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RESEARCH ARTICLE

The free spectra of varieties generated by idempotent semigroups

Gabriella Pluhár and Japheth Wood

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Abstract. We give an exact formula for the logarithm of the free spectra of band varieties. We show that the the logarithm of the size of a free algebra in a band variety is $\frac{4}{(k-3)!}n^{k-3}\log n - \frac{4}{(k-3)!}n^{k-3}\sum_{j=1}^{k-3}\frac{1}{j} + O(n^{k-4}\log n)$, where k is a constant depending on the variety.

1. Introduction

For finite algebras there are strong connections between the structural properties of the algebra and the free spectra. If $\mathbf{G}$ is a finite group, then the size of the n-generated relatively free group in the variety generated by $\mathbf{G}$ is exponential in n if and only if $\mathbf{G}$ is nilpotent, and doubly-exponential if $\mathbf{G}$ is not nilpotent ([12] and [18]).

Let $\mathbf{A}$ be an m-element finite algebra and let $\mathcal{V}$ denote the variety generated by $\mathbf{A}$. The free spectrum of a variety $\mathcal{V}$ is the sequence of cardinalities $|\mathbf{F}_{\mathcal{V}}(n)|$, $n = 0, 1, 2, \ldots$. We can think of the free spectrum as the number of n-ary operations over $\mathbf{A}$. By the p_n sequence of the variety we mean the number of essentially n-ary operations over $\mathbf{A}$. It is known that the size of the free algebra in $\mathcal{V}$ freely generated by n elements ($|\mathbf{F}_{\mathcal{V}}(n)|$) is at most m^{m^n}. If $m \geq 2$, then $|\mathbf{F}_{\mathcal{V}}(n)| \geq n$. For example, the free spectrum of Boolean algebras is $|\mathbf{F}_{\mathcal{V}}(n)| = 2^{2^n}$. The first important question about free spectra is the following: Within the above bounds, what are the possible sequences?

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Another theorem on free spectra is Theorem 12.3 in [13]: If $\mathcal{V}$ is a nontrivial locally finite congruence distributive variety, then for every constant c such that $0 < c < 1$ and for every large n, the free spectrum of $\mathcal{V}$ is bounded below by $2^{2^{cn}}$. There are the so-called gap theorems for the free spectra as well. At the lower end, for example, there is Theorem 12.2 in [13]: Let $\mathcal{V}$ be a variety generated by a k-element algebra. Then either $|\mathbf{F}_{\mathcal{V}}(n)| \leq cn^k$ for some finite c, or else $|\mathbf{F}_{\mathcal{V}}(n)| \geq 2^{n-k}$ for all n.

For simple algebras there is a characterization of possible free spectra using universal algebraic techniques [2].

There are very few results on the free spectra of semigroup varieties. For a basic reference on the general properties of p_n sequences for semigroups see [5]. A full description of finite semigroups for which the p_n sequence is bounded by a polynomial is presented in [6]. A semigroup S is called surjective if $S^2 = S$. The aim of paper [5] is to prove that p_n sequences of finite surjective semigroups are eventually strictly increasing, except in a few well-known cases, when they are bounded (by a constant). Semigroups with bounded p_n sequences are described first in terms of identities, then structurally as nilpotent extensions of semilattices, Boolean groups and rectangular bands [4]. Kátai and Szabó in [16] continued the investigation of p_n sequences from another direction. They investigated the p_n sequences and free spectra of varieties generated by 0-simple semigroups. Additionally, they showed in [15] that the free spectrum of the variety generated by all combinatorial 0-simple semigroups is asymptotically $n^2 2^{n^2}$.

In the present paper, we investigate the free spectra of varieties of idempotent semigroups, the so-called bands. An estimate for the growth of the logarithm of the free spectra of band monoid varieties was given in [19]. We show that the the logarithm of the size of a free algebra in a band variety is $\dfrac{4}{(k-3)!}n^{k-3}\log n - \dfrac{4}{(k-3)!}n^{k-3}\sum_{j=1}^{k-3}\dfrac{1}{j} + O(n^{k-4}\log n)$, where k is an invariant of the variety.

2. Terms and bands

For an algebra $\mathbf{A}$, the set of n-ary term operations is denoted by $\mathrm{Clo}_n\mathbf{A}$. Let $t = t(x_1, \ldots, x_n)$ be an n-ary term. Then a term operation $t^{\mathbf{A}}$ is said to be *essentially n-ary* if it depends on all of its variables. That is, if for all $1 \leq i \leq n$ there exist $a_1, \ldots, a_{i-1}, a, b, a_{i+1}, \ldots, a_n \in A$ such that

$$t(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \neq t(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

For $n \geq 1$, denote the set of essentially n-ary term operations over $\mathbf{A}$ by $E_n(\mathbf{A})$, while $E_0(\mathbf{A})$ denotes the set of all constant unary term operations of $\mathbf{A}$. Now we define $p_n(\mathbf{A}) = |E_n(\mathbf{A})|$.

The relation between the free spectrum of a variety and the number of essentially n-ary term operations is given by the formula $|\mathbf{F}_{\mathcal{V}}(n)| = |\mathrm{Clo}_n \mathbf{A}| = \sum_{k=0}^{n} \binom{n}{k} p_k(\mathbf{A})$.

A band is an idempotent semigroup, that is a semigroup satisfying the identity $x^2 = x$. Bands play an important role in semigroup theory. For a general reference about bands consult [14]. Varieties of bands were characterized from several aspects. The lattice of varieties of bands was independently described in [3], [7] and [8]. The lattice contains ten infinite sequences of varieties and seven varieties at the bottom of the lattice (see Figure 1). The varieties on the right and left sides of the figure are the join-irreducible varieties, and each one is generated by a single subdirectly irreducible algebra, called $\mathbf{A}_i, \mathbf{A}_i^*, \mathbf{B}_i, \mathbf{B}_i^*$. For their description see [10]. They are presented as a set of idempotent functions of the set {1, 2, …, i}. Every variety properly containing 3 is a join of (at most) two join-irreducible varieties. Each variety of bands is defined by a single identity (see [7]). For traditional reasons, the varieties in the middle are denoted by $3, 3', 4, 4', \ldots$, and the other two sequences are split into smaller sequences, which we denote by $\mathcal{U}_3, \mathcal{U}_4, \ldots, \mathcal{U}_3', \mathcal{U}_4', \ldots, \mathcal{U}_3^*, \mathcal{U}_4^*, \ldots$ and $\mathcal{U}_3^{*\prime}, \mathcal{U}_4^{*\prime}, \ldots$.

3. Free spectra

The free spectra of the ten varieties at the bottom of the lattice are well-known.

Proposition 3.1. *The free spectra of the bottom ten band varieties are the following*

- $|F_{\mathcal{T}}(n)| = 1$,
- $|F_{\mathcal{V}(A_2)}(n)| = |F_{\mathcal{V}(A_2^*)}(n)| = n$,
- $|F_{\mathcal{S}}(n)| = 2^n - 1$,
- $|F_{\mathcal{LNB}}(n)| = |F_{\mathcal{RNB}}(n)| = n2^{n-1}$,
- $|F_{\mathcal{LRN}}(n)| = n^2$,
- $|F_{\mathcal{V}(B_2)}(n)| = |F_{\mathcal{V}(B_2^*)}(n)| = \sum \binom{n}{k} k!$,

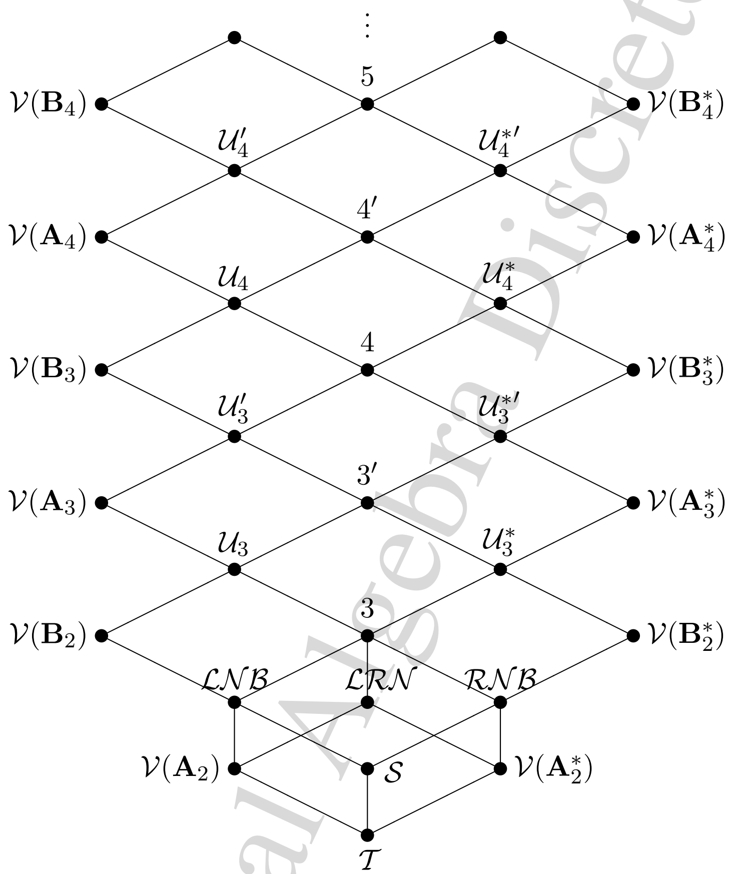

Figure 1: The Lattice of Band Varieties

- $|F_3(n)| = n(n+1)2^{n-2}$.

For band varieties above this level the p_n sequence of any equational class of bands, $\mathcal{V}$ can be calculated in the following way (see [9] Theorems 4.4 and 5.1).

Theorem 3.2. *Let $\mathcal{V}$ be a variety of bands containing 3. Then*

$$p_n(\mathcal{V}) = \sqrt{p_n(\overline{\mathcal{V}})p_n(\underline{\mathcal{V}})},$$

where $n \geq 0$, $\overline{\mathcal{V}}$ is the smallest variety of the form k or k' containing $\mathcal{V}$ and $\underline{\mathcal{V}}$ is the largest class of this kind contained in $\mathcal{V}$. Moreover, in case $k \geq 3$ and $n \geq 0$ we have

$$p_n(k') = \sqrt{p_n(k)}\sqrt{p_n(k+1)}.$$

Hence the following equations hold:

Corollary 3.3. *The p_n sequence of any band variety can be calculated with the $p_n(k)$ $(k \geq 3,\ k \in \mathbb{N})$ sequences.*

$p_n(\mathcal{V}(A_k)) = p_n(\mathcal{V}(A_k^*)) = \sqrt{p_n(k)}\sqrt{p_n(k+1)}$,

$p_n(\mathcal{V}(B_k)) = p_n(\mathcal{V}(B_k^*)) = \sqrt{p_n(k')}\sqrt{p_n((k+1)')} =$

$= \sqrt[4]{p_n(k)p_n^2(k+1)p_n(k+2)}$,

$p_n(\mathcal{U}_k) = p_n(\mathcal{U}_k^*) = \sqrt{p_n(k)}\sqrt{p_n(k')} = \sqrt[4]{p_n(k)^3 p_n(k+1)}$,

$p_n(\mathcal{U}_k') = p_n(\mathcal{U}_k^{*\prime}) = \sqrt{p_n(k')}\sqrt{p_n(k+1)} = \sqrt[4]{p_n(k)p_n(k+1)^3}$.

Therefore, it is enough to calculate the values of $p_n(k)$ and from this we can easily calculate the free spectra of all band varieties.

For the p_n sequence of the variety of (all) bands Gerhard introduced the notation $p_n(\infty)$. For $p_n(\infty)$ the following recurrence formula holds (see [11])

$$p_n(\infty) = n^2 p_{n-1}^2(\infty). \tag{1}$$

Moreover, $p_1(\infty) = 1$, hence $p_2(\infty) = 4$, $p_3(\infty) = 144$, etc. For $p_n(k)$ the formula looks more complicated (see [9], and for a corrected version see [19]).

$$p_n(k) = n^2 p_{k-2}^2(\infty) \prod_{j=k-1}^{n-1} j^2 p_j(k-1), \text{ for } n \geq k \geq 4. \tag{2}$$

The initial values are

$$p_n(3) = n^2 \quad \text{and} \quad p_n(k) = p_n(\infty) \text{ for } n < k.$$

Using (2) for k and $k-1$, after division we obtain that

$$\frac{p_n(k)}{p_{n-1}(k)} = n^2 p_{n-1}(k-1), \text{ for } n \geq k \geq 4.$$

We show that the formula holds for every n and k. Indeed, if $k > n$, we can substitute $p_n(k) = p_n(\infty)$ and $p_{n-1}(k) = p_{n-1}(\infty)$ for one occurrence of $p_{n-1}(\infty)$ and $p_{n-1}(k-1) = p_{n-1}(\infty)$ for the other occurrence of $p_{n-1}(\infty)$ in (1). We obtain that

$$p_n(k) = n^2 p_{n-1}(k-1) p_{n-1}(k), \text{ for } k > n, k \geq 4,$$

as well. Hence

$$p_n(k) = n^2 p_{n-1}(k) p_{n-1}(k-1) \tag{3}$$

holds in general for $n \geq 2$ and $k \geq 4$. Let $p(n,k) = \log p_n(k)$. Taking the logarithm of both sides of (3) we get

$$p(n,k) = 2\log n + p(n-1,k) + p(n-1,k-1) \tag{4}$$

for $n \geq 2$, $k \geq 4$ and

$$p(n,3) = 2 \log n \quad \text{and} \quad p(1,k) = 0. \tag{5}$$

The explicit form of the solution of recurrence formula (4) with initial values in (5) is

$$p(n,k) = 4 \sum_{m=1}^{n-1} \sum_{t=0}^{k-4} \binom{n-m-1}{t} \log m + 2\log n. \tag{6}$$

Theorem 3.4. *Let $k \geq 4$. For the p_n sequence of the band variety k*

$$\log p_n(k) = \frac{4}{(k-3)!} n^{k-3} \log n - \frac{4}{(k-3)!} n^{k-3} \sum_{j=1}^{k-3} \frac{1}{j} + O(n^{k-4} \log n)$$

holds.

Proof. We keep the leading terms of the second summations in (6). Rewrite (6) as:

$$p(n,k) = 4\sum_{m=1}^{n-1}\binom{n-m-1}{k-4}\log m + \\ + \left(4\sum_{m=1}^{n-1}\sum_{t=0}^{k-5}\binom{n-m-1}{t}\log m + 2\log n\right).$$

Write $\log n$ for $\log m$ in the error term, and use

$$\sum_{u=t}^{v}\binom{u}{t} = \binom{v+1}{t+1}. \tag{7}$$

Then we get

$$p(n,k) = 4\sum_{m=1}^{n-1}\binom{n-m-1}{k-4}\log m + O(n^{k-4}\log n)$$

Rewriting

$$\log m = \sum_{l=1}^{m}\frac{1}{l} - \gamma + O\left(\frac{1}{m}\right), \tag{8}$$

we get

$$p(n,k) = 4\sum_{m=1}^{n-1}\binom{n-m-1}{k-4}\left(\sum_{l=1}^{m}\frac{1}{l} - \gamma + O\left(\frac{1}{m}\right)\right) + O(n^{k-4}\log n).$$

Changing the summations, applying (7), and observing for the error term that $\sum \frac{1}{m}\binom{n-m-1}{k-4} \le n^{k-4}\sum\frac{1}{m}$, we get

$$p(n,k) = 4\sum_{l=1}^{n-1}\frac{1}{l}\sum_{m=1}^{n-1}\binom{n-m-1}{k-4} - 4\gamma\sum_{m=1}^{n-1}\binom{n-m-1}{k-4} + \\ + O(n^{k-4}\log n) = \\ = 4\sum_{l=1}^{n-1}\frac{1}{l}\binom{n-l}{k-3} - 4\gamma\binom{n-1}{k-3} + O(n^{k-4}\log n).$$

Write $\binom{n-l}{k-3} = \frac{(n-l)^{k-3}}{(k-3)!} + O(n^{k-4})$ to get

$$p(n,k) = \frac{4}{(k-3)!}\sum_{l=1}^{n-1}\frac{(n-l)^{k-3}}{l} - \frac{4\gamma}{(k-3)!}n^{k-3} + O(n^{k-4}\log n). \tag{9}$$

Expanding the numerators in the sum by the binomial theorem and keeping the leading terms only, (8) can be applied again, and we obtain

$$\log p_n(k) = \frac{4}{(k-3)!} n^{k-3} \log n + O(n^{k-3}).$$

If we want to arrive at a more accurate result, all coefficients have to be calculated in (9). Expanding the numerators by the binomial theorem, we get

$$\begin{aligned} \frac{4}{(k-3)!} \sum_{l=1}^{n-1} \frac{(n-l)^{k-3}}{l} &= \\ &= \frac{4}{(k-3)!} \sum_{l=1}^{n-1} \sum_{t=0}^{k-3} (-1)^{k-3-t} \binom{k-3}{t} \frac{n^t l^{k-3-t}}{l} = \\ &= \frac{4}{(k-3)!} \sum_{l=1}^{n-1} \left(\frac{1}{l} n^{k-3} + \sum_{t=0}^{k-4} (-1)^{k-3-t} \binom{k-3}{t} \frac{n^t l^{k-3-t}}{l} \right) = \\ &= \frac{4}{(k-3)!} n^{k-3} \sum_{l=1}^{n-1} \frac{1}{l} + \frac{4}{(k-3)!} \sum_{l=1}^{n-1} \sum_{t=0}^{k-4} (-1)^{k-3-t} \binom{k-3}{t} n^t l^{k-4-t}. \end{aligned}$$

Within the given error term, we obtain

$$\frac{4}{(k-3)!} n^{k-3} \log n + \frac{4}{(k-3)!} \sum_{t=0}^{k-4} \sum_{l=1}^{n-1} (-1)^{k-3-t} \binom{k-3}{t} n^t l^{k-4-t} =$$

$$= \frac{4}{(k-3)!} n^{k-3} \log n + \frac{4}{(k-3)!} \sum_{t=0}^{k-4} (-1)^{k-3-t} \binom{k-3}{t} n^t \sum_{l=1}^{n-1} l^{k-4-t}.$$

Now, let us substitute $\displaystyle\sum_{l=1}^{n-1} l^{k-4-t} = \frac{n^{k-3-t}}{k-3-t} + O(n^{k-4-t})$.

$$\frac{4}{(k-3)!}n^{k-3}\log n + \frac{4}{(k-3)!}\sum_{t=0}^{k-4}(-1)^{k-3-t}\binom{k-3}{t}n^t\frac{n^{k-3-t}}{k-3-t}+ \\ + O(n^{k-4}) =$$

$$= \frac{4}{(k-3)!}n^{k-3}\log n + \frac{4}{(k-3)!}\sum_{t=0}^{k-4}(-1)^{k-3-t}\binom{k-3}{t}\frac{n^{k-3}}{k-3-t}+ \\ + O(n^{k-4}) =$$

$$= \frac{4}{(k-3)!}n^{k-3}\log n + \frac{4}{(k-3)!}n^{k-3}\sum_{t=0}^{k-4}(-1)^{k-3-t}\binom{k-3}{t}\frac{1}{k-3-t}+ \\ + O(n^{k-4}).$$

Let us examine the sum in the coefficient of n^{k-3}.

Substituting $m = k-3$ and $s = k-3-t$ into the formula

$$\sum_{s=1}^{m}\frac{(-1)^s\binom{m}{s}}{s} = -\sum_{j=1}^{m}\frac{1}{j}$$

(see Exercise 1.43(e) [17]) we obtain

$$\log p_n(k) = \frac{4}{(k-3)!}n^{k-3}\log n - \frac{4}{(k-3)!}n^{k-3}\sum_{j=1}^{k-3}\frac{1}{j} + O(n^{k-4}\log n)$$

□

The p_n sequences and free spectra for the other band varieties can be calculated from Theorem 3.2 and Corollary 3.3.

Corollary 3.5. *For the free spectra of any band varieties the following hold ($k \geq 3$, $k \in \mathbb{N}$).*

- $\log|\mathbf{F}_k(n)| = \frac{4}{(k-3)!}n^{k-3}\log n - \frac{4}{(k-3)!}n^{k-3}\sum_{j=1}^{k-3}\frac{1}{j} + O(n^{k-4}\log n)$,
- $\log|\mathbf{F}_{k'}(n)| = \log|\mathbf{F}_{\mathcal{V}(A_k)}(n)| = |\log \mathbf{F}_{\mathcal{V}(A_k^*)}(n)| =$
$= \frac{2}{(k-2)!}n^{k-2}\log n - \frac{2}{(k-2)!}n^{k-2}\sum_{j=1}^{k-2}\frac{1}{j} + O(n^{k-3}\log n)$,
- $\log|\mathbf{F}_{\mathcal{V}(B_k)}(n)| = \log|\mathbf{F}_{\mathcal{V}(B_k^*)}(n)| =$
$= \frac{1}{(k-1)!}n^{k-1}\log n - \frac{1}{(k-1)!}n^{k-1}\sum_{j=1}^{k-1}\frac{1}{j} + O(n^{k-2}\log n)$,

- $\log|\mathbf{F}_{\mathcal{U}_k}(n)| = \log|\mathbf{F}_{\mathcal{U}_k^*}(n)| =$
  $= \frac{1}{(k-2)!}n^{k-2}\log n - \frac{1}{(k-2)!}n^{k-2}\sum_{j=1}^{k-2}\frac{1}{j} + O(n^{k-3}\log n)$,
- $\log|\mathbf{F}_{\mathcal{U}_k'}(n)| = |\log \mathbf{F}_{\mathcal{U}_k^{*\prime}}(n)| =$
  $= \frac{3}{(k-2)!}n^{k-2}\log n - \frac{3}{(k-2)!}n^{k-2}\sum_{j=1}^{k-2}\frac{1}{j} + O(n^{k-3}\log n)$.

Proof. From Theorem 3.2 and Corollary 3.3 it is straightforward to calculate the p_n sequences of band varieties.

$\log p_n(k') = \log p_n(\mathcal{V}(A_k)) = \log p_n(\mathcal{V}(A_k^*)) =$
$= \frac{1}{2}(\frac{4}{(k-2)!}n^{k-2}\log n - \frac{4}{(k-2)!}n^{k-2}\sum_{j=1}^{k-2}\frac{1}{j}) + O(n^{k-3}\log n)$,

$\log p_n(\mathcal{V}(B_k)) = \log p_n(\mathcal{V}(B_k^*)) =$
$= \frac{1}{4}(\frac{4}{(k-1)!}n^{k-1}\log n - \frac{4}{(k-1)!}n^{k-1}\sum_{j=1}^{k-1}\frac{1}{j}) + O(n^{k-2}\log n)$,

$\log p_n(\mathcal{U}_k) = \log p_n(\mathcal{U}_k^*) = \frac{1}{4}(\frac{4}{(k-2)!}n^{k-2}\log n - \frac{4}{(k-2)!}n^{k-2}\sum_{j=1}^{k-2}\frac{1}{j}) + O(n^{k-3}\log n)$,

$\log p_n(\mathcal{U}_k') = \log p_n(\mathcal{U}_k^{*\prime}) = \frac{3}{4}(\frac{4}{(k-2)!}n^{k-2}\log n - \frac{4}{(k-2)!}n^{k-2}\sum_{j=1}^{k-2}\frac{1}{j}) + O(n^{k-3}\log n)$.

The estimation $\log p_n(\mathcal{V}) < \log|\mathbf{F}_{\mathcal{V}}(n)| = logp_n(\mathcal{V}) + n\log 2$ is valid for any variety with monotone p_n sequence, so $\log \mathbf{F}_{\mathcal{V}}(n) = \log p_n(\mathcal{V}) + O(n)$. This suffices, unless $\mathcal{V}$ is contained in the band variety 4. The varieties contained in 4 have the following p_n sequences:

$p_n(4) = n!^4/n^2$
$p_n(\mathcal{U}_3') = p_n(\mathcal{U}_3^{*\prime}) = n!^3/n$
$p_n(\mathcal{V}(A_3)) = p_n(\mathcal{V}(A_3^*)) = p_n(3') = n!^2$
$p_n(\mathcal{U}_3) = p_n(\mathcal{U}_3^*) = n!n$

In each case we have $p_n \geq np_{n-1}$ for every $n \geq 1$. So we obtain

$$p_n \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq \sum_0^n \binom{n}{t} p_t \leq \sum_0^n \binom{n}{t} p_n \frac{1}{n(n-1)\cdots(t+1)} =$$
$$= p_n \sum_0^n \frac{1}{(n-t)!} \leq ep_n$$

Hence $\log|\mathbf{F}_{\mathcal{V}}(n)| = \log p_n(\mathcal{V}) + O(1)$, so the error terms for $\log p_n(\mathcal{V})$ and for $\log|\mathbf{F}_{\mathcal{V}}(n)|$ have the same order of magnitude in the remainig cases as well.

□

We have proved that for band varieties the logarithm of the free spectra is asymptotically polynomial times $\log n$. This is just one step towards the main goal, the characterization of free spectra of semigroup varieties.

Problem. *Classify the free spectra of semigroup varieties.*

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Contact information

G. Pluhár
Eötvös Loránd University, Department of Algebra and Number Theory, 1117 Budapest, Pázmány Péter sétány 1/c Hungary
E-Mail: `plugab@cs.elte.hu`

J. Wood
Bard College, MAT Program, PO Box 5000 Annandale-on-Hudson, NY 12504-5000
E-Mail: `jwood@bard.edu`

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