. Then one can consider α as a morphism $X \to Y$ in the category $\mathcal{A}\mathsf{G}(X, Y)$, and $\alpha x[1] = \sum_\sigma a_\sigma[\sigma]x[1] = \sum_\sigma a_\sigma x^\sigma[\sigma] = \sum_\sigma (y a_\sigma + \partial a_\sigma)[\sigma] = y[1]\alpha + \partial\alpha$, so α is a morphism $x[1] \to y[1]$ in the category $\mathsf{El}(\mathfrak{T}\mathsf{G})$ and one can set $\Phi(\alpha) = \alpha$.

Proposition 2.1. The functor Φ is fully faithful, i.e. for any objects x, y from $\mathsf{El}(\mathfrak{T})\mathsf{G}$ it induces the bijective map $\operatorname{Hom}_{\mathfrak{T}}\mathsf{G}(x, y) \to \operatorname{Hom}_{\mathfrak{T}\mathsf{G}}(x, y)$, where $\operatorname{Hom}_{\mathfrak{T}}\mathsf{G}$ denotes the morphisms in the category $\mathsf{El}(\mathfrak{T})\mathsf{G}$.

Proof. Obviously, this map is injective. Let $\alpha = \sum_\sigma a_\sigma[\sigma] : x[1] \to y[1]$, i.e. $\alpha x[1] = \sum_\sigma a_\sigma x^\sigma[\sigma] = y[1]\alpha + \partial\alpha = \sum_\sigma (y a_\sigma + \partial a_\sigma)[\sigma]$. Then $a_\sigma x^\sigma = y a_\sigma + \partial a_\sigma$ for all σ, so $a_\sigma : x^\sigma \to y$ in the category $\mathsf{El}(\mathfrak{T})$, thus $\alpha : x \to y$ in the category $\mathsf{El}(\mathfrak{T})\mathsf{G}$. Therefore, this map is also surjective. $\square$

If the group G is finite, one can also construct a functor $\Psi : \mathsf{El}(\mathfrak{T}\mathsf{G}) \to \mathsf{El}(\mathfrak{T})$. For every object $X \in \operatorname{Ob}\mathcal{A}$, set $\tilde{X} = \bigoplus_{\sigma\in\mathsf{G}} X^\sigma$ and for every element $\xi = \sum_\sigma x_\sigma[\sigma] \in \mathcal{B}\mathsf{G}(X, X)$, where $x_\sigma : X^\sigma \dashrightarrow X$, denote by $\tilde{\xi}$ the element from $\mathcal{B}(\tilde{X}, \tilde{X}) = \bigoplus_{\sigma,\tau} \mathcal{B}(X^\tau, X^\sigma)$ such that its component $\tilde{\xi}_{\sigma,\tau} \in \mathcal{B}(X^\tau, X^\sigma)$ equals $x^\sigma_{\sigma^{-1}\tau}\lambda_{\sigma,\sigma^{-1}\tau}$. Note that $x_{\sigma^{-1}\tau} : X^{\sigma^{-1}\tau} \dashrightarrow Y$, hence $x^\sigma_{\sigma^{-1}\tau} : (X^{\sigma^{-1}\tau})^\sigma \dashrightarrow Y^\sigma$, thus $x^\sigma_{\sigma^{-1}\tau}\lambda_{\sigma,\sigma^{-1}\tau} : X^\tau \dashrightarrow Y^\sigma$ indeed.

Let $\eta = \sum_\sigma y_\sigma[\sigma] \in \mathcal{B}\mathsf{G}(Y,Y)$, where $y_\sigma \in \mathcal{B}(Y^\sigma, Y)$ and $\alpha = \sum_\sigma a_\sigma[\sigma]$ be a morphism from ξ to η, where $a_\sigma \in \mathcal{A}(X^\sigma, Y)$. Since

$$\begin{aligned}\alpha\xi &= \sum_\rho\sum_\sigma a_\rho[\rho]x_\sigma[\sigma] = \sum_\rho\sum_\sigma a_\rho x_\sigma^\rho \lambda_{\rho,\sigma}[\rho\sigma] = \\ &= \sum_\tau \big(\sum_\rho a_\rho x^\rho_{\rho^{-1}\tau}\lambda_{\rho,\rho^{-1}\tau}\big)[\tau],\end{aligned}$$

and

$$\begin{aligned}\eta\alpha &= \sum_\rho\sum_\sigma y_\rho[\rho]a_\sigma[\sigma] = \sum_\rho\sum_\sigma y_\rho a_\sigma^\rho \lambda_{\rho,\sigma}[\rho\sigma] = \\ &= \sum_\tau \big(\sum_\rho y_\rho a^\rho_{\rho^{-1}\tau}\lambda_{\rho,\rho^{-1}\tau}\big)[\tau],\end{aligned}$$

it means that, for each τ,

$$\sum_\rho a_\rho x^\rho_{\rho^{-1}\tau}\lambda_{\rho,\rho^{-1}\tau} = \sum_\rho y_\rho a^\rho_{\rho^{-1}\tau}\lambda_{\rho,\rho^{-1}\tau} + \partial a_\tau. \tag{2.4}$$

Consider the morphism $\tilde\alpha : \tilde X \to \tilde Y$ such that

$$\tilde\alpha_{\sigma,\tau} = a^\sigma_{\sigma^{-1}\tau}\lambda_{\sigma,\sigma^{-1}\tau} : X^\tau \to Y^\sigma.$$

Then the (σ,τ)-component of the product $\tilde\alpha\tilde\xi$ equals

$$\mathrm{I} = \sum_\rho a^\sigma_{\sigma^{-1}\rho}\lambda_{\sigma,\sigma^{-1}\rho}x^\rho_{\rho^{-1}\tau}\lambda_{\rho,\rho^{-1}\tau} = \sum_\rho a^\sigma_{\sigma^{-1}\rho}\big(x^{\sigma^{-1}\rho}_{\rho^{-1}\tau}\big)^\sigma\lambda_{\sigma,\sigma^{-1}\rho}\lambda_{\rho,\rho^{-1}\tau},$$

while the (σ,τ)-component of the product $\tilde\eta\tilde\alpha$ equals

$$\mathrm{II} = \sum_\rho y^\sigma_{\sigma^{-1}\rho}\lambda_{\sigma,\sigma^{-1}\rho}a^\rho_{\rho^{-1}\tau}\lambda_{\rho,\rho^{-1}\tau} = \sum_\rho y^\sigma_{\sigma^{-1}\rho}\big(a^{\sigma^{-1}\rho}_{\rho^{-1}\tau}\big)^\sigma\lambda_{\sigma,\sigma^{-1}\rho}\lambda_{\rho,\rho^{-1}\tau}.$$

(In both cases we used the relation (2.2) replacing τ by $\sigma^{-1}\rho$). Since, by the condition (2.1) for the system of factors,

$$\lambda_{\sigma,\sigma^{-1}\rho}\lambda_{\rho,\rho^{-1}\tau} = \lambda^\sigma_{\sigma^{-1}\rho,\rho^{-1}\tau}\lambda_{\sigma,\sigma^{-1}\tau}, \quad \text{and} \quad \partial\lambda_{\sigma,\sigma^{-1}\tau} = 0,$$

we get from the relation (2.4) that $\mathrm{I} = \mathrm{II} + \partial\tilde\alpha_{\sigma,\tau}$ (we just replace ρ by $\sigma^{-1}\rho$, τ by $\sigma^{-1}\tau$, then apply the functor T_σ to both sides). Therefore, $\tilde\alpha$ is a morphism $\tilde\xi \to \tilde\eta$ and one can define the functor Ψ setting $\Psi(\xi) = \tilde\xi$ and $\Psi(\alpha) = \tilde\alpha$.

Proposition 2.2. The functors Φ and Ψ form an *adjoint pair*, i.e. there is a natural isomorphism $\mathrm{Hom}_{\mathfrak{T}}\mathsf{G}(\Phi x, \eta) \simeq \mathrm{Hom}_{\mathfrak{T}}(x, \Psi\eta)$ for each objects $x \in \mathsf{El}(\mathfrak{T})$ and $\eta \in \mathsf{El}(\mathfrak{T}\mathsf{G})$.

Proof. Let $x \in \mathcal{B}(X,X)$, $\eta \in \mathcal{B}\mathsf{G}(Y,Y)$, $\eta = \sum_\sigma y_\sigma[\sigma]$, where $y : Y^\sigma \dashrightarrow Y$, and $\alpha : \Phi(x) = x[1] \to \eta$ in the category $\mathsf{El}(\mathfrak{T}\mathsf{G})$. By definition, $\alpha = \sum_\sigma a_\sigma[\sigma]$, where $a_\sigma : X^\sigma \to Y$, and

$$\alpha x[1] = \sum_\sigma a_\sigma x^\sigma[\sigma] = \eta\alpha + \partial\alpha = \sum_\sigma \big(\sum_\rho y_\rho a^\rho_{\rho^{-1}\sigma}\lambda_{\rho,\rho^{-1}\sigma} + \partial a_\sigma\big)[\sigma],$$

i.e.

$$a_\sigma x^\sigma = \sum_\rho y_\rho a^\rho_{\rho^{-1}\sigma}\lambda_{\rho,\rho^{-1}\sigma} + \partial a_\sigma \tag{2.5}$$

for every σ. Consider the morphism $f(\alpha) = \beta : X^\tau \to \tilde{Y} = \bigoplus_\sigma Y^\sigma$ such that its component $\beta_\sigma : X \to Y^\sigma$ equals $a^\sigma_{\sigma^{-1}}\lambda_{\sigma,\sigma^{-1}}$. Compute the σ-components of the products βx and $\tilde{\eta}\beta$, where $\tilde{\eta} = \Psi\eta$. They equal, respectively,

$$\beta_\sigma x^\tau = a^\sigma_{\sigma^{-1}}\lambda_{\sigma,\sigma^{-1}}x = a^\sigma_{\sigma^{-1}}(x^{\sigma^{-1}})^\sigma\lambda_{\sigma,\sigma^{-1}}$$

and

$$\sum_\rho y^\sigma_{\sigma^{-1}\rho}\lambda_{\sigma,\sigma^{-1}\rho}a^\rho_{\rho^{-1}}\lambda_{\rho,\rho^{-1}} = \sum_\rho y^\sigma_{\sigma^{-1}\rho}(a^{\sigma^{-1}\rho}_{\rho^{-1}})^\sigma\lambda_{\sigma,\sigma^{-1}\rho}\lambda_{\rho,\rho^{-1}} =$$
$$= \sum_\rho y^\sigma_{\sigma^{-1}\rho}(a^{\sigma^{-1}\rho}_{\rho^{-1}})^\sigma\lambda^\sigma_{\sigma^{-1}\rho,\rho^{-1}}\lambda_{\sigma,\sigma^{-1}}.$$

The relation (2.5), where σ is replaced by σ^{-1} and ρ by $\sigma^{-1}\rho$, these two expressions differ exactly by $\partial\beta_\sigma = \partial a^\sigma_{\sigma^{-1}}\lambda_{\sigma,\sigma^{-1}}$, hence $\beta = f(\alpha)$ is a morphism $x \to \tilde{\eta}$ in the category $\mathsf{El}(\mathfrak{T})$. Obviously, if $\alpha \neq \alpha'$, then $f(\alpha) \neq f(\alpha')$ as well. Moreover, one easily checks that the correspondence $\alpha \mapsto f(\alpha)$ is functorial in x and η, i.e. $f(\alpha)b = f(\alpha\Phi b)$ and $f(\gamma\alpha) = (\Psi\gamma)f(\alpha)$ for any morphisms $b : x' \to x$ and $\gamma : \eta \to \eta'$.

On the contrary, let $\beta : x \to \tilde{\eta}$ be a morphism in the category $\mathsf{El}(\mathfrak{T})$. Denote by $\beta_\sigma : X \to Y^\sigma$ the corresponding component of β and consider the morphism $\alpha = \sum_\sigma a_\sigma[\sigma] : X \to Y$ in the category $\mathcal{A}\mathsf{G}$, where $a_\sigma = \lambda^{-1}_{\sigma,\sigma^{-1}}\beta^\sigma_{\sigma^{-1}} : X^\sigma \to Y$. Comparing the σ-components in the equality $\beta x = \tilde{\eta}\beta$, we get

$$\beta_\sigma x = \sum_\rho y^\sigma_{\sigma^{-1}\rho}\lambda_{\sigma,\sigma^{-1}\rho}\beta_\rho + \partial\beta_\sigma. \tag{2.6}$$

The coefficients near $[\sigma]$ in the products $\alpha(\Phi x) = \alpha x[1]$ and $\eta\alpha$ equal, respectively,

$$a_\sigma x^\sigma = \lambda^{-1}_{\sigma,\sigma^{-1}} \beta^\sigma_{\sigma^{-1}} x^\sigma$$

and

$$\begin{aligned}\sum_\rho y_\rho a^\rho \lambda_{\rho^{-1}\sigma} &= \sum_\rho y_\rho (\lambda^{-1}_{\rho^{-1}\sigma,\sigma^{-1}\rho})^\rho \beta^{\rho^{-1}\sigma}_{\sigma^{-1}\rho} \lambda_{\rho,\rho^{-1}\sigma} = \\ &= \sum_\rho y_\rho (\lambda^{-1}_{\rho^{-1}\sigma,\sigma^{-1}\rho})^\rho \lambda_{\rho,\rho^{-1}\sigma} \beta^\sigma_{\sigma^{-1}\rho}.\end{aligned}$$

The relation (2.6), with σ replaced by σ^{-1}, implies that

$$\begin{aligned}a_\sigma x^\sigma - \partial a_\sigma &= \sum_\rho \lambda^{-1}_{\sigma,\sigma^{-1}} (y^{\sigma^{-1}}_{\sigma\rho})^\sigma \lambda^\sigma_{\sigma^{-1},\rho} \beta^\sigma_{\sigma^{-1}\rho} = \\ &= \sum_\rho y_{\sigma\rho} \lambda^{-1}_{\sigma,\sigma^{-1}} \lambda^\sigma_{\sigma^{-1},\rho} \beta^\sigma_{\sigma^{-1}\rho} = \sum_\rho y_\rho \lambda^{-1}_{\sigma,\sigma^{-1}} \lambda^\sigma_{\sigma^{-1},\sigma\rho} \beta^\sigma_\rho = \\ &= \sum_\rho y_\rho \lambda^{-1}_{\sigma,\sigma^{-1}\rho} \beta^\sigma_{\sigma^{-1}\rho} = \sum_\rho y_\rho (\lambda^{-1}_{\rho^{-1}\sigma,\sigma^{-1}\rho})^\rho \lambda_{\rho,\rho^{-1}\sigma} \beta^\sigma_{\sigma^{-1}\rho}.\end{aligned}$$

(Passing from the second row to the third, we used the relation (2.1) for the triple $\sigma, \sigma^{-1}, \rho$, while in the third row we used the same relation for the triple $\rho, \rho^{-1}\sigma, \sigma^{-1}\rho$.) Therefore, $\alpha x[1] = \eta\alpha + \partial\alpha$, thus α is a morphism $\Phi x \to \eta$. Moreover, the σ-component of $f(\alpha)$ equals

$$a^\sigma_{\sigma^{-1}} \lambda_{\sigma,\sigma^{-1}} = (\lambda^{-1}_{\sigma^{-1},\sigma})^\sigma (\beta^{\sigma^{-1}}_\sigma)^\sigma \lambda_{\sigma,\sigma^{-1}} = (\lambda^{-1}_{\sigma^{-1},\sigma})^\sigma \lambda_{\sigma,\sigma^{-1}} \beta_\sigma = \beta_\sigma.$$

Hence $f(\alpha) = \beta$ and the map $\alpha \mapsto f(\alpha)$ is bijective. □

3. Separable actions

We call the *center* $\mathcal{Z}(\mathfrak{T})$ of a bimodule triple $\mathfrak{T} = (\mathcal{A}, \mathcal{B}, \partial)$ the endomorphism ring of the identity bifunctor $\mathrm{id}_\mathfrak{T}$. In other words, the elements of this center are the sets of morphisms

$$\alpha = \{\, \alpha_X : X \to X \mid X \in \operatorname{Ob}\mathcal{A} \,\},$$

such that $\alpha_Y a = a\alpha_X$ for every morphism $a : X \to Y$, $\alpha_Y x = x\alpha_X$ for every element $x : X \dashrightarrow Y$ and $\partial\alpha_X = 0$ for all X. In particular, the element α_X belongs to the center of the algebra $\mathcal{A}(X,X)$. One easily sees that if $\alpha = \{\, \alpha_X \,\}$ and $\beta = \{\, \beta_X \,\}$ are two such sets, then the sets $\alpha+\beta = \{\, \alpha_X + \beta_X \,\}$ and $\alpha\beta = \{\, \alpha_X\beta_X \,\}$ also belong to $\mathcal{Z}(\mathfrak{T})$. Hence, this center is a ring (even a **K**-algebra), commutative, since $\alpha_X\beta_X = \beta_X\alpha_X$. If $F = (F_0, F_1)$ is an equivalence of bimodule triples $\mathfrak{T} \to \mathfrak{T}' = (\mathcal{A}', \mathcal{B}', \partial')$, it induces an isomorphism $F_{\mathcal{Z}} : \mathcal{Z}(\mathfrak{T}) \xrightarrow{\sim} \mathcal{Z}(\mathfrak{T}')$. Namely, for any $X' \in \operatorname{Ob}\mathcal{A}'$, choose an isomorphism $\lambda : X' \to F_0X$ for some $X \in \operatorname{Ob}\mathcal{A}$, and, for each element $\alpha = \{\, \alpha_X \,\} \in \mathcal{Z}(\mathfrak{T})$, set $(F_{\mathcal{Z}}\alpha)_{X'} = \lambda^{-1}(F_0\alpha_X)\lambda$. Let Y' be another object from $\mathcal{A}$, $\mu : Y' \xrightarrow{\sim} F_0Y$ and $(F_{\mathcal{Z}}\alpha)_{Y'} = \mu^{-1}(F_0\alpha_Y)\mu$. If $a' \in \mathcal{A}'(X', Y')$, the morphism $\mu a' \lambda^{-1} : F_0X \to F_0Y$ is of the form F_0a for some $a : X \to Y$. It gives

$$
\begin{aligned}
(F_{\mathcal{Z}}\alpha)_{Y'}a' &= \mu^{-1}(F_0\alpha_Y)\mu \cdot \mu^{-1}(F_0a)\lambda = \\
&= \mu^{-1}(F_0\alpha_Y)(F_0a)\lambda = \mu^{-1}(F_0(\alpha_Y a))\lambda = \\
&= \mu^{-1}F_0(a\alpha_X)\lambda = \mu^{-1}(F_0a)(F_0\alpha_X)\lambda = \\
&= a'\lambda^{-1}(F_0\alpha_X)\lambda = a'(F_{\mathcal{Z}}\alpha)_{X'}.
\end{aligned} \tag{3.1}
$$

Especially, if $Y' = X'$ and $a' = 1_{X'}$, we see that $F_{\mathcal{Z}}(\alpha)_{X'}$ does not depend on the choice of X and λ. Just in the same way one checks that $(F_{\mathcal{Z}}\alpha)_{Y'}x' = x'(F_{\mathcal{Z}}\alpha)_{X'}$ for every $x' \in \mathcal{B}'(X', Y')$. Note that an isomorphism λ can always be chosen such that $\partial\lambda = 0$: for instance, one can use the isomorphism of bifunctors $\phi : \mathrm{id}_{\mathfrak{T}'} \to FG$ for some bifunctor G and set $X = G_0X'$, $\lambda = \phi(X')$. Therefore $\partial'(F_{\mathcal{Z}}\alpha)_{X'} = 0$, so the set $F_{\mathcal{Z}}\alpha = \{\, (F_{\mathcal{Z}}\alpha)_{X'} \,\}$ belongs to $\mathcal{Z}(\mathfrak{T}')$. Obviously, $F_{\mathcal{Z}}(\alpha+\beta) = F_{\mathcal{Z}}\alpha + F_{\mathcal{Z}}\beta$ and $F_{\mathcal{Z}}(\alpha\beta) = (F_{\mathcal{Z}}\alpha)(F_{\mathcal{Z}}\beta)$, and if $F' : \mathfrak{T}' \to \mathfrak{T}''$ is another equivalence, then $(F'F)_{\mathcal{Z}} = F'_{\mathcal{Z}}F_{\mathcal{Z}}$. Moreover, similarly to the equalities (3.1), one easily verifies that if $F \simeq F'$, then $F_{\mathcal{Z}} = F'_{\mathcal{Z}}$. In particular, if $G : \mathfrak{T}' \to \mathfrak{T}$ is such a bifunctor that $FG \simeq \mathrm{id}_{\mathfrak{T}'}$ and $GF \simeq \mathrm{id}_{\mathfrak{T}}$, then $G_{\mathcal{Z}} = F_{\mathcal{Z}}^{-1}$, thus $F_{\mathcal{Z}}$ is an isomorphism.

These considerations imply that every action T of a group G on a triple $\mathfrak{T}$ induces an action of the same group on the center of this triple with the trivial system of factors: if λ is a system of factors for the action T, then $(\alpha^\sigma)_X = \lambda^{-1}_{\sigma,\sigma^{-1}}\alpha^\sigma_{X^{\sigma^{-1}}}\lambda_{\sigma,\sigma^{-1}}$ for every $\alpha \in \mathcal{Z}(\mathfrak{T})$. Especially, if the group G is finite, for any element α from $\mathcal{Z}(\mathfrak{T})$ its *trace* is defined as $\operatorname{tr}\alpha = \operatorname{tr}_{\mathsf{G}}\alpha = \sum_\sigma \alpha^\sigma$, i.e. $(\operatorname{tr}\alpha)_X = \sum_\sigma \lambda^{-1}_{\sigma,\sigma^{-1}}\alpha^\sigma_{X^{\sigma^{-1}}}\lambda_{\sigma,\sigma^{-1}}$. Obviously, the center of the triple $\mathfrak{T}\mathsf{G}$ is a subalgebra of the center of $\mathfrak{T}$.

Proposition 3.1. The center $\mathcal{Z}(\mathfrak{T}\mathsf{G})$ coincides with the subalgebra $\mathcal{Z}(\mathfrak{T})^{\mathsf{G}}$ of elements of the center $\mathcal{Z}(\mathfrak{T})$ that are invariant under the action of G. In particular, if this group is finite, the trace of each element $\alpha \in \mathcal{Z}(\mathfrak{T})$ belongs to $\mathcal{Z}(\mathfrak{T}\mathsf{G})$.

Proof. Let $\alpha = \{\, \alpha_X \,\}$ be an element of the center $\mathcal{Z}(\mathfrak{T})$. Since $\alpha_Y a[\sigma] = a\alpha_{X^\sigma}[\sigma]$ and $a[\sigma]\alpha_X = a\alpha^\sigma_X[\sigma]$ for each morphism $a : X^\sigma \to Y$, this element belongs to the center of the triple $\mathfrak{T}\mathsf{G}$ if and only if $\alpha_{X^\sigma} = \alpha^\sigma_X$

for every X and every σ. But then

$$(\alpha^\sigma)_X = \lambda^{-1}_{\sigma,\sigma^{-1}} \alpha^\sigma_{X^{\sigma^{-1}}} \lambda_{\sigma,\sigma^{-1}} = \lambda^{-1}_{\sigma,\sigma^{-1}} (\alpha_X^{\sigma^{-1}})^\sigma \lambda_{\sigma,\sigma^{-1}} = \alpha_X,$$

so α is invariant under the action of G. Just in the same way one verifies that every invariant element from $\mathcal{Z}(\mathfrak{T})$ belongs to $\mathcal{Z}(\mathfrak{T}\mathsf{G})$. The last statement follows from the fact that $\operatorname{tr}\alpha$ is always invariant under the action of the group. $\square$

Definition. We call an action of a finite group G on a bimodule triple $\mathfrak{T}$ *separable*, if there is an element of the center $\alpha \in \mathcal{Z}(\mathfrak{T})$ such that $\operatorname{tr}\alpha = 1$.

Certainly, it is enough $\operatorname{tr}\alpha$ to be invertible. For instance, if the order of the group G is invertible in the ring $\mathbf{K}$, any action of this group is separable. Another important case is when the center of the triple $\mathfrak{T}$ contains a subring $\mathbf{R}$ such that it is G-invariant, the group G acts *effectively* (i.e. for any $\sigma \neq 1$ there is $r \in \mathbf{R}$ such that $r^\sigma \neq r$) and $\mathbf{R}$ is a *separable extension* of its subring of invariants $\mathbf{R}^\mathsf{G}$ [4]. If $\mathbf{R}$ is a field and G acts effectively on $\mathbf{R}$, the last condition always holds. In general case it is necessary and sufficient that every element $\sigma \neq 1$ induce a non-identity automorphism of the residue field $\mathbf{R}/\mathfrak{m}$ for each maximal ideal $\mathfrak{m} \subset \mathbf{R}$ such that $\mathfrak{m}^\sigma = \mathfrak{m}$ [4, Theorem 1.3]. For an action of a group on a category (that is, on a principle triple) the notion of separability was introduced in [8]. Obviously, if an action of a group on a bimodule triple is separable, so is also its induced action on the corresponding bimodule category. We also note that if an action of a group G is separable, so is the action of every subgroup $\mathsf{H} \subseteq \mathsf{G}$: if $\operatorname{tr}_\mathsf{G}\alpha = 1$ and $\beta = \sum_{\sigma\in R}\alpha^\sigma$, where R is a set of representatives of right cosets $\mathsf{H}\backslash\mathsf{G}$, then $\operatorname{tr}_\mathsf{H}\beta = 1$.

Recall that a ring homomorphism $\mathbf{A} \to \mathbf{A}'$ is called *separable* if the natural homomorphism of $\mathbf{A}'$-bimodules $\mathbf{A}' \otimes_\mathbf{A} \mathbf{A}' \to \mathbf{A}'$ sending $a \otimes b$ to ab *splits*, i.e. there is an element $\sum_i b_i \otimes c_i$ in $\mathbf{A}' \otimes_\mathbf{A} \mathbf{A}'$ such that $\sum_i b_i c_i = 1$ and $\sum_i ab_i \otimes c_i = \sum_i b_i c_i a$ for all $a \in \mathbf{A}'$.

Lemma 3.2. An action of a finite group G on a triple $\mathfrak{T}$ is separable if and only if so is the ring homomorphism $\mathcal{Z} \to \mathcal{Z}\mathsf{G}$, where $\mathcal{Z} = \mathcal{Z}(\mathfrak{T})$.

Proof. Suppose that the action is separable, $\alpha = \{\,\alpha_X\,\}$ is such an element of the center that $\operatorname{tr}\alpha = 1$. Let $t = \sum_\sigma \alpha^\sigma[\sigma] \otimes [\sigma^{-1}] \in \mathcal{Z}\mathsf{G} \otimes_\mathcal{Z} \mathcal{Z}\mathsf{G}$. Then $\sum_\sigma \alpha^\sigma[\sigma][\sigma^{-1}] = \operatorname{tr}\alpha = 1$ and, for any $\beta \in \mathcal{Z},\ \tau \in \mathsf{G}$,

$$\begin{aligned} \beta[\tau]\cdot t &= \sum_\sigma \beta\alpha^{\tau\sigma}[\tau\sigma] \otimes [\sigma^{-1}] = \sum_\sigma \alpha^{\tau\sigma}\beta[\tau\sigma] \otimes [\sigma^{-1}] = \\ &= \sum_\sigma \alpha^\sigma \beta[\sigma] \otimes [\sigma^{-1}\tau] = \sum_\sigma \alpha^\sigma[\sigma] \otimes [\sigma^{-1}]\beta[\tau] = t\cdot\beta[\tau], \end{aligned}$$

so the homomorphism $\mathcal{Z} \to \mathcal{Z}\mathsf{G}$ is separable.

Now let the homomorphism $\mathcal{Z} \to \mathcal{Z}\mathsf{G}$ be separable. Note that every element from $\mathcal{Z}\mathsf{G} \otimes_{\mathcal{Z}} \mathcal{Z}\mathsf{G}$ is of the form $\sum_{\sigma,\tau} z_{\sigma,\tau}[\sigma] \otimes [\tau]$ for some $z_{\sigma,\tau} \in \mathcal{Z}$. Hence there are elements $z_{\sigma,\tau}$ such that $\sum_{\sigma,\tau} z_{\sigma,\tau}[\sigma\tau] = \sum_{\tau}\big(\sum_{\sigma} z_{\sigma,\sigma^{-1}\tau}\big)[\tau] = 1$, i.e. $\sum_{\sigma} z_{\sigma,\sigma^{-1}} = 1$, and $\sum_{\sigma} z_{\sigma,\sigma^{-1}\tau} = 0$ if $\tau \neq 1$, moreover, for every $\rho \in \mathsf{G}$ we have:

$$[\rho]\big(\sum_{\sigma,\tau} z_{\sigma,\tau}[\sigma] \otimes [\tau]\big) = \sum_{\sigma,\tau} z^{\rho}_{\sigma,\tau}[\rho\sigma] \otimes [\tau] = \sum_{\sigma,\tau} z^{\rho}_{\rho^{-1}\sigma,\tau}[\sigma] \otimes [\tau] =$$
$$= \big(\sum_{\sigma,\tau} z_{\sigma,\tau}[\sigma] \otimes [\tau]\big)[\rho] = \sum_{\sigma,\tau} z_{\sigma,\tau}[\sigma] \otimes [\tau\rho] = \sum_{\sigma,\tau} z_{\sigma,\tau\rho^{-1}}[\sigma] \otimes [\tau].$$

Thus $z^{\rho}_{\rho^{-1}\sigma,\tau} = z_{\sigma,\tau\rho^{-1}}$ for ρ, σ, τ. Especially, for $\sigma = \rho$, $\tau = 1$ we get $z_{\sigma,\sigma^{-1}} = z^{\sigma}_{1,1}$. Therefore, $\operatorname{tr} z_{1,1} = 1$ and the action is separable. □

Corollary 3.3. If an action of a group G on a triple $\mathfrak{T} = (\mathcal{A}, \mathcal{B}, \partial)$ is separable, so is also the embedding functor $\mathcal{A} \to \mathcal{A}\mathsf{G}$, i.e. the homomorphism of $\mathcal{A}\mathsf{G}$-bimodules $\phi : \mathcal{A}\mathsf{G} \otimes_{\mathcal{A}} \mathcal{A}\mathsf{G} \to \mathcal{A}\mathsf{G}$ splits, or, the same, for every object $X \in \operatorname{Ob}\mathcal{A}$ there is an element $t_X \in (\mathcal{A}\mathsf{G} \otimes_{\mathcal{A}} \mathcal{A}\mathsf{G})(X, X)$ such that $\phi(t_X) = 1_X$ and $at_X = t_Y a$ for each $a \in \mathcal{A}\mathsf{G}(X, Y)$. In particular, the action of a group G on a category $\mathcal{A}$ is separable if and only if so is the embedding functor $\mathcal{A} \to \mathcal{A}\mathsf{G}$.

Theorem 3.4. If an action of finite group G on a bimodule triple $\mathfrak{T} = (\mathcal{A}, \mathcal{B}, \partial)$ is separable, the functor $\Phi : \mathsf{El}(\mathfrak{T})\mathsf{G} \to \mathsf{El}(\mathfrak{T}\mathsf{G})$ induces an equivalence of the categories $\operatorname{add}\mathsf{El}(\mathfrak{T})\mathsf{G} \to \mathsf{El}(\mathfrak{T}\mathsf{G})$.

Proof. First we prove a lemma about fully additive categories.

Lemma 3.5. Let $\mathcal{C}$ be a fully additive category, $F : \mathcal{C} \to \mathcal{C}'$ be a fully faithful functor. F is an equivalence of categories if and only if every object $X' \in \mathcal{C}'$ is isomorphic to a direct summand of an object of the form FY, where $Y \in \operatorname{Ob}\mathcal{C}$.

Proof. The necessity of this condition is obvious, so we only have to prove the sufficiency. If X' is a direct summand of FY, there are morphisms $\iota' : X' \to FY$ and $\pi' : FY \to X'$ such that $\pi'\iota' = 1_{X'}$. Then $e' = \iota'\pi'$ is an idempotent endomorphism of the object FY. Since the functor F is fully faithful, $e' = Fe$ for an idempotent endomorphism $e : Y \to Y$. Since the category $\mathcal{C}$ is fully additive, there are an object X and morphisms $\iota : X \to Y$ and $\pi : Y \to X$ such that $e = \iota\pi$ and $\pi\iota = 1_X$. Then $(F\iota)(F\pi) = e'$ and $(F\pi)(F\iota) = 1_{FX}$. Let $u = \pi' F(\iota)$, $v = (F\pi)\iota'$; then we immediately get that $uv = 1_{X'}$ and $vu = 1_{FX}$, i.e. $X' \simeq FX$, the functor F is also dense, so it is an equivalence of categories. □

We prove now that every object ξ of the category $\mathsf{El}(\mathfrak{T}\mathsf{G})$ is isomorphic to a direct summand of $\Phi\Psi\xi$. Since Φ is fully faithful (Proposition 2.1), Theorem 3.4 follows then from Lemma 3.5. Let $\xi = \sum_\sigma x_\sigma[\sigma] \in \mathcal{B}\mathsf{G}(X,X)$, where $x_\sigma \in \mathcal{B}(X^\sigma, X)$. Then $\Psi\xi = \tilde{\xi} \in \mathcal{B}(\tilde{X},\tilde{X})$, where $\tilde{X} = \bigoplus_\sigma X^\sigma$ and $\tilde{\xi}_{\sigma,\tau} = x^\sigma_{\sigma^{-1}\tau}\lambda_{\sigma,\sigma^{-1}\tau}$, and $\Phi\Psi\xi = \tilde{\xi}[1]$. Choose an element $\alpha \in \mathcal{Z}(\mathfrak{T})$ such that $\operatorname{tr}\alpha = 1$. Consider the morphism $\pi : \tilde{X} \to X$ such that its σ-component equals $\pi_\sigma = \lambda^{-1}_{\sigma^{-1},\sigma}[\sigma^{-1}] : X^\sigma \to X$. Then the σ-component of the element $\xi\pi$ equals

$$\sum_\rho x_\rho(\lambda^\rho_{\sigma^{-1},\sigma})^{-1}\lambda_{\rho,\sigma^{-1}}[\rho\sigma^{-1}] = \sum_\rho x_\rho\lambda^{-1}_{\rho\sigma^{-1},\sigma}[\rho\sigma^{-1}]$$

(we use the relation (2.1) for the triple ρ,σ^{-1},σ), while the σ-component of the element $\pi\tilde{\xi}[1]$ equals

$$\sum_\rho \lambda^{-1}_{\rho^{-1},\rho}(x^\rho_{\rho^{-1}\sigma})^{\rho^{-1}}\lambda^{\rho^{-1}}_{\rho,\rho^{-1}\sigma}[\rho^{-1}] = \sum_\rho x_{\rho^{-1}\sigma}\lambda^{-1}_{\rho,\rho^{-1}}\lambda^{\rho^{-1}}_{\rho,\rho^{-1}\sigma}[\rho^{-1}] =$$
$$= \sum_\rho x_{\rho^{-1}\sigma}\lambda^{-1}_{\rho^{-1},\sigma}[\rho^{-1}] = \sum_\rho x_\rho\lambda^{-1}_{\rho\sigma^{-1},\sigma}[\rho\sigma^{-1}].$$

Here we used first the relation (2.1) for the triple $\rho^{-1},\rho,\rho^{-1}\sigma$ and then replaced ρ by $\sigma\rho^{-1}$. So $\xi\pi = \pi\tilde{\xi}[1]$ and, since $\partial\pi = 0$, π is a morphism $\tilde{\xi}[1] \to \xi$. Now consider the morphism $\iota : X \to \tilde{X}$ such that its σ-component equals $\alpha_{X^\sigma}[\sigma]$. The σ-component of the element $\iota\xi$ equals

$$\sum_\rho \alpha_{X^\sigma}x^\sigma_\rho\lambda_{\sigma,\rho}[\sigma\rho] = \sum_\rho \alpha_{X^\sigma}x^\sigma_{\sigma^{-1}\rho}\lambda_{\sigma,\sigma^{-1}\rho}[\rho],$$

and the σ-component of the element $\tilde{\xi}[1]\iota$ equals

$$\sum_\rho x^\sigma_{\sigma^{-1}\rho}\lambda_{\sigma,\sigma^{-1}\rho}\alpha_{X^\rho}[\rho] = \sum_\rho \alpha_{X^\sigma}x^\sigma_{\sigma^{-1}\rho}\lambda_{\sigma,\sigma^{-1}\rho}[\rho],$$

since $\alpha \in \mathcal{Z}(\mathfrak{T})$. Therefore, $\tilde{\xi}[1]\iota = \iota\xi$, thus ι is a morphism $\xi \to \tilde{\xi}[1]$. But $\pi\iota = \sum_\sigma \lambda^{-1}_{\sigma^{-1},\sigma}\alpha^{\sigma^{-1}}_{X^\sigma}\lambda_{\sigma^{-1},\sigma} = (\operatorname{tr}\alpha)_X = 1_X = 1_\xi$, which just means that the element ξ is a direct summand of the element $\tilde{\xi}[1]$. $\square$

One can get more information if the group G is finite abelian and the ring $\mathbf{K}$ is a field containing a *primitive n-th root of unit*, where $n = \#(\mathsf{G})$, i.e. such an element ζ that $\zeta^n = 1$ and $\zeta^k \neq 1$ for $0 < k < n$. Then certainly $\operatorname{char}\mathbf{K} \nmid n$, so any action of the group G on a bimodule triple $\mathfrak{T} = (\mathcal{A},\mathcal{B},\partial)$ is separable.

Let $\hat{\mathsf{G}}$ be the *group of characters* of the group G, i.e. the group of its homomorphisms to the multiplicative group $\mathbf{K}^\times$ of the field $\mathbf{K}$. This group acts on the triple $\mathfrak{T}\mathsf{G}$ (with the trivial system of factors) by the rules:

$$X^\chi = X \quad \text{for every} \quad X \in \operatorname{Ob}\mathcal{A},$$
$$\Big(\sum_\sigma x_\sigma[\sigma]\Big)^\chi = \sum_\sigma \chi(\sigma)x_\sigma[\sigma],$$

where $\chi \in \hat{\mathsf{G}}$ and $\sum_\sigma x_\sigma[\sigma]$ is a morphism from $\mathcal{A}\mathsf{G}$ or an element from $\mathcal{B}\mathsf{G}$. Recall that also $\#(\hat{\mathsf{G}}) = n$, so this action is separable as well. We denote by χ_0 the *unit character*, i.e. such that $\chi_0(\sigma) = 1$ for all $\sigma \in \mathsf{G}$. By definition, morphisms from $\mathcal{A}\mathsf{G}\hat{\mathsf{G}}$ and elements of $\mathcal{B}\mathsf{G}\hat{\mathsf{G}}$ are of the form $\sum_{\sigma,\chi} x_{\sigma,\chi}[\sigma][\chi]$. We write $[\chi]$ instead of $[1][\chi]$ and σ instead of $[\sigma][\chi_0]$. In particular an element $x[1][\chi_0]$ is denoted by x.

Theorem 3.6. The bimodule triples $\operatorname{add}\mathfrak{T}$ and $\operatorname{add}\mathfrak{T}\mathsf{G}\hat{\mathsf{G}}$ are equivalent.

Proof. Consider the elements $e_\sigma = \dfrac{1}{n}\sum_\chi \chi(\sigma)[\chi]$ from the endomorphism ring $\mathcal{A}\mathsf{G}\hat{\mathsf{G}}(X,X)$. The formulae of orthogonality for characters [7, Theorem 3.5] immediately imply that e_σ are mutually orthogonal idempotents and $\sum_\sigma e_\sigma = 1$. Moreover, $e_\sigma[\tau] = [\tau]e_{\sigma\tau}$, so all these idempotents are conjugate, thus define isomorphic direct summands X_σ of the object X in the category $\operatorname{add}\mathcal{A}\mathsf{G}\hat{\mathsf{G}}$, and $X = \bigoplus_\sigma X_\sigma$. We define the bifunctor $\Theta : \operatorname{add}\mathfrak{T} \to \operatorname{add}\mathfrak{T}\mathsf{G}\hat{\mathsf{G}}$ setting $\Theta X = X_1$ and $\Theta x = xe_1 = e_1x$, where x is a morphism $X \to Y$ or an element from $\mathcal{B}(X,Y)$. Obviously, the functor $\Theta_0 : \operatorname{add}\mathcal{A} \to \operatorname{add}\mathcal{A}\mathsf{G}\hat{\mathsf{G}}$ satisfies the conditions of Lemma 3.5, so it defines an equivalence of categories. Since every map $\Theta_1(X,Y)$ is also bijective, the bifunctor Θ is an equivalence by Lemma 1.1. □

Corollary 3.7. The categories $\mathsf{El}(\mathfrak{T})$ and $\operatorname{add}\mathsf{El}(\mathfrak{T})\mathsf{G}\hat{\mathsf{G}}$ are equivalent.

Proof. Indeed, $\operatorname{add}\mathsf{El}(\mathfrak{T})\mathsf{G}\hat{\mathsf{G}} \simeq \mathsf{El}(\mathfrak{T}\mathsf{G}\hat{\mathsf{G}})$ by Theorem 3.4. □

4. Radical and decomposition

In this section we suppose that the ring $\mathbf{K}$ is *noetherian, local and henselian* [3] (for instance, *complete*). We denote by $\mathfrak{m}$ its maximal ideal and by $\mathfrak{k} = \mathbf{K}/\mathfrak{m}$ its residue field. We call a $\mathbf{K}$-category $\mathcal{A}$ *piecewise finite* if all $\mathbf{K}$-modules $\mathcal{A}(X,Y)$ are finitely generated. Then its additive hull $\operatorname{add}\mathcal{A}$ is piecewise finite as well. Moreover, each endomorphism ring $A = \mathcal{A}(X,X)$ is *semiperfect*, i.e. possesses a unit decomposition $1 = \sum_{i=1}^n e_i$, where e_i are mutually orthogonal idempotents and all rings e_iAe_i are local. Hence the category $\operatorname{add}\mathcal{A}$ is *local*, i.e. every object in it decomposes into a finite direct sum of objects with local endomorphism rings. Therefore this category is a *Krull–Schmidt category*, i.e. every object X in it decomposes into a finite direct sum of indecomposables: $X = \bigoplus_{i=1}^{m} X_i$ and such a decomposition is unique, i.e. if also $X = \bigoplus_{i=1}^{n} X'_i$, where all X'_i are indecomposable, then $m = n$ and there is a permutation ε of the set $\{1, 2, \dots, m\}$ such that $X_i \simeq X'_{\varepsilon i}$ for all i [2, Theorem I.3.6]. Recall that the *radical* of a local category $\mathcal{A}$ is the ideal $\operatorname{rad}\mathcal{A}$ consisting of all such morphisms $a : X \to Y$ that all components of a with respect to some (then any) decompositions of X and Y into a direct sum of indecomposables are non-invertible. We denote $\overline{\mathcal{A}} = \mathcal{A}/\operatorname{rad}\mathcal{A}$. In particular, $\operatorname{rad}\mathcal{A}(X,X)$ is the radical of the ring $\mathcal{A}(X,X)$ and $\overline{\mathcal{A}}(X,X)$ is a semisimple artinian ring [9]. In the case of a piecewise finite category always $\operatorname{rad}\mathcal{A} \supseteq \mathfrak{m}\mathcal{A}$, in particular, $\overline{\mathcal{A}}(X,X)$ is a finite dimensional $\mathfrak{k}$-algebra. The category $\overline{\mathcal{A}}$ is *semisimple*, i.e. every object in it decomposes into a finite direct sum of indecomposables and $\overline{\mathcal{A}}(X,Y) = 0$ if X and Y are non-isomorphic indecomposables, while $\overline{\mathcal{A}}(X,X)$ is a skewfield for every indecomposable object X. (Note that an object X is indecomposable in the category $\mathcal{A}$ if and only if it is so in the category $\overline{\mathcal{A}}$). Moreover, $\operatorname{rad}\mathcal{A}$ is the biggest among the $\mathcal{I} \subset \mathcal{A}$ such that the factor-category $\mathcal{A}/\mathcal{I}$ is semisimple.

If a finite group G acts on a piecewise finite category $\mathcal{A}$ with a system of factors λ, the category $\mathcal{A}\mathsf{G}$ is piecewise finite as well. Moreover, the radical is a G-invariant ideal, i.e. $(\operatorname{rad}\mathcal{A})^\sigma = \operatorname{rad}\mathcal{A}$ for all $\sigma \in \mathsf{G}$, and the ideal $(\operatorname{rad}\mathcal{A})\mathsf{G}$ is contained in the radical of the category $\mathcal{A}\mathsf{G}$.

Proposition 4.1. If the action of a group G on a category $\mathcal{A}$ is separable, so is also its induced action on the category $\overline{\mathcal{A}}\mathsf{G}$. In this case $\operatorname{rad}(\mathcal{A}\mathsf{G}) = (\operatorname{rad}\mathcal{A})\mathsf{G}$ and the category $\overline{\mathcal{A}}\mathsf{G}$ is semisimple.

Proof is evident. □

From now on, we suppose that $\mathcal{A}$ is a piecewise finite local $\mathbf{K}$-category, $\mathcal{R} = \operatorname{rad}\mathcal{A}$, $X \in \operatorname{Ob}\mathcal{A}$ is an indecomposable object from $\mathcal{A}$, $\mathbf{A} = \mathcal{A}(X,X)$ and G is a finite group acting on $\mathcal{A}$ with a system of factors λ so that its action is separable. We are interested in the decomposition of the object X in the category $\mathcal{A}\mathsf{G}$ into a direct sum of indecomposables, especially, the number $\nu_{\mathsf{G}}(X)$ of non-isomorphic summands in such a decomposition. Recall that such decomposition comes from a decomposition of the ring $\mathcal{A}\mathsf{G}(X,X)$ or, equivalently, of the ring $\overline{\mathcal{A}}\mathsf{G}(X,X)$ into a direct sum of indecomposable modules.

Proposition 4.2. Let $\mathsf{H} = \{\sigma \in \mathsf{G} \mid X^\sigma \simeq X\}$. Then

$$\mathcal{A}\mathsf{G}(X,X)/\mathcal{R}\mathsf{G}(X,X) \simeq \mathcal{A}\mathsf{H}(X,X)/\mathcal{R}\mathsf{H}(X,X),$$

in particular, $\nu_{\mathsf{G}}(X) = \nu_{\mathsf{H}}(X)$.

Proof is evident, since $a_\sigma \in \mathcal{R}$ for every morphism $a_\sigma : X^\sigma \to X$ if $\sigma \notin \mathsf{H}$. $\square$

Corollary 4.3. If $X^\sigma \not\simeq X$ for all $\sigma \in \mathsf{G}$, the object X remains indecomposable in the category $\mathcal{A}\mathsf{G}$.

Therefore, dealing with the decomposition of X, we can only consider the action of the subgroup H. For every $\sigma \in \mathsf{H}$ we fix an isomorphism $\phi_\sigma : X^\sigma \to X$ and consider the action T' of the group H on the ring $\mathbf{A}$ given by the rule $T'_\sigma(a) = \phi_\sigma a^\sigma \phi_\sigma^{-1}$. One easily verifies that the elements $\lambda'_{\sigma,\tau} = \phi_\sigma \phi_\tau^\sigma \lambda_{\sigma,\tau} \phi_{\sigma\tau}^{-1}$ form a system of factors for this action, moreover, the map $a[\sigma] \mapsto a\phi_\sigma[\sigma]$ establishes an isomorphism $\mathbf{A}(\mathsf{H}, T', \lambda') \simeq \mathcal{A}\mathsf{H}(X,X)$. Thus, in what follows, we investigate the algebras $\mathbf{A}(\mathsf{H}, T', \lambda')$ and $\mathbf{D}(\mathsf{H}, T', \bar\lambda)$, where $\mathbf{D} = \mathbf{A}/\operatorname{rad}\mathbf{A}$ and $\bar\lambda_{\sigma,\tau}$ denotes the image of $\lambda'_{\sigma,\tau}$ in the skewfield $\mathbf{D}$. The latter factor-ring is finite dimensional skewfield (division algebra) over the field $\mathfrak{k}$. We denote by $\mathbf{F}$ the center if this algebra (it is a field). Let N be the subgroup of H consisting of all elements σ such that the automorphism T'_σ induces an inner automorphism of the skewfield $\mathbf{D}$, or, equivalently, the identity automorphism of the field $\mathbf{F}$ [7, Corollary IV.4.3]. It is a normal subgroup in H. For every element $\rho \in \mathsf{N}$ we choose an element $d_\rho \in \mathbf{D}$ such that $T'_\rho(a) = d_\rho a d_\rho^{-1}$ for all $a \in \mathbf{D}$. We also choose a set $\mathcal{S}$ of representatives of cosets H/N and, for every $\sigma \in \mathsf{H}$, denote by $\bar\sigma$ the element from $\mathcal{S}$ such that $\sigma\mathsf{N} = \bar\sigma\mathsf{N}$, and by $\rho(\sigma)$ the element from N such that $\sigma = \rho(\sigma)\bar\sigma$. Now we set $D_\sigma(a) = d_{\rho(\sigma)}^{-1} T'_\sigma(a) d_{\rho(\sigma)}$. An immediate verification shows that we get in this way an action of the group H on the skewfield $\mathbf{D}$ with the system of factors $\mu_{\sigma,\tau} = d_{\rho(\sigma)}^{-1}(d_{\rho(\tau)}^\sigma)^{-1}\bar\lambda_{\sigma,\tau} d_{\rho(\sigma\tau)}$ and, besides, the map $[\sigma] \mapsto d_{\rho(\sigma)}[\sigma]$ induces an isomorphism $\mathbf{D}(\mathsf{H}, T', \bar\lambda) \simeq \mathbf{D}(\mathsf{H}, D, \mu)$. Note that now

$$\mathsf{N} = \{\, \sigma \in \mathsf{H} \mid D_\sigma = \mathrm{id} \,\} = \{\, \sigma \in \mathsf{H} \mid D_\sigma|_{\mathbf{F}} = \mathrm{id} \,\}.$$

Moreover, one easily sees that $\mu_{\sigma,\tau} \in \mathbf{F}$ if $\sigma, \tau \in \mathsf{H}$.

Further on we denote $\mathbf{D}\mathsf{H} = \mathbf{D}(\mathsf{H}, D, \mu)$. The number of non-isomorphic indecomposable summands in the decomposition of $\mathbf{D}\mathsf{H}$ equals the number of simple components of this algebra [7, Theorem II.6.2], or, the same, the number of simple components of its center.

Proposition 4.4. The center of the algebra $\mathbf{D}\mathsf{H}$ coincides with the set

$$(\mathbf{F}\mathsf{N})^{\mathsf{H}} = \{\, \alpha \in \mathbf{F}\mathsf{H} \mid \forall\tau\, [\tau]\alpha = \alpha[\tau] \,\} =$$
$$= \Big\{ \sum_{\sigma\in\mathsf{N}} a_\sigma[\sigma] \,\Big|\, \forall\sigma\big(a_\sigma \in \mathbf{F} \,\&\, \forall\tau(\tau \in \mathsf{H} \Rightarrow a_\sigma^\tau \mu_{\tau,\sigma} = a_{\tau\sigma\tau^{-1}}\mu_{\tau\sigma\tau^{-1},\tau})\big) \Big\}.$$

Especially, if $\mathsf{N} = \{1\}$, then $\mathbf{D}\mathsf{H}$ is a central simple algebra over the field of invariants $\mathbf{F}^{\mathsf{H}}$, hence, $\nu_{\mathsf{G}}(X) = 1$.[1]

Proof. If an element $\alpha = \sum_\sigma a_\sigma[\sigma]$ belongs to the center of $\mathbf{D}\mathsf{H}$, then $\sum_\sigma ba_\sigma[\sigma] = \sum_\sigma a_\sigma[\sigma]b = \sum_\sigma a_\sigma b^\sigma[\sigma]$, so if $a_\sigma \neq 0$, then $b^\sigma = a_\sigma^{-1}ba_\sigma$, hence, $\sigma \in \mathsf{N}$, $b^\sigma = b$ and $a_\sigma \in \mathbf{F}$. Finally, the equalities $[\tau]\alpha = \sum_\sigma a_\sigma^\tau \mu_{\tau,\sigma}[\tau\sigma] = \alpha[\tau] = \sum_\sigma a_\sigma \mu_{\sigma,\tau}[\sigma\tau] = \sum_\sigma a_{\tau\sigma\tau^{-1}}\mu_{\tau\sigma\tau^{-1},\tau}[\tau\sigma]$ complete the proof. □

Corollary 4.5. If $\mathbf{F} = \mathfrak{k}$ (for instance, the residue field $\mathfrak{k}$ is algebraically closed) and the group H is abelian, the center of the algebra $\mathbf{D}\mathsf{H}$ coincides with $\mathfrak{k}\mathsf{H}_0$, where H_0 is the subgroup of H consisting of all elements σ such that $\mu_{\sigma,\tau} = \mu_{\tau,\sigma}$ for all $\tau \in \mathsf{H}$. In particular, $\nu_{\mathsf{G}}(X) = \#(\mathsf{H}_0)$.

Proof. In this case $\mathsf{N} = \mathsf{H}$, so the center of $\mathbf{D}\mathsf{H}$ coincides with $\mathfrak{k}\mathsf{H}_0$ (one easily checks that H_0 is indeed a subgroup). Since the latter algebra is commutative and semisimple, it is isomorphic to $\mathfrak{k}^m$, where $m = \#(\mathsf{H}_0)$, therefore, the number of its simple components equals m. □

Corollary 4.6. If $\mathbf{F} = \mathfrak{k}$ and the group H is cyclic, the center of the algebra $\mathbf{D}\mathsf{H}$ coincides with $\mathfrak{k}\mathsf{H}$ and $\nu_{\mathsf{G}}(X) = \#(\mathsf{H})$.

Proof. Actually, in this case it is well-known that $\mu_{\sigma,\tau} = \mu_{\tau,\sigma}$ for all $\sigma, \tau \in \mathsf{H}$. □

Note that all these corollaries hold if the group G itself is abelian or cyclic.

If $\mathbf{K}$-category $\mathcal{A}$ is piecewise finite, so is every bimodule category $\mathsf{El}(\mathfrak{T})$ as well, where $\mathfrak{T} = (\mathcal{A}, \mathcal{B}, \partial)$. If a group G acts separably on the triple $\mathfrak{T}$, it acts separably on the category $\mathsf{El}(\mathfrak{T})$ as well, and, according to Theorem 3.4, add $\mathsf{El}(\mathfrak{T})\mathsf{G} \simeq \mathsf{El}(\mathfrak{T}\mathsf{G})$, this equivalence being induced by the functor $\Phi : x \mapsto x[1]$. Therefore, all the results above can be applied to the study of the decomposition of an element $x[1]$ in the category $\mathsf{El}(\mathfrak{T}\mathsf{G})$. We only quote explicitly the reformulations of Corollaries 4.5 and 4.6 for this case.

Corollary 4.7. Let the residue field $\mathfrak{k}$ be algebraically closed and the group $\mathsf{H} = \{\sigma \mid x^\sigma \simeq x\}$ be abelian. Choose isomorphisms $\phi_\sigma : x^\sigma \to x$ for every element $\sigma \in \mathsf{H}$ and denote by $\mu_{\sigma,\tau}$ the image of a morphism $\phi_\sigma\phi_\tau\lambda_{\sigma,\tau}\phi_{\sigma\tau}^{-1}$ in $\mathfrak{k} \simeq \mathrm{Hom}_{\mathfrak{T}}(x,x)/\mathrm{rad}_{\mathfrak{T}}(x,x)$. Then the number of non-isomorphic indecomposable direct summands in the decomposition of the object $x[1]$ in the category $\mathsf{El}(\mathfrak{T}\mathsf{G})$ equals the order of the group $\mathsf{H}_0 = \{\sigma \mid \forall\tau\, \mu_{\sigma,\tau} = \mu_{\tau,\sigma}\}$. Especially, if the group H is cyclic, this number equals the order of H.

[1] The last statement is well-known, see [10, Theorem 4.50].

Remark 4.8. It is evident that all these statements also hold if separable is the action of the group H on the skewfield $\mathbf{D}$, or, equivalently, on its center $\mathbf{F}$. It is known [10, Section 4.18] that one only has to verify that separable is the action of the subgroup N, i.e. that $\operatorname{char}\mathfrak{k} \nmid \#(\mathsf{N})$, since the action of N on $\mathbf{F}$ is trivial.

Proposition 4.1 evidently implies some more corollaries concerning the structure of the radical of the category $\mathcal{A}\mathsf{G}$ (for instance, bimodule category $\mathsf{El}(\mathfrak{T}\mathsf{G})$).

Corollary 4.9. Let the action of the group G is separable. If a set of morphisms $\{a_i\}$ is a set of generators of the $\mathcal{A}$-module $(\operatorname{rad}\mathcal{A})(X, \_)$ (or $\mathcal{A}^{\mathrm{op}}$-module $(\operatorname{rad}\mathcal{A})(\_\,, X)$), its image $\{a_i[1]\}$ in $\mathcal{A}\mathsf{G}$ is a set of generators of the $\mathcal{A}\mathsf{G}$-module $(\operatorname{rad}\mathcal{A}\mathsf{G})(X, \_)$ (respectively, $\mathcal{A}^{\mathrm{op}}$-module $(\operatorname{rad}\mathcal{A}\mathsf{G})(\_\,, X)$).

We call a morphism $a : Y \to X$ *left almost split* (respectively, *right almost split*) if it generates the $\mathcal{A}$-module $(\operatorname{rad}\mathcal{A})(\_\,, X)$ (respectively, $\mathcal{A}^{\mathrm{op}}$-module $(\operatorname{rad}\mathcal{A})(Y, \_)$), and an equality $a = bf$ implies that the morphism f is left invertible, or, the same, is a split epimorphism (respectively, the equality $a = fb$ implies that g is right invertible, or, the same, is a split monomorphism).[2]

Corollary 4.10. Let the action of G is separable. If a morphism $a : Y \to X$ is left (right) almost split, so is $a[1]$ as well.

A sequence $X \xrightarrow{a} Y \xrightarrow{b} X'$ is called *almost split* if the morphism a is left almost split, the morphism b is right almost split and, besides, $a = \operatorname{Ker} b$ and $b = \operatorname{Cok} a$, i.e., for every object Z, the induced sequences of groups

$$
\begin{aligned}
&0 \to \mathcal{A}(Z, X) \to \mathcal{A}(Z, Y) \to \mathcal{A}(Z, X'),\\
&0 \to \mathcal{A}(X', Z) \to \mathcal{A}(Y, Z) \to \mathcal{A}(X, Z)
\end{aligned}
$$

are exact.

Corollary 4.11. Let the action of G is separable. If a sequence $X \xrightarrow{a} Y \xrightarrow{b} X'$ is almost split in the category $\mathcal{A}$, the sequence $X \xrightarrow{a[1]} Y \xrightarrow{b[1]} X'$ is almost split in the category $\mathcal{A}\mathsf{G}$.

[2] In the book [1] one only uses these notions in the case when X (respectively, Y) is indecomposable. However, one can easily see that a left (right) almost split morphism in our sense is just a direct sum of those in the sense of [1]. The same also concerns the notion of the *almost split sequences* used below.

Since, under the separability condition, every object from add $\mathcal{A}\mathsf{G}$ is a direct summand of an object that has come from the category $\mathcal{A}$, Corollaries 4.10 and 4.11 describe almost split morphisms and sequences in the category add $\mathcal{A}\mathsf{G}$ as soon as they are known in the category $\mathcal{A}$. In particular, these results can be applied to the bimodule categories $\mathsf{El}(\mathfrak{T}\mathsf{G})$ due to Theorem 3.4.

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Contact information

Yu. A. Drozd Institute of Mathematics, National Academy of Sciences of Ukraine, Tereschenkivska 3, 01601 Kiev, Ukraine
E-Mail: drozd@imath.kiev.ua

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