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Algebra in superextensions of groups, I: zeros and commutativity

T. Banakh, V. Gavrylkiv, O. Nykyforchyn

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Abstract. Given a group X we study the algebraic structure of its superextension $\lambda(X)$. This is a right-topological semigroup consisting of all maximal linked systems on X endowed with the operation

$$\mathcal{A}\circ\mathcal{B}=\{C\subset X:\{x\in X:x^{-1}C\in\mathcal{B}\}\in\mathcal{A}\}$$

that extends the group operation of X. We characterize right zeros of $\lambda(X)$ as invariant maximal linked systems on X and prove that $\lambda(X)$ has a right zero if and only if each element of X has odd order. On the other hand, the semigroup $\lambda(X)$ contains a left zero if and only if it contains a zero if and only if X has odd order $|X|\le 5$. The semigroup $\lambda(X)$ is commutative if and only if $|X|\le 4$. We finish the paper with a complete description of the algebraic structure of the semigroups $\lambda(X)$ for all groups X of cardinality $|X|\le 5$.

Introduction

After the topological proof of the Hindman theorem [H_1] given by Galvin and Glazer[1], topological methods become a standard tool in the modern combinatorics of numbers, see [HS], [P]. The crucial point is that any semigroup operation $*$ defined on a discrete space X can be extended to a right-topological semigroup operation on $\beta(X)$, the Stone-Čech compactification of X. The extension of the operation from X to $\beta(X)$ can

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[1]Unpublished, see [HS, p.102], [H_2]

be defined by the simple formula:

$$\mathcal{U} \circ \mathcal{V} = \{A \subset X : \{x \in X : x^{-1}A \in \mathcal{V}\} \in \mathcal{U}\}, \tag{1}$$

where $\mathcal{U}, \mathcal{V}$ are ultrafilters on X and $x^{-1}A = \{y \in X : xy \in A\}$. Endowed with the so-extended operation, the Stone-Čech compactification $\beta(X)$ becomes a compact right-topological semigroup. The algebraic properties of this semigroup (for example, the existence of idempotents or minimal left ideals) have important consequences in combinatorics of numbers, see [HS], [P].

The Stone-Čech compactification $\beta(X)$ of X is the subspace of the double power-set $\mathcal{P}(\mathcal{P}(X))$, which is a complete lattice with respect to the operations of union and intersection. In [G_2] it was observed that the semigroup operation extends not only to $\beta(X)$ but also to the complete sublattice $G(X)$ of $\mathcal{P}(\mathcal{P}(X))$ generated by $\beta(X)$. This complete sublattice consists of all inclusion hyperspaces over X.

By definition, a family $\mathcal{F}$ of non-empty subsets of a discrete space X is called an *inclusion hyperspace* if $\mathcal{F}$ is monotone in the sense that a subset $A \subset X$ belongs to $\mathcal{F}$ provided A contains some set $B \in \mathcal{F}$. On the set $G(X)$ there is an important transversality operation assigning to each inclusion hyperspace $\mathcal{F} \in G(X)$ the inclusion hyperspace

$$\mathcal{F}^{\perp} = \{A \subset X : \forall F \in \mathcal{F}\ (A \cap F \neq \emptyset)\}.$$

This operation is involutive in the sense that $(\mathcal{F}^{\perp})^{\perp} = \mathcal{F}$.

It is known that the family $G(X)$ of inclusion hyperspaces on X is closed in the double power-set $\mathcal{P}(\mathcal{P}(X)) = \{0,1\}^{\mathcal{P}(X)}$ endowed with the natural product topology. The induced topology on $G(X)$ can be described directly: it is generated by the sub-base consisting of the sets

$$U^{+} = \{\mathcal{F} \in G(X) : U \in \mathcal{F}\} \text{ and } U^{-} = \{\mathcal{F} \in G(X) : U \in \mathcal{F}^{\perp}\}$$

where U runs over subsets of X. Endowed with this topology, $G(X)$ becomes a Hausdorff supercompact space. The latter means that each cover of $G(X)$ by the sub-basic sets has a 2-element subcover.

The extension of a binary operation $*$ from X to $G(X)$ can be defined in the same way as for ultrafilters, i.e., by the formula (1) applied to any two inclusion hyperspaces $\mathcal{U}, \mathcal{V} \in G(X)$. In [$G_2$] it was shown that for an associative binary operation $*$ on X the space $G(X)$ endowed with the extended operation becomes a compact right-topological semigroup. The algebraic properties of this semigroups were studied in details in [G_2].

Besides the Stone-Čech compactification $\beta(X)$, the semigroup $G(X)$ contains many important spaces as closed subsemigroups. In particular, the space

$$\lambda(X) = \{\mathcal{F} \in G(X) : \mathcal{F} = \mathcal{F}^{\perp}\}$$

of maximal linked systems on X is a closed subsemigroup of $G(X)$. The space $\lambda(X)$ is well-known in General and Categorial Topology as the *superextension* of X, see [vM], [TZ]. Endowed with the extended binary operation, the superextension $\lambda(X)$ of a semigroup X is a supercompact right-topological semigroup containing $\beta(X)$ as a subsemigroup.

The space $\lambda(X)$ consists of maximal linked systems on X. We recall that a system of subsets $\mathcal{L}$ of X is *linked* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{L}$. An inclusion hyperspace $\mathcal{A} \in G(X)$ is linked if and only if $\mathcal{A} \subset \mathcal{A}^{\perp}$. The family of all linked inclusion hyperspace on X is denoted by $N_2(X)$. It is a closed subset in $G(X)$. Moreover, if X is a semigroup, then $N_2(X)$ is a closed subsemigroup of $G(X)$. The superextension $\lambda(X)$ consists of all maximal elements of $N_2(X)$, see [G_1], [G_2].

In this paper we start a systematic investigation of the algebraic structure of the semigroup $\lambda(X)$. This program will be continued in the forthcoming papers [BG_2] – [BG_4]. The interest to studying the semigroup $\lambda(X)$ was motivated by the fact that for each maximal linked system $\mathcal{L}$ on X and each partition $X = A \cup B$ of X into two sets A, B either A or B belongs to $\mathcal{L}$. This makes possible to apply maximal linked systems to Combinatorics and Ramsey Theory.

In this paper we concentrate on describing zeros and commutativity of the semigroup $\lambda(X)$. In Proposition 3.1 we shall show that a maximal linked system $\mathcal{L} \in \lambda(X)$ is a right zero of $\lambda(X)$ if and only if $\mathcal{L}$ is invariant in the sense that $xL \in \mathcal{L}$ for all $L \in \mathcal{L}$ and all $x \in X$. In Theorem 3.2 we shall prove that a group X admits an invariant maximal linked system (equivalently, $\lambda(X)$ contains a right zero) if and only if each element of X has odd order. The situation with (left) zeros is a bit different: a maximal linked system $\mathcal{L} \in \lambda(X)$ is a left zero in $\lambda(X)$ if and only if $\mathcal{L}$ is a zero in $\lambda(X)$ if and only if $\mathcal{L}$ is a unique invariant maximal linked system on X. The semigroup $\lambda(X)$ has a (left) zero if and only if X is a finite group of odd order $|X| \leq 5$ (equivalently, X is isomorphic to the cyclic group C_1, C_3 or C_5). The semigroup $\lambda(X)$ rarely is commutative: this holds if and only if the group X has finite order $|X| \leq 4$.

We start the paper studying self-linked subsets of groups. By definition, a subset A of a group X is called *self-linked* if $A \cap xA \neq \emptyset$ for all $x \in X$. In Proposition 1.1 we shall give lower and upper bounds for the smallest cardinality $sl(X)$ of a self-linked subset of X. We use those bounds to characterize groups X with $sl(X) \geq |X|/2$ in Theorem 1.2.

In Section 2 we apply self-linked sets to evaluating the cardinality of the (rectangular) semigroup $\overset{\leftrightarrow}{\lambda}(X)$ of maximal invariant linked systems on a group X. In Theorem 2.2 we show that for an infinite group X the cardinality of $\overset{\leftrightarrow}{\lambda}(X)$ equals $2^{2^{|X|}}$. In Proposition 2.3 and Theorem 2.6 we calculate the cardinality of $\overset{\leftrightarrow}{\lambda}(X)$ for all finite groups X of order $|X| \le 8$ and also detect groups X with $|\overset{\leftrightarrow}{\lambda}(X)| = 1$. In Sections 4 and 5 these results are applied for characterizing groups X whose superextensions have zeros or are commutative.

We finish the paper with a description of the algebraic structure of the superextensions of groups X of order $|X| \le 5$.

Now a couple of words about notations. Following the algebraic tradition, by C_n we denote the cyclic group of order n and by D_{2n} the dihedral group of cardinality $2n$, that is, the isometry group of the regular n-gon. For a group X by e we denote the neutral element of X. For a real number x we put

$$\lceil x \rceil = \min\{n \in \mathbb{Z} : n \ge x\} \text{ and } \lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \le x\}.$$

1. Self-linked sets in groups

In this section we study self-linked subsets in groups. By definition, a subset A of a group G is *self-linked* if $A \cap xA \ne \emptyset$ for each $x \in G$. In fact, this notion can be defined in the more general context of G-spaces.

By a *G-space* we understand a set X endowed with a left action $G \times X \to X$ of a group G. Each group G will be considered as a G-space endowed with the left action of G. An important example of a G-space is the homogeneous space $G/H = \{xH : x \in G\}$ of a group G by a subgroup $H \subset G$.

A subset $A \subset X$ of a G-space X defined to be *self-linked* if $A \cap gA \ne \emptyset$ for all $g \in G$. Let us observe that a subset $A \subset G$ of a group G is self-linked if and only if $AA^{-1} = G$.

For a G-space X by $sl(X)$ we denote the smallest cardinality $|A|$ of a self-linked subset $A \subset X$. Some lower and upper bounds for $sl(G)$ are established in the following proposition.

Proposition 1.1. *Let G be a finite group and H be a subgroup of G. Then*

1) $sl(G) \ge (1 + \sqrt{4|G| - 3})/2$;

2) $sl(G) \le sl(H) \cdot sl(G/H) \le sl(H) \cdot \lceil (|G/H| + 1)/2 \rceil$.

3) $sl(G) < |H| + |G/H|$.

Proof. 1. Take any self-linked set $A \subset G$ of cardinality $|A| = sl(G)$ and consider the surjective map $f : A \times A \to G$, $f : (x, y) \mapsto xy^{-1}$. Since $f(x, y) = xy^{-1} = e$ for all $(x, y) \in \Delta_A = \{(x, y) \in A^2 : x = y\}$, we get

$|G| = |G \setminus \{e\}| + 1 \le |A^2 \setminus \Delta_A| + 1 = sl(G)^2 - sl(G) + 1$, which just implies that $sl(G) \ge (1 + \sqrt{4|G| - 3})/2$.

2a. Let H be a subgroup of G. Take self-linked sets $A \subset H$ and $\mathcal{B} \subset G/H = \{xH : x \in G\}$ having sizes $|A| = sl(H)$ and $|\mathcal{B}| = sl(G/H)$. Fix any subset $B \subset G$ such that $|B| = |\mathcal{B}|$ and $\{xH : x \in B\} = \mathcal{B}$. We claim that the set $C = BA$ is self-linked. Given arbitrary $x \in G$ we should prove that the intersection $C \cap xC$ is not empty. Since $\mathcal{B}$ is self-linked, the intersection $\mathcal{B} \cap x\mathcal{B}$ contains the coset $bH = xb'H$ for some $b, b' \in B$. It follows that $b^{-1}xb' \in H = AA^{-1}$. The latter equality follows from the fact that the set $A \subset H$ is self-linked in H. Consequently, $b^{-1}xb' = a'a^{-1}$ for some $a, a' \in A$. Then $xC \ni xb'a = ba' \in C$ and thus $C \cap xC \ne \emptyset$. The self-linkedness of C implies the desired upper bound

$$sl(G) \le |C| \le |A| \cdot |B| = sl(H) \cdot sl(G/H).$$

2b. Next, we show that $sl(G/H) \le \lceil (|G/H| + 1)/2 \rceil$. Take any subset $A \subset G/H$ of size $|A| = \lceil (|G/H| + 1)/2 \rceil$ and note that $|A| > |G/H|/2$. Then for each $x \in G$ the shift xA has size $|xA| = |A| > |G/H|/2$. Since $|A| + |xA| > |G/H|$, the sets A and xA meet each other. Consequently, A is self-linked and $sl(G/H) \le |A| = \lceil (|G/H| + 1)/2 \rceil$.

3. Pick a subset $B \subset G$ of size $|B| = |G/H|$ such that $BH = G$ and observe that the set $A = H \cup B$ is self-linked and has size $|A| \le |H| + |B| - 1$ (because $B \cap H$ is a singleton). □

Theorem 1.2. *For a finite group G*

(i) $sl(G) = \lceil (|G|+1)/2 \rceil > |G|/2$ *if and only if G is isomorphic to one of the groups:* C_1, C_2, C_3, C_4, $C_2 \times C_2$, C_5, D_6, $(C_2)^3$;

(ii) $sl(G) = |G|/2$ *if and only if G is isomorphic to one of the groups:* C_6, C_8, $C_4 \times C_2$, D_8, Q_8.

Proof. I. First we establish the inequality $sl(G) < |G|/2$ for all groups G not isomorphic to the groups appearing in the items (i), (ii). Given such a group G we should find a self-linked subset $A \subset G$ with $|A| < |G|/2$.

We consider 8 cases.

1) G contains a subgroup H of order $|H| = 3$ and index $|G/H| = 3$. Then $sl(H) = 2$ and we can apply Proposition 1.1(2) to conclude that

$$sl(G) \le sl(H) \cdot sl(G/H) \le 2 \cdot 2 < 9/2 = |G|/2.$$

2) $|G| \notin \{9, 12, 15\}$ and G contains a subgroup H of order $n = |H| \ge 3$ and index $m = |G/H| \ge 3$. It this case $n + m - 1 < nm/2$ and $sl(G) \le |H| + |G/H| - 1 = n + m - 1 < nm/2$ by Proposition 1.1(3).

3) G is cyclic of order $n = |G| \geq 9$. Given a generator a of G, construct a sequence $(x_i)_{2 \leq i \leq n/2}$ letting $x_2 = a^0$, $x_3 = a$, $x_4 = a^3$, $x_5 = a^5$, and $x_i = x_{i-1}a^i$ for $5 < i \leq n/2$. Then the set $A = \{x_i : 2 \leq i \leq n/2\}$ has size $|A| < n/2$ and is self-linked.

4) G is cyclic of order $|G| = 7$. Given a generator a of G observe that $A = \{e, a, a^3\}$ is a 3-element self-linked subset and thus $sl(G) \leq 3 < |G|/2$.

5) G contains a cyclic subgroup $H \subset G$ of prime order $|H| \geq 7$. By the preceding two cases, $sl(H) < |H|/2$ and then $sl(G) \leq sl(H) \cdot sl(G/H) < \frac{|H|}{2} \cdot \frac{|G|}{|H|} = |G|/2$.

6) $|G| > 6$ and $|G| \notin \{8, 10, 12\}$. If $|G|$ is prime or $|G| = 15$, then G is cyclic of order $|G| \geq 7$ and thus has $sl(G) < |G|/2$ by the items (3), (4). If $|G| = 2p$ for some prime number p, then G contains a cyclic subgroup of order $p \geq 7$ and thus has $sl(G) < |G|/2$ by the item (5). If $|G| = 4n$ for some $n \geq 4$, then by Sylow's Theorem (see [OA, p.74]), G contains a subgroup $H \subset G$ of order $|H| = 4$ and index $|G/H| \geq 4$. Then $sl(G) < |G|/2$ by the item (2). If the above conditions do not hold, then $|G| = nm \neq 15$ for some odd numbers $n, m \geq 3$ and we can apply the items (1) and (2) to conclude that $sl(G) < |G|/2$.

7) If $|G| = 8$, then G is isomorphic to one of the groups: C_8, $C_2 \times C_4$, $(C_2)^3$, D_8, Q_8. All those groups appear in the items (i), (ii) and thus are excluded from our consideration.

8) If $|G| = 10$, then G is isomorphic to C_{10} or D_{10}. If G is isomorphic to C_{10}, then $sl(G) < |G|/2$ by the item (3). If G is isomorphic to D_{10}, then G contains an element a of order 5 and an element b of order 2 such that $bab^{-1} = a^{-1}$. Now it is easy to check that the 4-element set $A = \{e, a, b, ba^2\}$ is self-linked and hence $sl(G) \leq 4 < |G|/2$.

9) In this item we consider groups G with $|G| = 12$. It is well-known that there are five non-isomorphic groups of order 12: the cyclic group C_{12}, the direct sum of two cyclic groups $C_6 \oplus C_2$, the dihedral group D_{12}, the alternating group A_4, and the semidirect product $C_3 \rtimes C_4$ with presentation $\langle a, b | a^4 = b^3 = 1, aba^{-1} = b^{-1} \rangle$.

If G is isomorphic to C_{12}, $C_6 \oplus C_2$ or A_4, then G contains a normal 4-element subgroup H. By Sylow's Theorem, G contains also an element a of order 3. Taking into account that $a^2 \notin H$ and $Ha^{-1} = a^{-1}H$, we conclude that the 5-element set $A = \{a\} \cup H$ is self-linked and hence $sl(G) \leq 5 < |G|/2$.

If G is isomorphic to $C_3 \rtimes C_4$, then G contains a normal subgroup H of order 3 and an element $a \in G$ such that $a^2 \notin H$. Observe that the 5-element set $A = H \cup \{a, a^2\}$ is self-linked. Indeed, $AA^{-1} \supset H \cup aH \cup a^2H \cup Ha^{-1} = G$. Consequently, $sl(G) \leq 5 < |G|/2$.

Finally, consider the case of the dihedral group D_{12}. It contains an element a generating a cyclic subgroup of order 6 and an element b of order 2 such that $bab^{-1} = a^{-1}$. Consider the 5-element set $A = \{e, a, a^3, b, ba\}$ and note that $AA^{-1} = \{e, a, a^3, b, ba\} \cdot \{e, a^5, a^3, b, ba\} = G$. This yields the desired inequality $sl(G) \le 5 < 6 = |G|/2$.

Therefore we have completed the proof of the inequality $sl(G) < |G|/2$ for all groups not appearing in the items (i),(ii) of the theorem.

II. Now we shall prove the item (i).

The lower bound from Proposition 1.1(1) implies that $sl(G) = \lceil (|G|+1)/2 \rceil > |G|/2$ for all groups G with $|G| \le 5$.

It remains to check that $sl(G) > |G|/2$ if G is isomorphic to D_6 or C_2^3. First we consider the case $G = D_6$. In this case G contains a normal 3-element subgroup T. Assuming that $sl(G) \le |G|/2 = 3$, find a self-linked 3-element subset A. Without loss of generality we can assume that the neutral element e of G belongs to A (otherwise replace A by a suitable shift xA). Taking into account that $AA^{-1} = G$, we conclude that $A \not\subset T$ and thus we can find an element $a \in A \setminus T$. This element has order 2. Then

$$AA^{-1} = \{e, a, b\} \cdot \{e, a, b^{-1}\} = \{e, a, b, a, e, ba, b^{-1}, ba, e\} \ne G,$$

which is a contradiction.

Now assume that G is isomorphic to C_2^3. In this case G is the 3-dimensional linear space over the field C_2. Assuming that $sl(A) \le 4 = |G|/2$, find a 4-element self-linked subset $A \subset G$. Replacing A by a suitable shift, we can assume that A contains a neutral element e of G. Since $AA^{-1} = G$, the set A contains three linearly independent points a, b, c. Then

$$AA^{-1} = \{e, a, b, c\} \cdot \{e, a, b, c\} = \{e, a, b, c, ab, ac, bc\} \ne G,$$

which contradicts the choice of A.

III. Finally, we prove the equality $sl(G) = |G|/2$ for the groups appearing in the item (ii).

If $G = C_6$, then $sl(G) \ge 3$ by Proposition 1.1(1). On the other hand, we can check that for any generator a of G the 3-element subset $A = \{e, a, a^3\}$ is self-linked in G, which yields $sl(G) = 3 = |G|/2$.

If $|G| = 8$, then $sl(G) \ge 4$ by Proposition 1.1(1).

If G is cyclic of order 8 and a is a generator of G, then the set $A = \{e, a, a^3, a^4\}$ is self-linked and thus $sl(C_8) = 4$.

If G is isomorphic to $C_4 \oplus C_2$, then G has two commuting generators a, b such that $a^4 = b^2 = 1$. One can check that the set $A = \{e, a, a^2, b\}$ is self-linked and thus $sl(C_4 \oplus C_2) = 4$.

If G is isomorphic to the dihedral group D_8, then G has two generators a, b connected by the relations $a^4 = b^2 = 1$ and $bab^{-1} = a^{-1}$. One can check that the 4-element subset $A = \{e, a, b, ba^2\}$ is self-linked.

If G is isomorphic to the group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ of quaternion units, then we can check that the 4-element subset $A = \{-1, 1, i, j\}$ is self-linked and thus $sl(Q_8) = 4$. $\square$

In the following proposition we complete Theorem 1.2 calculating the values of the cardinal $sl(G)$ for all groups G of cardinality $|G| \le 13$.

Proposition 1.3. *The number $sl(G)$ for a group G of size $|G| \le 13$ can be found from the table:*

G	C_2	C_3	C_5	C_4	$C_2 \oplus C_2$	C_6	D_6	C_8	$C_2 \oplus C_4$	D_8	Q_8	C_2^3
$sl(G)$	2	2	3	3	3	3	4	4	4	4	4	5

G	C_7	C_{11}	C_{13}	C_9	$C_3 \oplus C_3$	C_{10}	D_{10}	C_{12}	$C_2 \oplus C_6$	D_{12}	A_4	$C_3 \rtimes C_4$
$sl(G)$	3	4	4	4	4	4	4	4	5	5	5	5

Proof. For groups G of order $|G| \le 10$ the value of $sl(G)$ is uniquely determined by the lower bound $sl(G) \ge \frac{1+\sqrt{4|G|-3}}{2}$ from Proposition 1.1(1) and the upper bound from Theorem 1.2. It remains to consider the groups G of order $11 \le |G| \le 13$.

1. If G is cyclic of order 11 or 13, then take a generator a of G and check that the 4-element set $A = \{e, a^4, a^5, a^7\}$ is self-linked, witnessing that $sl(C_{12}) = 4$.

2. If G is cyclic of order 12, then take a generator a for G and check that the 4-element subset $A = \{e, a, a^3, a^7\}$ is self-linked witnessing that $sl(G) = 4$.

It remains to consider all other groups of order 12. Theorem 1.2 gives us an upper bound $sl(G) \le 5$. So, we need to show that $sl(G) > 4$ for all non-cyclic groups G with $|G| = 12$.

3. If G is isomorphic to $C_6 \oplus C_2$ or A_4, then G contains a normal subgroup H isomorphic to $C_2 \oplus C_2$. Assuming that $sl(G) = 4$, we can find a 4-element self-linked subset $A \subset G$. Since $AA^{-1} = G$, we can find a suitable shift xA such that $xA \cap H$ contains the neutral element e of G and some other element a of H. Replacing A by xA, we can assume that $e, a \in A$. Since $A \not\subset H$, there is a point $b \in A \setminus H$. Since the quotient group G/H has order 3, $bH \cap Hb^{-1} = \emptyset$.

Concerning the forth element $c \in A \setminus \{e, a, b\}$ there are three possibilities: $c \in H$, $c \in b^{-1}H$, and $c \in bH$. If $c \in H$, then $bH = bH \cap AA^{-1} = b(A \cap H)^{-1}$ consists of 3 elements which is a contradiction. If $c \in b^{-1}H$, then $H = H \cap AA^{-1} = \{e, a\}$, which is absurd. So, $c \in bH$ and thus $c = bh$ for some $h \in H$. Since $h = h^{-1}$, we get $cb^{-1} = bhb^{-1} =$

$bh^{-1}b^{-1} = bc^{-1}$. Then $H = H \cap AA^{-1} = \{e, a, cb^{-1}, bc^{-1}\}$ has cardinality $|H| = |\{e, a, cb^{-1} = bc^{-1}\}| \le 3$, which is not true. This contradiction completes the proof of the inequality $sl(G) > 4$ for the groups $C_6 \oplus C_2$ and A_4.

4. Assume that G is isomorphic to the dihedral group D_{12}. Then G contains a normal cyclic subgroup H of order 6, and for each $b \in G \setminus H$ and $a \in H$ we get $b^2 = e$ and $bab^{-1} = a^{-1}$. Assuming that $sl(D_{12}) = 4$, we can find a 4-element self-linked subset $A \subset G$. Let a be a generator of the group H. Since $a \in AA^{-1} = G$, we can find two element $x, y \in A$ such that $a = xy^{-1}$. Then the shift Ay^{-1} contains e and a. Replacing A by Ay^{-1}, if necessary, we can assume that $e, a \in A$. Since $A \not\subset H$, there is an element $b \in A \setminus H$. Concerning the forth element $c \in A \setminus \{e, a, b\}$ there are two possibilities: $c \in H$ and $c \notin H$. If $c \in H$, then the set $A_H = A \cap H = \{e, a, c\}$ contains three elements and is equal to $bA_H^{-1}b^{-1}$, which implies $bA_H^{-1} = A_H b^{-1} = bA_H^{-1} \cup A_H b^{-1} = AA^{-1} \cap bH = bH$. This is a contradiction, because $|H| = 4 > 3 = |bA_H^{-1}|$. Then $c \in bH$ and hence $H = H \cap AA^{-1} = \{e, a, a^{-1}, bc^{-1}, cb^{-1}\}$ which is not true because $|H| = 6 > 5$.

5. Assume that G is isomorphic to the semidirect product $C_3 \rtimes C_4$ and hence has a presentation $\langle a, b | a^4 = b^3 = 1, aba^{-1} = b^{-1}\rangle$. Then the cyclic subgroup H generated by b is normal in G and the quotient G/H is cyclic of order 4. Assuming that $sl(G) = 4$, take any 4-element self-linked subset $A \subset G$.

After a suitable shift of A, we can assume that $e, b \in A$. Since $A \not\subset H$, there is an element $c \in A \setminus H$. We claim that the fourth element $d \in A \setminus \{e, b, c\}$ does not belong to $H \cup cH \cup c^{-1}H$. Otherwise, $AA^{-1} \subset H \cup cH \cup c^{-1}H \ne G$. This implies that one of the elements, say c belongs to the coset a^2H and the other to aH or $a^{-1}H$. We lose no generality assuming that $d \in aH$. Then $c = a^2b^i$, $d = ab^j$ for some $i, j \in \{-1, 0, 1\}$. It follows that

$$\begin{aligned} aH = aH \cap AA^{-1} &= \{d, db^{-1}, cd^{-1}\} = \\ &= \{ab^j, ab^{j-1}, a^2b^{i-j}a^{-1}\} = \{ab^j, ab^{j-1}, ab^{j-i}\} \end{aligned}$$

which implies that $i = -1$ and thus $c = a^2b^{-1}$. In this case we arrive to a contradiction looking at

$$a^2H \cap AA^{-1} = \{c, cb^{-1}, c^{-1}, bc^{-1}\} = \{a^2b^{-1}, a^2b^{-2}, ba^2, b^2a^2\} \not\ni a^2.$$

□

Problem 1.4. *What is the value of $sl(G)$ for other groups G of small cardinality? Is $sl(G) = \lceil(1 + \sqrt{4|G| - 3})/2\rceil$ for all finite cyclic groups G?*

2. Maximal invariant linked systems

In this section we study (maximal) invariant linked systems on groups. An inclusion hyperspace $\mathcal{A}$ on a group X is called *invariant* if $x\mathcal{A} = \mathcal{A}$ for all $x \in X$. The set of all invariant inclusion hyperspaces on X is denoted by $\overleftrightarrow{G}(X)$. By [G$_2$], $\overleftrightarrow{G}(X)$ is a closed rectangular subsemigroup of $G(X)$ coinciding with the minimal ideal of $G(X)$. The *rectangularity* of $\overleftrightarrow{G}(X)$ means that $\mathcal{A} \circ \mathcal{B} = \mathcal{B}$ for all $\mathcal{A}, \mathcal{B} \in \overleftrightarrow{G}(X)$.

Let $\overleftrightarrow{N}_2(X) = N_2(X) \cap \overleftrightarrow{G}(X)$ denote the set of all invariant linked systems on X and $\overleftrightarrow{\lambda}(X) = \max \overleftrightarrow{N}_2(X)$ be the family of all maximal elements of $\overleftrightarrow{N}_2(X)$. Elements of $\overleftrightarrow{\lambda}(X)$ are called *maximal invariant linked systems*. The reader should be conscious of the fact that maximal invariant linked systems need not be maximal linked!

Theorem 2.1. *For every group X the set $\overleftrightarrow{\lambda}(X)$ is a non-empty closed rectangular subsemigroup of $G(X)$.*

Proof. The rectangularity of $\overleftrightarrow{\lambda}(X)$ implies from the rectangularity of $\overleftrightarrow{G}(X)$ established in [G$_2$, §5] and the inclusion $\overleftrightarrow{\lambda}(X) \subset \overleftrightarrow{G}(X)$.

The Zorn Lemma implies that each invariant linked system on X (in particular, $\{X\}$) can be enlarged to a maximal invariant linked system on X. This observation implies the set $\overleftrightarrow{\lambda}(X)$ is not empty. Next, we show that the subsemigroup $\overleftrightarrow{\lambda}(X)$ is closed in $G(X)$. Since the set $\overleftrightarrow{N}_2(X) = N_2(X) \cap \overleftrightarrow{G}(X)$ is closed in $G(X)$, it suffices to show that $\overleftrightarrow{\lambda}(X)$ is closed in $\overleftrightarrow{N}_2(X)$. Take any invariant linked system $\mathcal{L} \in \overleftrightarrow{N}_2(X) \setminus \overleftrightarrow{\lambda}(X)$. Being not maximal invariant, the linked system $\mathcal{L}$ can be enlarged to a maximal invariant linked system $\mathcal{M}$ that contains a subset $B \in \mathcal{M} \setminus \mathcal{L}$. Since $\mathcal{M} \ni B$ is invariant, the system $\{xB : x \in X\} \subset \mathcal{M}$ is linked. Observe that $B \notin \mathcal{L}$ and $B \in \mathcal{M} \supset \mathcal{L}$ implies $X \setminus B \in \mathcal{L}^\perp$ and $B \in \mathcal{L}^\perp$. We claim that $O(\mathcal{L}) = B^- \cap (X \setminus B)^- \cap \overleftrightarrow{N}_2(X)$ is a neighborhood of $\mathcal{L}$ in $\overleftrightarrow{N}_2(X)$ that misses the set $\overleftrightarrow{\lambda}(X)$. Indeed, for any $\mathcal{A} \in O(\mathcal{L})$, we get that $\mathcal{A}$ is an invariant linked system such that $B \in \mathcal{A}^\perp$. Observe that for every $x \in X$ and $A \in \mathcal{A}$ we get $x^{-1}A \in \mathcal{A}$ by the invariantness of $\mathcal{A}$ and hence the set $B \cap x^{-1}A$ and its shift $xB \cap A$ both are not empty. This witnesses that $xB \in \mathcal{A}^\perp$ for every $x \in X$. Then the maximal invariant linked system generated by $\mathcal{A} \cup \{xB : x \in X\}$ is an invariant linked enlargement of $\mathcal{A}$, which shows that $\mathcal{A}$ is not maximal invariant linked. □

Next, we shall evaluate the cardinality of $\overleftrightarrow{\lambda}(X)$.

Theorem 2.2. *For any infinite group X the semigroup $\overset{\leftrightarrow}{\lambda}(X)$ has cardinality $|\overset{\leftrightarrow}{\lambda}(X)| = 2^{2^{|X|}}$.*

Proof. The upper bound $|\overset{\leftrightarrow}{\lambda}(X)| \le 2^{2^{|X|}}$ follows from the chain of inclusions:

$$\overset{\leftrightarrow}{\lambda}(X) \subset G(X) \subset \mathcal{P}(\mathcal{P}(X)).$$

Now we prove that $|\overset{\leftrightarrow}{\lambda}(X)| \ge 2^{2^{|X|}}$. Let $|X| = \kappa$ and $X = \{x_\alpha : \alpha < \kappa\}$ be an injective enumeration of X by ordinals $< \kappa$ such that x_0 is the neutral element of X. For every $\alpha < \kappa$ let $B_\alpha = \{x_\beta, x_\beta^{-1} : \beta < \alpha\}$. By transfinite induction, choose a transfinite sequence $(a_\alpha)_{\alpha<\kappa}$ such that $a_0 = x_0$ and

$$a_\alpha \notin B_\alpha^{-1} B_\alpha A_{<\alpha}$$

where $A_{<\alpha} = \{a_\beta : \beta < \alpha\}$.

Consider the set $A = \{a_\alpha : \alpha < \kappa\}$. By [HS, 3.58], the set $U_\kappa(A)$ of κ-uniform ultrafilters on A has cardinality $|U_\kappa(A)| = 2^{2^\kappa}$. We recall that an ultrafilter $\mathcal{U}$ is κ-uniform if for every set $U \in \mathcal{U}$ and any subset $K \subset U$ of size $|K| < \kappa$ the set $U \setminus K$ still belongs to $\mathcal{U}$.

To each κ-uniform ultrafilter $\mathcal{U} \in U_\kappa(A)$ assign the invariant filter $\mathcal{F}_\mathcal{U} = \bigcap_{x\in X} x\mathcal{U}$. This filter can be extended to a maximal invariant linked system $\mathcal{L}_\mathcal{U}$. We claim that $\mathcal{L}_\mathcal{U} \ne \mathcal{L}_\mathcal{V}$ for two different κ-uniform ultrafilters $\mathcal{U}, \mathcal{V}$ on A. Indeed, $\mathcal{U} \ne \mathcal{V}$ yields a subset $U \subset A$ such that $U \in \mathcal{U}$ and $U \notin \mathcal{V}$. Let $V = A \setminus U$. Since $\mathcal{U}$, $\mathcal{V}$ are κ-uniform, $|U| = |V| = \kappa$.

For every $\alpha < \kappa$ consider the sets $U_\alpha = \{a_\beta \in U : \beta > \alpha\} \in \mathcal{U}$ and $V_\alpha = \{a_\beta \in V : \beta > \alpha\} \in \mathcal{V}$.

It is clear that

$$F_U = \bigcup_{\alpha<\kappa} x_\alpha U_\alpha \in \mathcal{F}_\mathcal{U} \text{ and } F_V = \bigcup_{\alpha<\kappa} x_\alpha V_\alpha \in \mathcal{F}_\mathcal{V}.$$

Let us show that $F_U \cap F_V = \emptyset$. Otherwise there would exist two ordinals α, β and points $u \in U_\alpha$, $v \in V_\beta$ such that $x_\alpha u = x_\beta v$. It follows from $u \ne v$ that $\alpha \ne \beta$. Write the points u, v as $u = a_\gamma$ and $v = a_\delta$ for some $\gamma > \alpha$ and $\delta > \beta$. Then we have the equality $x_\alpha a_\gamma = x_\beta a_\delta$. The inequality $u \ne v$implies that $\gamma \ne \delta$. We lose no generality assuming that $\delta > \gamma$. Then

$$a_\delta = x_\beta^{-1} x_\alpha a_\gamma \in B_\delta^{-1} B_\delta A_{<\delta}$$

which contradicts the choice of a_δ.

Therefore, $F_U \cap F_V = \emptyset$. Taking into account that the linked systems $\mathcal{L}_\mathcal{U} \supset \mathcal{F}_\mathcal{U} \ni F_U$ and $\mathcal{L}_\mathcal{V} \supset \mathcal{F}_\mathcal{V} \ni F_V$ contain disjoint sets F_U, F_V, we conclude that $\mathcal{L}_\mathcal{U} \neq \mathcal{L}_\mathcal{V}$. Consequently,

$$|\overleftrightarrow{\lambda}(X)| \geq |\{\mathcal{L}_\mathcal{U} : \mathcal{U} \in \mathcal{U}_\kappa(A)\}| = |U_\kappa(A)| = 2^{2^\kappa}.$$

□

The preceding theorem implies that $|\overleftrightarrow{\lambda}(G)| = 2^\mathfrak{c}$ for any countable group G. Next, we evaluate the cardinality of $\overleftrightarrow{\lambda}(G)$ for finite groups G.

Given a finite group G consider the invariant linked system

$$\mathcal{L}_0 = \{A \subset X : 2|A| > |G|\}$$

and the subset

$$\uparrow\!\mathcal{L}_0 = \{\mathcal{A} \in \overleftrightarrow{\lambda}(G) : \mathcal{A} \supset \mathcal{L}_0\}$$

of $\overleftrightarrow{\lambda}(G)$.

Proposition 2.3. *Let G be a finite group. If $sl(G) \geq |G|/2$, then $\overleftrightarrow{\lambda}(G) = \uparrow\!\mathcal{L}_0$.*

Proof. We should prove that each maximal invariant linked system $\mathcal{A} \in \overleftrightarrow{\lambda}(G)$ contains $\mathcal{L}_0$. Take any set $L \in \mathcal{L}_0$. Taking into account that $sl(G) \geq |G|/2$ and each set $A \in \mathcal{A}$ is self-linked, we conclude that $|A| \geq |G|/2$ and hence A intersects each shift xL of L (because $|A|+|xL| > |G|$). Since the set L is self-linked, we get that the invariant linked system $\mathcal{A} \cup \{xL : x \in G\}$ is equal to $\mathcal{A}$ by the maximality of $\mathcal{A}$. Consequently, $L \in \mathcal{A}$ and hence $\mathcal{L}_0 \subset \mathcal{A}$. □

In light of Proposition 2.3 it is important to evaluate the cardinality of the set $\uparrow\!\mathcal{L}_0$. In $|G|$ is odd, then the invariant linked system $\mathcal{L}_0$ is maximal linked and thus $\uparrow\!\mathcal{L}_0$ is a singleton. The case of even $|G|$ is less trivial.

Given an group G of finite even order $|G|$, consider the family

$$\mathcal{S} = \{A \subset G : AA^{-1} = G,\ |A| = |G|/2\}$$

of self-linked subsets $A \subset G$ of cardinality $|A| = |G|/2$. On the family $\mathcal{S}$ consider the equivalence relation $\sim$ letting $A \sim B$ for $A, B \in \mathcal{S}$ if there is $x \in G$ such that $A = xB$ or $X \setminus A = xB$. Let $\mathcal{S}/_\sim$ the quotient set of $\mathcal{S}$ by this equivalence relation and $s = |\mathcal{S}/_\sim|$ stand for the cardinality of $S/_\sim$.

Proposition 2.4. $|\overleftrightarrow{\lambda}(G)| \geq |\uparrow\!\mathcal{L}_0| = 2^s$.

Proof. First we show that $\sim$ indeed is an equivalence relation on $\mathcal{S}$. So, assume that $\mathcal{S} \neq \emptyset$. Let us show that $G \setminus A \in \mathcal{S}$ for every $A \in \mathcal{S}$. Let $B = G \setminus A$. Assuming that $B \notin \mathcal{S}$, we conclude that $B \cap xB = \emptyset$ for some $x \in G$. Since $|B| = |A| = |G|/2$, we conclude that $xB = A$ and $G \setminus A = B = x^{-1}A$. The equality $A \cap x^{-1}A = \emptyset$ implies $x^{-1} \notin AA^{-1} = G$, which is a contradiction.

Taking into account that $A = eA$ for every $A \in \mathcal{S}$, we conclude that $\sim$ is a reflexive relation on $\mathcal{S}$. If $A \sim B$, then there is $x \in X$ such that $A = xB$ or $G \setminus A = xB$. This implies that $B = x^{-1}A$ or $X \setminus B = x^{-1}A$, that is $B \sim A$ and $\sim$ is symmetric. It remains to prove that the relation $\sim$ is transitive on $\mathcal{S}$. So let $A \sim B \sim C$. This means that there exist $x, y \in G$ such that $A = xB$ or $G \setminus A = xB$ and $B = yC$ or $G \setminus B = yC$. It is easy to check that in these cases $A = xyC$ or $X \setminus A = xyC$.

Choose a subset $\mathcal{T}$ of $\mathcal{S}$ intersecting each equivalence class of $\sim$ at a single point. Observe that $|\mathcal{T}| = |\mathcal{S}/_\sim| = s$. Now for every function $f : \mathcal{T} \to 2 = \{0, 1\}$ consider the maximal invariant linked system

$$\mathcal{L}_f = \mathcal{L}_0 \cup \{xT : x \in G,\ T \in f^{-1}(0)\} \cup \{x(G \setminus T) : x \in G,\ T \in f^{-1}(1)\}.$$

It can be shown that

$$|{\uparrow}\mathcal{L}_0| = |\{\mathcal{L}_f : f \in 2^{\mathcal{T}}\}| = 2^{|\mathcal{T}|} = 2^s.$$

$\square$

This proposition will help us to calculate the cardinality of the set $\overleftrightarrow{\lambda}(G)$ for all finite groups G of order $|G| \leq 8$:

Theorem 2.5. *The cardinality of $\overleftrightarrow{\lambda}(G)$ for a group G of size $|G| \leq 8$ can be found from the table:*

G	C_2	C_3	C_4	$C_2 \oplus C_2$	C_5	D_6	C_6	C_7	C_2^3	D_8	$C_4 \oplus C_2$	C_8	Q_8
$sl(G)$	2	2	3	3	3	4	3	3	5	4	4	4	4
$\overleftrightarrow{\lambda}(X)$	1	1	1	1	1	1	2	3	1	2	4	8	8

Proof. We divide the proof into 5 cases.

1. If $sl(G) > |G|/2$, then $\mathcal{L}_0$ is a unique maximal invariant linked system and thus $|\overleftrightarrow{\lambda}(X)| = 1$. By Theorem 1.2, $sl(G) > |G|/2$ if and only if $|G| \leq 5$ or G is isomorphic to D_6 or C_2^3.

2. If $sl(G) = |G|/2$, then $|\overleftrightarrow{\lambda}(G)| = 2^s$ where $s = |\mathcal{S}/_\sim|$. So it remains to calculate the number s for the groups C_6, D_8, $C_4 \oplus C_2$, C_8, and Q_8.

2a. If G is cyclic of order 6, then we can take any generator a on G and by routine calculations, check that

$$\mathcal{S} = \{xT, x(G \setminus T) : x \in G\}$$

where $T = \{e, a, a^3\}$. It follows that $s = |\mathcal{S}/_\sim| = 1$ and thus

$$|\overset{\leftrightarrow}{\lambda}(G)| = |{\uparrow}\mathcal{L}_0| = 2^s = 2.$$

2b. If G is cyclic of order 8, then we can take any generator a on G and by routine verification check that

$$\mathcal{S} = \{xA, G \setminus xA, xB, G \setminus xB, C, G \setminus xC : x \in G\}$$

where $A = \{e, a, a^2, a^4\}$, $B = \{e, a, a^2, a^5\}$, and $C = \{e, a, a^3, a^5\}$. It follows that $s = |\mathcal{S}/_\sim| = 3$ and thus

$$|\overset{\leftrightarrow}{\lambda}(G)| = |{\uparrow}\mathcal{L}_0| = 2^s = 8.$$

2c. Assume that the group G is isomorphic to $C_4 \oplus C_2$ and let $G_2 = \{x \in G : xx = e\}$ be the Boolean subgroup of G. We claim that a 4-element subset $A \subset G$ is self-linked if and only if $|A \cap G_2|$ is odd.

To prove the "if" part of this claim, assume that $|A \cap G_2| = 3$. We claim that A is self-linked. Let $A_2 = A \cap G_2$ and note that $G_2 = A_2 A_2^{-1} \subset AA^{-1}$ because $|A_2| = 3 > 2 = |G_2|/2$. Now take any element $a \in A \setminus G_2$ and note that $AA^{-1} \supset aA_2^{-1} \cup A_2 a^{-1}$. Observe that both $aA_2^{-1} = aA_2$ and $A_2 a^{-1} = a^{-1} A_2$ are 3-element subsets in the 4-element coset aG_2. Those 3-element sets are different. Indeed, assuming that $aA_2^{-1} = A_2 a^{-1}$ we would obtain that $a^2 A_2 = A_2$ which implies that $|A_2| = 3$ is even. Consequently, $aG_2 = aA_2^{-1} \cup A_2 a^{-1} \subset AA^{-1}$ and finally $G = AA^{-1}$.

If $|A \cap G_2| = 1$, then we can take any $a \in A \setminus G_2$ and consider the shift Aa^{-1} which has $|Aa^{-1} \cap G_2| = 3$. Then the preceding case implies that Aa^{-1} is self-linked and so is A.

To prove the "only if" part of the claim assume that $|A \cap G_2|$ is even. If $|A \cap G_2| = 4$, then $A = G_2$ and $AA^{-1} = G_2 G_2^{-1} = G_2 \neq G$. If $|A \cap G_2| = 0$, then $A = G_2 a$ for any $a \in A$ and hence $AA^{-1} = G_2 a a^{-1} G_2^{-1} = G_2 \neq G$. If $|A \cap G_2| = 2$, then $|G_2 \cap AA^{-1}| \le 3$ and again $AA^{-1} \neq G$.

Thus

$$\mathcal{S} = \{A \subset G : |A| = 4 \text{ and } |A \cap G_2| \text{ is odd}\}.$$

Each set $A \in \mathcal{S}$ has a unique shift aA with $aA \cap G_2 = \{e\}$. There are exactly four subsets $A \in \mathcal{S}$ with $A \cap G_2 = \{e\}$ forming two equivalence classes with respect to the relation $\sim$. Therefore $s = 2$ and

$$|\overset{\leftrightarrow}{\lambda}(G)| = |{\uparrow}\mathcal{L}_0| = 2^s = 4.$$

2d. Assume that G is isomorphic to the dihedral group D_8 of isometries of the square. Then G contains an element a of order 4 generating a normal cyclic subgroup H. The element a^2 commutes with all the elements of the group G.

We claim that for each self-linked 4-element subset $A \subset G$ we get $|A \cap H| = 2$. Indeed, if $|A \cap H|$ equals 0 or 4, then $A = Hb$ for some $b \in G$ and then $AA^{-1} = Abb^{-1}A^{-1} = H \neq G$. If $|A \cap H|$ equals 1 or 3, then replacing A by a suitable shift, we can assume that $A \cap H = \{e\}$ and hence $A = \{e\} \cup B$ for some 3-element subset $B \subset G \setminus H$. It follows that $G \setminus H = AA^{-1} \setminus H = (B \cup B^{-1}) = B \neq G \setminus H$. This contradiction shows that $|A \cap H| = 2$. Without loss of generality, we can assume that $A \cap H = \{e, a^2\}$ (if it is not the case, replace A by its shift Ax^{-1} where $x, y \in A$ are such that $yx^{-1} = a^2$). Now take any element $b \in A \setminus H$. Since G is not commutative, we get $ab = ba^3$. Observe that $ba^2 \notin A$ (otherwise $A = \{e, b, a^2, ba^2\}$ would be a subgroup of G with $AA^{-1} = A \neq G$). Consequently, the 4-th element $c \in A \setminus \{e, a^2, b\}$ of A should be of the form $c = ba$ or $c = ba^3 = ab$. Observe that both the sets $A_1 = \{e, a^2, b, ba\}$ and $A_2 = \{e, a^2, b, ab\}$ are self-linked. Observe also that

$$a^3(G \setminus A_1) = a^3 \cdot \{a, a^3, ba^2, ba^3\} = \{e, a^2, ab, b\} = A_2.$$

Consequently, $s = |\mathcal{S}/_\sim| = 1$ and $|\overset{\leftrightarrow}{\lambda}(G)| = 2^s = 2$.

2e. Finally assume that G is isomorphic to the group

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

of quaternion units. The two-element subset $H = \{-1, 1\}$ is a normal subgroup in X. Let $\mathcal{S}_\pm = \{A \in S : H \subset A\}$ and observe that each set $A \in \mathcal{S}$ has a left shift in $\mathcal{S}$. Take any set $A \in \mathcal{S}_\pm$ and pick a point $a \in A \setminus \{1, -1\}$. Observe that the 4-th element $b \in A \setminus \{1, -1, a\}$ of A is not equal to $-a$ (otherwise, A is a subgroup of G).

Conversely, one can easily check that each set $A = \{1, -1, a, b\}$ with $a, b \in G \setminus H$ and $a \neq -b$ is self-linked. This means that

$$\mathcal{S}_\pm = \{\{-1, 1, a, b\} : a \neq -b \text{ and } a, b \in G \setminus H\}$$

and thus $|\mathcal{S}_\pm| = C_6^2 - 3 = 12$. Observe that for each $A \in \mathcal{S}_2$ the set $-A \in \mathcal{S}_2$ and there axactly two shifts of $X \setminus A$ that belong to $\mathcal{S}_2$. This means that the equivalence class $[A]_\sim$ of any set $A \in \mathcal{S}$ intersects $\mathcal{S}_2$ in four sets. Consequently, $s = |\mathcal{S}/_\sim| = |\mathcal{S}_\pm|/4 = 12/4 = 3$ and

$$|\overset{\leftrightarrow}{\lambda}(G)| = |{\uparrow}\mathcal{L}_0| = 2^s = 8.$$

3. If $|G| = 7$, then $\mathcal{L}_0$ is one of three elements of $\overset{\leftrightarrow}{\lambda}(G)$. The other two elements can be found as follows. Consider the invariant linked system

$$\mathcal{L}_1 = \{A \subset G : |A| \geq 5\}$$

and observe that $\mathcal{L}_1 \subset \mathcal{A}$ for each $\mathcal{A} \in \overset{\leftrightarrow}{\lambda}(G)$. Indeed, assuming that some $A \in \mathcal{L}_1$ does not belong to $\mathcal{A}$, we would conclude that $B = G \setminus A \in \mathcal{A}$ by the maximality of $\mathcal{A}$. Since $|G \setminus B| \le 2$ we can find $x \in G \setminus BB^{-1}$. It follows that B, xB are two disjoint sets in $\mathcal{A}$ which is not possible. Thus $\mathcal{L}_1 \subset \mathcal{A}$.

Observe that $\mathcal{L}_1 \subset \mathcal{A} \subset \mathcal{L}_0 \cup \mathcal{L}_3$, where

$$\mathcal{L}_3 = \{A \subset G : |A| = 3,\ AA^{-1} = G\}.$$

Given a generator a of the cyclic group G, consider the 3-element set $T = \{a, a^2, a^4\}$ and note that $TT^{-1} = G$ and $T^{-1} \cap T = \emptyset$. By a routine calculation, one can check that

$$\mathcal{L}_3 = \{xT, xT^{-1} : x \in G\}.$$

Since T and T^{-1} are disjoint, the invariant linked system $\mathcal{A}$ cannot contain both the sets T and T^{-1}. If $\mathcal{A}$ contains none of the sets T, T^{-1}, then $\mathcal{A} = \mathcal{L}_0$. If $\mathcal{A}$ contains T, then

$$\mathcal{A} = (\mathcal{L}_0 \cup \{xT : x \in G\}) \setminus \{y(G \setminus T) : y \in G\}.$$

If $T^{-1} \in \mathcal{A}$, then

$$\mathcal{A} = (\mathcal{L}_0 \cup \{xT^{-1} : x \in G\}) \setminus \{y(G \setminus T^{-1}) : y \in G\}.$$

And those are the unique 3 maximal invariant systems in $\overset{\leftrightarrow}{\lambda}(G)$. $\square$

In the following theorem we characterize groups possessing a unique maximal invariant linked system.

Theorem 2.6. *For a finite group G the following conditions are equivalent:*

1) $|\overset{\leftrightarrow}{\lambda}(G)| = 1$;

2) $sl(G) > |G|/2$;

3) $|G| \le 5$ *or else G is isomorphic to* D_6 *or* C_2^3.

Proof. (2) $\Rightarrow$ (1). If $sl(G) > |G|/2$, then $\mathcal{L}_0 = \{A \subset G : |A| > |G|/2\}$ is a unique maximal invariant linked system on G (because invariant linked systems compose of self-linked sets).

(1) $\Rightarrow$ (2) Assume that $sl(G) \le |G|/2$ and take a self-linked subset $A \subset G$ with $|A| \le |G|/2$. If $|G|$ is odd, then $\mathcal{L}_0$ is maximal linked and

then any maximal invariant linked system $\mathcal{A}$ containing the self-linked set A is distinct from $\mathcal{L}_0$, witnessing that $|\overset{\leftrightarrow}{\lambda}(G)| > 1$.

If G is even, then we can enlarge A, if necessary, and assume that $|A| = |G|/2$. We claim that the complement $B = G \setminus A$ of A is self-linked too. Assuming the converse, we would find some $x \notin BB^{-1}$ and conclude that $B \cap xB = \emptyset$, which implies that $A = G \setminus B = xB$ and hence $x^{-1}A = B$. Then the sets A and $x^{-1}A$ are disjoint which contradicts $x^{-1} \in AA^{-1} = G$. Thus $BB^{-1} = G$ which implies that $\{xB : x \in G\}$ is an invariant linked system. Since $|G| = 2|A|$ is even, the unions $\mathcal{A} = \{xA : x \in G\} \cup \mathcal{L}_0$ and $\mathcal{B} = \{xB : x \in G\} \cup \mathcal{L}_0$ are invariant linked systems that can be enlarged to maximal linked systems $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$, respectively. Since the sets $A \in \mathcal{A} \subset \tilde{\mathcal{A}}$ and $B \in \mathcal{B} \subset \tilde{\mathcal{B}}$ are disjoint, $\tilde{\mathcal{A}} \neq \tilde{\mathcal{B}}$ are two distinct maximal invariant systems on G and thus $|\overset{\leftrightarrow}{\lambda}(G)| \geq 2$.

The equivalence (2) $\Leftrightarrow$ (3) follows from Theorem 1.2(i). $\square$

3. Right zeros in $\lambda(X)$

In this section we return to studying the superextensions of groups and shall detect groups X whose superextensions $\lambda(X)$ have right zeros. We shall show that for every group X the right zeros of $\lambda(X)$ coincide with invariant maximal linked systems.

We recall that an element z of a semigroup S is called a *right* (resp. *left*) *zero* in S if $xz = z$ (resp. $zx = z$) for every $x \in S$. This is equivalent to saying that the singleton $\{x\}$ is a left (resp. right) ideal of S.

By [G_2, 5.1] an inclusion hyperspace $\mathcal{A} \in G(X)$ is a right zero in $G(X)$ if and only if $\mathcal{A}$ is invariant. This implies that the minimal ideal of the semigroup $G(X)$ coincides with the set $\overset{\leftrightarrow}{G}(X)$ of invariant inclusion hyperspaces and is a compact rectangular topological semigroup. We recall that a semigroup S is called *rectangular* if $xy = y$ for all $x, y \in S$.

A similar characterization of right zeros holds also for the semigroup $\lambda(X)$.

Proposition 3.1. *A maximal linked system $\mathcal{L}$ is a right zero of the semigroup $\lambda(X)$ if and only if $\mathcal{L}$ is invariant.*

Proof. If $\mathcal{L}$ is invariant, then by proposition 5.1 of [G_2], $\mathcal{L}$ is a right zero in $G(X)$ and consequently, a right zero in $\lambda(X)$.

Assume conversely that $\mathcal{L}$ is a right zero in $\lambda(X)$. Then for every $x \in X$ we get $x\mathcal{L} = \mathcal{L}$, which means that $\mathcal{L}$ is invariant. $\square$

Unlike the semigroup $G(X)$ which always contains right zeros, the semigroup $\lambda(X)$ contains right zeros only for so-called odd groups. We define a group X to be *odd* if each element $x \in X$ has odd order. We recall that the *order* of an element x is the smallest integer number $n \geq 1$ such that x^n coincides with the neutral element e of X.

Theorem 3.2. *For a group X the following conditions are equivalent:*

1) *the semigroup $\lambda(X)$ has a right zero;*

2) *some maximal invariant linked system on X is maximal linked (which can be written as $\overset{\leftrightarrow}{\lambda}(X) \cap \lambda(X) \neq \emptyset$);*

3) *each maximal invariant linked system is maximal linked (which can be written as $\overset{\leftrightarrow}{\lambda}(X) \subset \lambda(X)$);*

4) *for any partition $X = A \cup B$ either $AA^{-1} = X$ or $BB^{-1} = X$;*

5) *each element of X has odd order.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 3.1.

(2) $\Rightarrow$ (4) Assume that $\lambda(X)$ contains an invariant maximal linked system $\mathcal{A}$. Given any partition $X = A_1 \cup A_2$, use the maximality of $\mathcal{A}$ to find $i \in \{1, 2\}$ with $A_i \in \mathcal{A}$. We claim that $A_i A_i^{-1} = X$. Indeed, for every $x \in X$ the invariantness of $\mathcal{A}$ implies that $xA_i \in \mathcal{A}$ and hence $A_i \cap xA_i \neq \emptyset$, which implies $x \in A_i A_i^{-1}$.

(4) $\Rightarrow$ (3) Assume that for every partition $X = A \cup B$ either $AA^{-1} = X$ or $BB^{-1} = X$. We need to check that each maximal invariant linked system $\mathcal{L}$ is maximal linked. In the other case, there would exist a set $A \in \mathcal{L}^\perp \setminus \mathcal{L}$. Since $\mathcal{L} \not\ni A$ is maximal invariant linked system, some shift xA of A does not intersect A and thus $x \notin AA^{-1}$. Then our assumption implies that $B = X \setminus A$ has property $BB^{-1} = X$, which means that the family $\{xB : x \in X\}$ is linked. We claim that $B \in \mathcal{L}^\perp$. Assuming the converse, we would find a set $L \in \mathcal{L}$ with $L \cap B = \emptyset$ and conclude that $A \in \mathcal{L}$ because $L \subset X \setminus B = A$. But this contradicts the choice of $A \in \mathcal{L}^\perp \setminus \mathcal{L}$. Therefore $B \in \mathcal{L}^\perp$ and

$$\mathcal{L} \cup \{L \subset X : \exists x \in X \ (xB \subset L)\}$$

is an invariant linked system that enlarges $\mathcal{L}$. Since $\mathcal{L}$ is a maximal invariant linked system, we conclude that $B \in \mathcal{L}$, which is not possible because B does not intersect $A \in \mathcal{L}^\perp$. The obtained contradiction shows that $\mathcal{L}^\perp \setminus \mathcal{L} = \emptyset$, which means that $\mathcal{L}$ belongs to $\lambda(X)$ and thus is an invariant maximal linked system.

The implication (3) $\Rightarrow$ (2) is trivial.

$\neg(5) \Rightarrow \neg(4)$ Assume that $X \setminus \{e\}$ contains a point a whose order is even or infinity. Then the cyclic subgroup $H = \{a^n : n \in \mathbb{Z}\}$ generated by a decomposes into two disjoint sets $H_1 = \{a^n : n \in 2\mathbb{Z}+1\}$ and $H_2 = \{a^n : n \in 2\mathbb{Z}\}$ such that $aH_1 = H_2$. Take a subset $S \subset X$ meeting each coset Hx, $x \in X$, in a single point. Consider the disjoint sets $A_1 = H_1S$ and $A_2 = H_2S$ and note that $aA_1 = A_2 = X \setminus A_1$ and $aA_2 = X \setminus A_2$, which implies that $a \notin A_iA_i^{-1}$ for $i \in \{1,2\}$. Since $A_1 \cup A_2 = X$, we get a negation of (4).

$(5) \Rightarrow (4)$ Assume that each element of X has odd order and assume that X admits a partition $X = A \sqcup B$ such that $a \notin AA^{-1}$ and $b \notin BB^{-1}$ for some $a, b \in X$. Then $aA \subset X \setminus A = B$ and $bB \subset X \setminus B = A$. Observe that

$$baA \subset bB \subset A$$

and by induction, $(ba)^iA \subset A$ for all $i > 0$. Since all elements of X have finite order, $(ba)^n = e$ for some $n \in \mathbb{N}$. Then $(ba)^{n-1}A \subset A$ implies

$$A = (ba)^nA \subset baA \subset bB \subset A$$

and hence $bB = A$. It follows from

$$X = bA \sqcup bB = bA \sqcup A = B \sqcup A$$

that $bA = B$. Thus $x \in A$ if and only if $bx \in B$.

Let $H = \{b^n : n \in \mathbb{Z}\} \subset X$ be the cyclic subgroup generated by b. By our assumption it is of odd order. On the other hand, the equality $bB = A = b^{-1}B$ implies that the intersections $H \cap A$ and $H \cap B$ have the same cardinality because $b(B \cap H) = A \cap H$. But this is not possible because of the odd cardinality of H. □

4. (Left) zeros of the semigroup $\lambda(X)$

An element z of a semigroup S is called a *zero* in S if $xz = z = zx$ for all $x \in S$. This is equivalent to saying that z is both a left and right zero in S.

Proposition 4.1. *Let X be a group. For a maximal linked system $\mathcal{L} \in \lambda(X)$ the following conditions are equivalent:*

1) $\mathcal{L}$ is a left zero in $\lambda(X)$;

2) $\mathcal{L}$ is a zero in $\lambda(X)$;

3) $\mathcal{L}$ is a unique invariant maximal linked system on X.

Proof. (1) $\Rightarrow$ (3) Assume that $\mathcal{Z}$ is a left zero in $\lambda(X)$. Then $\mathcal{Z}x = \mathcal{Z}$ for all $x \in X$ and thus

$$\mathcal{Z}^{-1} = \{Z^{-1} : Z \in \mathcal{Z}\}$$

is an invariant maximal linked system on X, which implies that the group X is odd according to Theorem 3.2. Note that for every right zero $\mathcal{A}$ of $\lambda(X)$ we get

$$\mathcal{Z} = \mathcal{Z} \circ \mathcal{A} = \mathcal{A}$$

which implies that $\mathcal{Z}$ is a unique right zero in $\lambda(X)$ and by Proposition 3.1 a unique invariant maximal linked system on X.

(3) $\Rightarrow$ (2) Assume that $\mathcal{Z}$ is a unique invariant maximal linked system on X. We claim that $\mathcal{Z}$ is a left zero of $\lambda(X)$. Indeed, for every $\mathcal{A} \in \mathcal{A}$ and $x \in X$ we get $x\mathcal{Z} \circ \mathcal{A} = \mathcal{Z} \circ \mathcal{A}$, which means that $\mathcal{Z} \circ \mathcal{A}$ is an invariant maximal linked system. By Proposition 3.1, $\mathcal{Z} \circ \mathcal{A}$ is a right zero and hence $\mathcal{Z} \circ \mathcal{A} = \mathcal{Z}$ because $\mathcal{Z}$ is a unique right zero. This means that $\mathcal{Z}$ is a left zero, and being a right zero, a zero in $\lambda(X)$.

(2) $\Rightarrow$ (1) is trivial. $\square$

Theorem 4.2. *The superextension $\lambda(X)$ of a group X has a zero if and only if X is isomorphic to C_1, C_3 or C_5.*

Proof. If X is a group of odd order $|X| \leq 5$, then $\overset{\leftrightarrow}{\lambda}(X) \subset \lambda(X)$ because X is odd and $|\overset{\leftrightarrow}{\lambda}(X)| = 1$ by Theorem 2.6. This means that $\lambda(X)$ contains a unique invariant maximal linked system, which is the zero of $\lambda(X)$ by Proposition 4.1.

Now assume conversely that the semigroup $\lambda(X)$ has a zero element $\mathcal{Z}$. By Proposition 3.1 and Theorem 3.2, X is odd and thus $\overset{\leftrightarrow}{\lambda}(X) \subset \lambda(X)$. Since the zero $\mathcal{Z}$ of $\lambda(X)$ is a unique invariant maximal linked system on X, we get $|\overset{\leftrightarrow}{\lambda}(X)| \leq 1$. By Theorem 2.6, X has order $|X| \leq 5$ or is isomorphic to D_3 or C_2^3. Since X is odd, X must be isomorphic to C_1, C_3 or C_5. $\square$

5. The commutativity of $\lambda(X)$

In this section we detect groups X with commutative superextension.

Theorem 5.1. *The superextension $\lambda(X)$ of a group X is commutative if and only if $|X| \leq 4$.*

Proof. The commutativity of the superextensions $\lambda(X)$ of groups X of order $|X| \le 4$ will be established in Section 6.

Now assume that a group X has commutative superextension $\lambda(X)$. Then X is commutative. We need to show that $|X| \le 4$. First we show that $|\overleftrightarrow{\lambda}(X)| = 1$.

Assume that $\overleftrightarrow{\lambda}(X)$ contains two distinct maximal invariant linked systems $\mathcal{A}$ and $\mathcal{B}$. Taking into account that $\mathcal{A}, \mathcal{B} \in \overleftrightarrow{\lambda}(X) \subset \overleftrightarrow{G}(X)$ and each element of $\overleftrightarrow{G}(X)$ is a right zero in $G(X)$ (see [G$_2$, 5.1]) we conclude that

$$\mathcal{A} \circ \mathcal{B} = \mathcal{B} \ne \mathcal{A} = \mathcal{B} \circ \mathcal{A}.$$

Extend the linked systems systems $\mathcal{A}, \mathcal{B}$ to maximal linked systems $\tilde{\mathcal{A}} \supset \mathcal{A}$ and $\tilde{\mathcal{B}} \supset \mathcal{B}$. Because of the commutativity of $\lambda(X)$, we get

$$\mathcal{A} = \mathcal{B} \circ \mathcal{A} \subset \tilde{\mathcal{B}} \circ \tilde{\mathcal{A}} = \tilde{\mathcal{A}} \circ \tilde{\mathcal{B}} \supset \mathcal{A} \circ \mathcal{B} = \mathcal{B}.$$

This implies that the union $\mathcal{A} \cup \mathcal{B} \ne \mathcal{A}$ is an invariant linked system extending $\mathcal{A}$, which is not possible because of the maximality of $\mathcal{A}$. This contradiction shows that $|\overleftrightarrow{\lambda}(X)| = 1$. Applying Theorem 2.6, we conclude that $|X| \le 5$ or X is isomorphic to C_2^3.

It remains to show that the semigroups $\lambda(C_5)$ and $\lambda(C_2^3)$ are not commutative. The non-commutativity of $\lambda(C_5)$ will be shown in Section 6.

To see that $\lambda(C_2^3)$ is not commutative, take any 3 generators a, b, c of C_2^3 and consider the sets $A = \{e, a, b, abc\}$, $H_1 = \{e, a, b, ab\}$, $H_2 = \{e, a, bc, abc\}$. Observe that H_1, H_2 are subgroups in C_2^3. For every $i \in \{1, 2\}$ consider the linked system $\mathcal{A}_i = \langle \{H_1, H_2\} \cup \{xA : x \in H_i\} \rangle$ and extend it to a maximal linked system $\tilde{\mathcal{A}}_i$ on C_2^3.

We claim that the maximal linked systems $\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_2$ do not commute. Indeed,

$$\tilde{\mathcal{A}}_2 \circ \tilde{\mathcal{A}}_1 \ni \bigcup_{x \in H_1} x * (x^{-1} bA) = bA = \{e, b, ba, ac\},$$

$$\tilde{\mathcal{A}}_1 \circ \tilde{\mathcal{A}}_2 \ni \bigcup_{x \in H_2} x * (x^{-1} bcA) = bcA = \{a, c, bc, abc\}.$$

It follows from $bA \cap bcA = \emptyset$ that $\tilde{\mathcal{A}}_1 \circ \tilde{\mathcal{A}}_2 \ne \tilde{\mathcal{A}}_2 \circ \tilde{\mathcal{A}}_1$. □

6. The superextensions of finite groups

In this section we shall describe the structure of the superextensions $\lambda(G)$ of finite groups G of small cardinality (more precisely, of cardinality $|G| \le 5$). It is known that the cardinality of $\lambda(G)$ growth very quickly as $|G|$ tends to infinity. The calculation of the cardinality of $|\lambda(G)|$ seems to be a difficult combinatorial problem related to the still unsolved Dedekind's problem of calculation of the number $M(n)$ of inclusion hyperpspaces on an n-element subset, see [De]. We were able to calculate the cardinalities of $\lambda(G)$ only for groups G of cardinality $|G| \leq 6$. The results of (computer) calculations are presented in the following table:

$\lvert G\rvert$	1	2	3	4	5	6
$\lvert\lambda(G)\rvert$	1	2	4	12	81	2646
$\lvert\lambda(G)/G\rvert$	1	1	2	3	17	447

Before describing the structure of superextensions of finite groups, let us make some remarks concerning the structure of a semigroup S containing a group G. In this case S can be thought as a G-space endowed with the left action of the group G. So we can consider the orbit space $S/G = \{Gs : s \in S\}$ and the projection $\pi : S \to S/G$. If G lies in the center of the semigroup S (which means that the elements of G commute with all the elements of S), then the orbit space S/G admits a unique semigroup operation turning S/G into a semigroup and the orbit projection $\pi : S \to S/G$ into a semigroup homomorphism. A subsemigroup $T \subset S$ will be called a *transversal semigroup* if the restriction $\pi : T \to S/G$ is an isomorphism of the semigroups. If S admits a transversal semigroup T, then it is a homomoprhic image of the product $G \times T$ under the semigroup homomorphism

$$h : G \times T \to S, \quad h : (g, t) \mapsto gt.$$

This helps to recover the algebraic structure of S from the structure of a transversal semigroup.

For a system $\mathcal{B}$ of subsets of a set X by

$$\langle \mathcal{B} \rangle = \{A \subset X : \exists B \in \mathcal{B} \ \ (B \subset A)\}$$

we denote the inclusion hyperspace generated by $\mathcal{B}$.

Now we shall analyse the entries of the above table. First note that each group G of size $|G| \leq 5$ is abelian and is isomorphic to one of the groups: C_1, C_2, C_3, C_4, $C_2 \oplus C_2$, C_5. It will be convenient to think of the cyclic group C_n as the multiplicative subgroups $\{z \in \mathbb{C} : z^n = 1\}$ of the complex plane.

6.1. The semigroups $\lambda(C_1)$ and $\lambda(C_2)$

For the groups C_n with $n \in \{1, 2\}$ the semigroup $\lambda(C_n)$ coincides with C_n while the orbit semigroup $\lambda(C_n)/C_n$ is trivial.

6.2. The semigroup $\lambda(C_3)$

For the group C_3 the semigroup $\lambda(C_3)$ contains the three principal ultrafilters $1, z, -z$ where $z = e^{2\pi i/3}$ and the maximal linked inclusion hyperspace $\triangleright = \langle\{1,z\},\{1,-z\},\{z,-z\}\rangle$ which is the zero in $\lambda(C_3)$. The superextension $\lambda(C_3)$ is isomorphic to the multiplicative semigroup $C_3^0 = \{z \in \mathbb{C} : z^4 = z\}$ of the complex plane. The latter semigroup has zero 0 and unit 1 which are the unique idempotents.

The transversal semigroup $\lambda(C_3)/C_3$ is isomorphic to the semilattice $2 = \{0,1\}$ endowed with the min-operation.

6.3. The semigroups $\lambda(C_4)$ and $\lambda(C_2 \oplus C_2)$

The semigroup $\lambda(C_4)$ contains 12 elements while the orbit semigroup $\lambda(C_4)/C_4$ contains 3 elements. The semigroup $\lambda(C_4)$ contains a transversal semigroup

$$\lambda_T(G) = \{1, \triangle, \square\}$$

where 1 is the neutral element of $C_4 = \{1, -1, i, -i\}$,

$$\begin{aligned}\triangle &= \langle\{1,i\},\{1,-i\},\{i,-i\}\rangle \text{ and}\\ \square &= \langle\{1,i\},\{1,-i\},\{1,-1\},\{i,-i,-1\}\rangle.\end{aligned}$$

The transversal semigroup is isomorphic to the extension $C_2^1 = C_2 \cup \{e\}$ of the cyclic group C_2 by an external unit $e \notin C_2$ (such that $ex = x = xe$ for all $x \in C_2^1$). The action of the group C_4 on $\lambda(C_4)$ is free so, $\lambda(C_4)$ is isomorphic to $\lambda_T(C_4) \oplus C_4$.

The semigroup $\lambda(C_2 \oplus C_2)$ has a similar algebraic structure. It contains a transversal semigroup

$$\lambda_T(C_2 \oplus C_2) = \{e, \triangle, \square\} \subset \lambda(C_2 \oplus C_2)$$

where e is the principal ultrafilter supported by the neutral element $(1,1)$ of $C_2 \oplus C_2$ and the maximal linked inclusion hyperspaces $\triangle$ and $\square$ are defined by analogy with the case of the group C_4:

$$\begin{aligned}\triangle &= \langle\{(1,1),(1,-1)\},\{(1,1),(-1,1)\},\{(1,-1),(-1,1)\}\rangle \text{ and}\\ \square &= \langle\{(1,1),(1,\text{-}1)\},\{(1,1),(\text{-}1,1)\},\{(1,1),(\text{-}1,\text{-}1)\},\{(1,\text{-}1),(\text{-}1,1),(\text{-}1,\text{-}1)\}\rangle.\end{aligned}$$

The transversal semigroup $\lambda_T(C_2 \oplus C_2)$ is isomorphic to C_2^1 and $\lambda(C_2 \oplus C_2)$ is isomorphic to $C_2^1 \oplus C_2 \oplus C_2$.

We summarize the obtained results on the algebraic structure of the semigroups $\lambda(C_4)$ and $\lambda(C_2 \oplus C_2)$ in the following proposition.

Proposition 6.1. *Let G be a group of cardinality $|G| = 4$.*

1. *The semigroup* $\lambda(G)$ *is isomorphic to* $C_2^1 \oplus G$ *and thus is commutative;*

2. $\lambda(G)$ *contains two idempotents;*

3. $\lambda(G)$ *has a unique proper ideal* $\lambda(G) \setminus G$ *isomorphic to the group* $C_2 \oplus G$.

6.4. The semigroup $\lambda(C_5)$.

Unlike to $\lambda(C_4)$, the semigroup $\lambda(C_5)$ has complicated algebraic structure. It contains 81 elements. One of them is zero

$$\mathcal{Z} = \{A \subset C_5 : |A| \geq 3\},$$

which is invariant under any bijection of C_5. All the other 80 elements have 5-element orbits under the action of C_5, which implies that the orbit semigroup $\lambda(C_5)/C_5$ consists of 17 elements. Let $\pi : \lambda(C_5) \to \lambda(C_5)/C_5$ denote the orbit projection.

It will be convenient to think of C_5 as the field $\{0, 1, 2, 3, 4\}$ with the multiplicative subgroup $C_5^* = \{1, -1, 2, -2\}$ of invertible elements (here -1 and -2 are identified with 4 and 3, respectively). Also for elements $x, y, z \in C_5$ we shall write xyz instead of $\{x, y, z\}$.

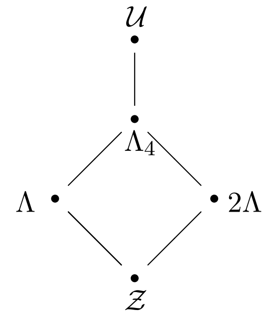
The semigroup $\lambda(C_5)$ contains 5 idempotents:

$$\begin{aligned}
\mathcal{U} =&\langle 0\rangle,\ \mathcal{Z},\\
\Lambda_4 =&\langle 01, 02, 03, 04, 1234\rangle,\\
\Lambda =&\langle 02, 03, 123, 014, 234\rangle,\\
2\Lambda =&\langle 04, 01, 124, 023, 143\rangle,
\end{aligned}$$

which commute and thus form an abelian subsemigroup $E(\lambda(C_5))$. Being a semilattice, $E(\lambda(C_5))$ carries a natural partial order: $e \leq f$ iff $e \circ f = e$. The partial order

$$\mathcal{Z} \leq \Lambda, 2\Lambda \leq \Lambda_4 \leq \mathcal{U}$$

on the set $E(\lambda(C_5))$ is designed at the picture:



The other distinguished subset of $\lambda(C_5)$ is

$$\begin{aligned}\sqrt{E(\lambda(C_5))} = &\{\mathcal{L} \in \lambda(C_5) : \mathcal{L} \circ \mathcal{L} \in E(\lambda(C_5))\} = \\ = &\{\mathcal{L} \in \lambda(C_5) : \mathcal{L} \circ \mathcal{L} \circ \mathcal{L} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{L}\}.\end{aligned}$$

We shall show that this set contains a point from each C_5-orbit in $\lambda(C_5)$.

First we show that this set has at most one-point intersection with each orbit. Indeed, if $\mathcal{L} \in \sqrt{E(\lambda(C_5))}$ and $\mathcal{L} \circ \mathcal{L} \neq \mathcal{Z}$, then for every $a \in C_5 \setminus \{0\}$, we get

$$\begin{aligned}(\mathcal{L} + a) \circ (\mathcal{L} + a) \circ (\mathcal{L} + a) \circ (\mathcal{L} + a) = \mathcal{L} \circ \mathcal{L} \circ \mathcal{L} \circ \mathcal{L} + 4a = \\ = \mathcal{L} \circ \mathcal{L} + 4a \neq \mathcal{L} \circ \mathcal{L} + 2a = (\mathcal{L} + a) \circ (\mathcal{L} + a).\end{aligned}$$

witnessing that $\mathcal{L} + a \notin \sqrt{\lambda_T(C_5)}$.

By a direct calculation one can check that the set $\lambda_T(C_5)$ contains the following four maximal linked systems:

$$\begin{aligned}\Delta = &\langle 02, 03, 23\rangle, \\ \Lambda_3 = &\langle 02, 03, 04, 234\rangle, \\ \Theta = &\langle 14, 012, 013, 123, 024, 034, 234\rangle, \\ \Gamma = &\langle 02, 04, 013, 124, 234\rangle.\end{aligned}$$

For those systems we get

$$\begin{aligned}&\Delta \circ \Delta = \Delta \circ \Delta \circ \Delta = \Lambda, \\ &\Lambda_3 \circ \Lambda_3 = \Lambda_3 \circ \Lambda_3 \circ \Lambda_3 = \Lambda, \\ &\mathcal{F} \circ \Theta = \mathcal{F} \circ \Gamma = \mathcal{Z} \ \text{ for every } \mathcal{F} \in \lambda(C_5) \setminus C_5.\end{aligned}$$

All the other elements of $\lambda(C_5)$ can be found as images of $\Delta, \Theta, \Gamma, \Lambda_3$ under the affine transformations of the field C_5. Those are maps of the form

$$f_{a,b} : x \mapsto ax + b \mod 5,$$

where $a \in \{1, -1, 2, -2\} = C_5^*$ and $b \in C_5$. The image of a maximal linked system $\mathcal{L} \in \lambda(C_5)$ under such a transformation will be denoted by $a\mathcal{L} + b$.

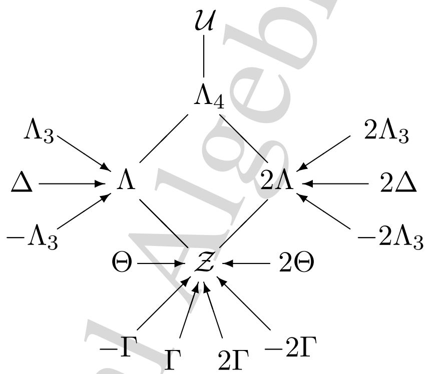
One can check that $a\Lambda_4 = \Lambda_4$ for each $a \in C_5^*$ while $\Lambda = -\Lambda$, and $\Theta = -\Theta$. Since the linear transformations of the form $f_{a,0} : C_5 \to C_5$, $a \in C_5^*$, are authomorphisms of the group C_5 the induced transformations $\lambda f_{a,0} : \lambda(C_5) \to \lambda(C_5)$ are authomorphisms of the semigroup $\lambda(C_5)$. This implies that those transformations do not move the subsets $E(\lambda(C_5))$ and $\sqrt{E(\lambda(C_5))}$. Consequently, the set $\sqrt{E(\lambda(C_5)}$ contains the maximal linked systems:

$$a\Delta, a\Theta, a\Lambda_3, a\Gamma,\ a \in \mathbb{Z}_5^*,$$

which together with the idempotents form a 17-element subset

$$T_{17} = E(\lambda(C_5)) \cup \big\{a\Delta, a\Theta : a \in \{1, 2\}\big\} \cup \{a\Lambda_3, a\Gamma : a \in \mathbb{Z}_5^*\}$$

that projects bijectively onto the orbit semigroup $\lambda(C_5)/C_5$. The set T_{17} looks as follows (we connect an element $x \in T_{17}$ with an idempotent $e \in T_{17}$ by an arrow if $x \circ x = e$):



The set $\sqrt{E(\lambda(C_5))}$ includes 24 elements more and coincides with the union $T_{17} \cup \sqrt{\mathcal{Z}}$ where

$$\sqrt{\mathcal{Z}} = \{a\Theta + b, a\Gamma + b : a \in \mathbb{Z}_5^*,\ b \in C_5\}.$$

Since each element of $\lambda(C_5)$ can be uniquely written as the sum $\mathcal{L} + b$ for some $\mathcal{L} \in T_{17}$ and $b \in C_5$, the multiplication table for the semigroup $\lambda(C_5)$ can be recovered from the Cayley table for multiplication of the elements from T_{17}:

$\circ$	Λ_4	Λ	Δ	Λ_3	$-\Lambda_3$	2Λ	2Δ	$2\Lambda_3$	$-2\Lambda_3$	$a\Theta$, $a\Gamma$
Λ_4	Λ_4	Λ	Λ	Λ	Λ	2Λ	2Λ	2Λ	2Λ	$\mathcal{Z}$
Λ	Λ	Λ	Λ	Λ	Λ	$\mathcal{Z}$	$\mathcal{Z}$	$\mathcal{Z}$	$\mathcal{Z}$	$\mathcal{Z}$
Δ	Δ	Λ	Λ	Λ	Λ	2Θ	2Θ	2Θ	2Θ	$\mathcal{Z}$
Λ_3	Λ_3	Λ	Λ	Λ	Λ	$2\Theta+2$	$2\Theta+2$	$2\Theta+2$	$2\Theta+2$	$\mathcal{Z}$
$-\Lambda_3$	$-\Lambda_3$	Λ	Λ	Λ	Λ	$2\Theta-2$	$2\Theta-2$	$2\Theta-2$	$2\Theta-2$	$\mathcal{Z}$
2Λ	2Λ	$\mathcal{Z}$	$\mathcal{Z}$	$\mathcal{Z}$	$\mathcal{Z}$	2Λ	2Λ	2Λ	2Λ	$\mathcal{Z}$
2Δ	2Δ	Θ	Θ	Θ	Θ	2Λ	2Λ	2Λ	2Λ	$\mathcal{Z}$
$2\Lambda_3$	$2\Lambda_3$	$\Theta-1$	$\Theta-1$	$\Theta-1$	$\Theta-1$	2Λ	2Λ	2Λ	2Λ	$\mathcal{Z}$
$-2\Lambda_3$	$-2\Lambda_3$	$\Theta+1$	$\Theta+1$	$\Theta+1$	$\Theta+1$	2Λ	2Λ	2Λ	2Λ	$\mathcal{Z}$
Θ	Θ	Θ	Θ	Θ	Θ	$\mathcal{Z}$	$\mathcal{Z}$	$\mathcal{Z}$	$\mathcal{Z}$	$\mathcal{Z}$
2Θ	2Θ	$\mathcal{Z}$	$\mathcal{Z}$	$\mathcal{Z}$	$\mathcal{Z}$	2Θ	2Θ	2Θ	2Θ	$\mathcal{Z}$
Γ	Γ	$\Theta+1$	$\Theta+1$	$\Theta+1$	$\Theta+1$	$2\Theta+2$	$2\Theta+2$	$2\Theta+2$	$2\Theta+2$	$\mathcal{Z}$
$-\Gamma$	$-\Gamma$	$\Theta-1$	$\Theta-1$	$\Theta-1$	$\Theta-1$	$2\Theta-2$	$2\Theta-2$	$2\Theta-2$	$2\Theta-2$	$\mathcal{Z}$
2Γ	2Γ	$\Theta-1$	$\Theta-1$	$\Theta-1$	$\Theta-1$	$2\Theta+2$	$2\Theta+2$	$2\Theta+2$	$2\Theta+2$	$\mathcal{Z}$
-2Γ	-2Γ	$\Theta+1$	$\Theta+1$	$\Theta+1$	$\Theta+1$	$2\Theta-2$	$2\Theta-2$	$2\Theta-2$	$2\Theta-2$	$\mathcal{Z}$

Looking at this table we can see that T_{17} is not a subsemigroup of $\lambda(C_5)$ and hence is not a transversal semigroup for $\lambda(C_5)$. This is not occasional.

Proposition 6.2. *The semigroup $\lambda(C_5)$ contains no transversal semigroup.*

Proof. Assume conversely that $\lambda(C_5)$ contains a subsemigroup T that projects bijectively onto the orbit semigroup $\lambda(C_5)/C_5$. Then T must include the set $E(\lambda(C_5))$ of idempotents and also the subset $\sqrt{E(\lambda(C_5))}\setminus\sqrt{\mathcal{Z}}$. Consequently,

$$T \supset \{\mathcal{U}, \mathcal{Z}, \Lambda, -\Lambda, \Delta, 2\Delta, \Lambda_3, -\Lambda_3, 2\Lambda_3, -2\Lambda_3\}.$$

Since $2\Lambda_3 \circ \Lambda = \Theta - 1 \neq \Theta = 2\Delta \circ \Lambda$, then there are two different points in the intersection $T \cap (\Theta + C_5)$ which should be a singleton. This contradiction completes the proof.

$\square$

Analysing the Cayley table for the set T_{17} we can establish the following properties of the semigroup $\lambda(C_5)$.

Proposition 6.3. *1. The maximal linked system $\mathcal{Z}$ is the zero of $\lambda(\mathbb{Z})$.*

2. $\lambda(C_5)$ contains 5 idempotents: $\mathcal{U}$, $\mathcal{Z}$, Λ_4, Λ, 2Λ, which commute.

3. The set of central elements of $\lambda(C_5)$ coincides with $C_5 \cup \{\mathcal{Z}\}$.

4. All non-trivial subgroups of $\lambda(C_5)$ are isomorphic to C_5.

6.5. Summary table

The obtained results on the superextensions of groups G with $|G| \leq 5$ are summed up in the following table in which $K(\lambda(G))$ stands for the minimal ideal of $\lambda(G)$.

$\|G\|$	$\|\lambda(G)\|$	$\lambda(G)$	$\|E(\lambda(G))\|$	$K(\lambda(G))$	maximal group
2	2	C_2	1	C_2	C_2
3	4	$C_3 \cup \{\triangleright\}$	2	$\{\triangleright\}$	C_3
4	12	$C_2^1 \times G$	2	$C_2 \times G$	$C_2 \times G$
5	81	$T_{17} \cdot C_5$	5	$\{\mathcal{Z}\}$	C_5

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Contact information

Taras Banakh
Ivan Franko National University of Lviv, Universytetska 1, 79000, Ukraine
E-Mail: tbanakh@yahoo.com
URL: www.franko.lviv.ua/faculty/mechmat/Departments/Topology/bancv.html

V. Gavrylkiv, O. Nykyforchyn
Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine
E-Mail: vgavrylkiv@yahoo.com

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