Algebra and Discrete Mathematics
Number 2. **(2008).** pp. 50 – 64

RESEARCH ARTICLE

On well p-embedded subgroups of finite groups

Nataliya V. Hutsko and Alexander N. Skiba

Communicated by L. A. Shemetkov

Abstract. Let G be a finite group, H a subgroup of G and H_{sG} the subgroup of H genarated by all those subgroups of H which are s-permutable in G. Then we say that H is well p-embedded in G if G has a quasinormal subgroup T such that $HT = G$ and $T \cap H \leq H_{sG}$. In the present article we use the well p-embedded groups to obtain new characterizations for some class of finite soluble, supersoluble, metanilpotent and dispersive groups.

Introduction

All groups under study in this article are finite. Ore considered [10] two generalizations of normality that still pique the unwaning interest of researchers. Note first of all that quasinormal subgroups were introduced in [10] into the practice of mathematicians for the first time. Following [10], we say that a subgroup H of a groups G is *quasinormal in* G if H commutes with every subgroup of G (i.e. $HT = TH$ for all subgroups T of G). It turned out that quasinormal subgroups possess a series of interesting properties [2, 6, 9, 10, 11, 16, 17] and that actually they are not much different from normal subgroups. Note, in particular, that according to [9] for each quasinormal subgroups H we have $H^G/H_G \subseteq Z_\infty(G/H_G)$, and by [12, Theorem 2.1.3], quasinormal subgroups are precisely those subnormal subgroups of G that are modular elements in the lattice of all subgroups of G.

2000 Mathematics Subject Classification: *20D10, 20D20, 20E28.*
Key words and phrases: *quasinormal group, s-permutable group, well p-embedded group.*

It is clear that if a subgroup H of G is normal in G, then G must have some subgroup T that satisfies the condition

$$G = HT \text{ and both subgroups } T \text{ and } T \cap H \text{ are normal in } G. \qquad (*)$$

Therefore, $(*)$ is another generalization of normality. This idea appeared firstly in [10] too, where it is shown in particular that *G is soluble if and only if all maximal subgroups of G satisfy* $(*)$ (in this regard, also see the article of Baer [1]). Later the subgroups satisfying $(*)$ were called *c-normal* in [18]. In this article a nice theory of c-normal subgroups was presented and some of its applications were given to the questions of classification of groups with some distinguished systems of subgroups.

Recall that a subgroup H of G is said to be s-permutable or s-quasinormal [10] in G if $HP = PH$ for all Sylow subgroups P of G.

In the present article we exemine the following concept which generalizes the conditions of quasinormality as well as c-normality for subgroups.

Definiton 1. Let H be a subgroup of G. Then we say that H is well p-embedded in G if G has a quasinormal subgroup T such that $HT = G$ and $T \cap H \leq H_{sG}$.

In this definition H_{sG} denotes the s-core of H [14], that is the subgroup of H genarated by all those subgroups of H which are s-permutable in G.

It is clear that every s-permutable subgroup and c-normal subgroup are well p-embedded. The following simple example shows that, in general, a well p-embedded subgroup need not be quasinormal or c-normal.

Example 1. Consider $P = M_m(2) =< x, y | x^{2^{m-1}} = y^2 = 1, x^y = x^{1+2^{m-2}} >$, where $m > 3$, and take $A =< x >$ and $B =< y >$. Then $P = [A]B$ and $|B| = 2$. Since $Z(P)$ is a cyclic group of order 2^{m-2}, it follows that B is normal in $Z(P)B$. Given a group Z_3 of prime order 3, take $G = Z_3 \wr P = [K]P$, where K is the base of the regular wreath product G. Since $G = (KB)A$, so $A \cap KB = 1$ and P is a modular group. It follows that KB is quasinormal in G. Hence A is well p-embedded in G, but not quasinormal and not c-normal in G.

In the present article we use the well p-embedded groups to obtain new characterizations for some class of finite soluble, supersoluble, metanilpotent and dispersive groups.

1. Preliminaries

Let G be a group and $p_1 > p_2 > \ldots > p_t$ are different prime divisors of the order of G. Then the group G is said to be *dispersive* (in sence Ore [10]) if there are subgroups $P_1, P_2, \ldots, P_t$ such that P_k is a Sylow p_k-subgroup of G and the subgroup $P_1 P_2 \ldots P_k$ is normal in G for all $k = 1, 2, \ldots, t$.

The following known results about subnormal subgroups will be used in the paper several times.

Lemma 1.1. *Let G be a group and $A \leq K \leq G$, $B \leq G$. Then*

(1) If A and B are subnormal in G, then $\langle A, B\rangle$ is subnormal in G [3, A, Lemma 14.4].

(2) Suppose that A is normal in G. Then K/A is subnormal in G/A if and only if K is subnormal in G [3, A, Lemma 14.1].

(3) If A is subnormal in G, then $A \cap B$ is subnormal in B [3, A, Lemma 14.1].

(4) If A is a subnormal Hall subgroup of G, then A is normal in G [19].

(5) If A is subnormal in G and B is a Hall π-subgroup of G, then $A \cap B$ is a Hall π-subgroup of A [19].

(6) If A is subnormal in G and A is a π-subgroup of G, then $A \leq O_\pi(G)$ [19].

(7) If A is subnormal in G and B is a minimal normal subgroup of G, then $B \leq N_G(A)$ [3, A, Lemma 14.5].

(8) If A is a subnormal soluble (nilpotent) subgroup of G, then A is contained in some soluble (respectively in some nilpotent) normal subgroup of G [19].

We will need to know a few facts about s-permutable subgroups.

Lemma 1.2. *[8] Let G be a group and $H \leq K \leq G$. Then*

(1) If H is s-permutable in G, then H is s-permutable in K.

(2) Suppose that H is normal in G. Then K/H is s-permutable in G if and only if K is s-permutable in G.

(3) If H is s-permutable in G, then H is subnormal in G.

From Lemma 1.2 we directly have.

Lemma 1.3. *Let G be a group and $H \leq K \leq G$. Then the following statements hold:*

(1) H_{sG} is a s-permutable subgroup of G and $H_G \leq H_{sG}$.

(2) $H_{sG} \leq H_{sK}$.

(3) Suppose that H is normal in G. Then $(K/H)_{s(G/H)} = K_{sG}/H$.

(4) If H is either a Sylow subgroup of G or a maximal subgroup of G, then $H_{sG} = H_G$.

Proof. Statements (1-3) are evident. By Lemmas 2(1) and 3(1), H_{sG} is subnormal in G and so in the case when H is a Sylow subgroup of G, $H_{sG} = H_G$, by Lemma 1(6).

Now assume that H is a maximal subgroup of G. If $D = H_G \neq 1$, then by induction $(H/D)_{\pi(G/D)} = (H/D)_{(G/D)} = D/D$. Hence $H_{sG} = D$. Let $D = 1$ and let N be a minimal normal subgroup of G. Then by [3], we know that either N is the only minimal normal subgroup of G and $C = C_G(N) \leq N$ or G has precisely two minimal normal subgroups N and R say, $N \simeq R$ is non-abelian, $R = C$ and $N \cap H = 1 = R \cap H$. Let L be a minimal subnormal subgroup of G contained in H. If $L \leq N$, then $L^G = L^{NH} = L^H \leq D = 1$, a contradiction. Hence $L \nsubseteq N$ and analogously $L \nsubseteq R$. Hence $L \cap N = 1 = L \cap R$. But by Lemma 1(7), $NL = N \times L$, so $L \leq C$, a contradiction. Thus $H_{sG} = 1 = D$. □

Lemma 1.4. *Let G be a group and $H \leq K \leq G$. Then*

(1) Suppose that H is normal in G. Then K/H is well p-embedded in G/H if and only if K is well p-embedded in G.

(2) If H is well p-embedded in G, then H is well p-embedded in K.

(3) Suppose that H is normal in G. Then the subgroup HE/H is well p-embedded in G/H for every well p-embedded in G subgroup E satisfying $(|H|, |E|) = 1$.

Proof. (1) *Necessity.* Suppose first that K/H is well p-embedded in G/H and let T/H be a quasinormal subgroup of G/H such that $(K/H)(T/H) = G/H$ and $(T/H) \cap (K/H) \leq (K/H)_{s(G/H)}$. By Lemma 2(3), T/H is subnormal in G/H. By Lemma 1(2), T is subnormal in G . Besides, we have $KT = G$ and $T \cap K \leq K_{sG}$, by Lemma 3(3). Hence K is well p-embedded in G.

Sufficiency. Now assume that for some quasinormal subgroup T of G we have $KT = G$ and $T \cap K \leq K_{sG}$. Then by Lemma 1(1), HT is subnormal in G, so by Lemma 1(2), HT/H is subnormal in G/H. Besides, we have $(HT/H)(K/H) = G/H$ and $(HT/H) \cap (K/H) = (HT \cap K)/H = H(T \cap K)/H \leq HK_{sG}/H = K_{sG}/H = (K/H)_{s(G/H)}$, by Lemma 3(3). Thus K/H is well p-embedded in G/H.

(2) Let T be a quasinormal subgroup of G such that $HT = G$ and $T \cap H \leq H_{sG}$. Then $K = K \cap HT = H(K \cap T)$ and $K \cap T$ is quasinormal in K. By Lemma 3(2), we also see that $(K \cap T) \cap H \leq H_{sG} \leq H_{sK}$. Hence H is well p-embedded in K.

(3) Assume that E is well p-embedded in G and let T be a quasinormal subgroup of G such that $ET = G$ and $T \cap E \leq E_{sG}$. Clearly, $H \leq T$,

so $T \cap HE = H(T \cap E) \leq H(E_{sG}) \leq (HE)_{sG}$. Hence HE is well p-embedded in G. By (2), HE/H is well p-embedded in G/H. □

The following Lemmas will be necessary for the proof of theorems in Section 2.

Lemma 1.5. *If every maximal subgroup of group G has complement, which is a quasinormal subgroup in G, then G is nilpotent.*

Proof. Suppose that this is false and that G is a counterexample of minimal order. Then $|G|$ is not prime, so G is not simple group. Let N be any proper normal subgroup of G and M/N a maximal subgroup in G/N. And let T be a permutable subgroup in G such that $G = MT$ and $M \cap T = 1$. Then TN/N is permutable in G/N, $(TN/N)(M/N) = G/N$ and $(TN/N) \cap (M/N) = (TN \cap M)/N = N(T \cap M)/N = N/N$. As the class of all nilpotent groups is the saturated formation, we see that G has only minimal normal subgroup. Let N be only minimal normal subgroup of G. Then $C_G(N) = N$. Let M be a maximal subgroup of group G such that $N \leq M$. And let T be permutable in G such that $G = TM$ and $T \cap M = 1$. By Lemma 1(7), $N \leq N_G(T)$ and $NT = N \times T$. Then $T \leq C_G(N) = N$. The received contradiction finishes the proof of lemma. □

Lemma 1.6. *Suppose that $G = AB$ and A is a subnormal subgroup of G, B a nilpotent subgroup. If every Sylow subgroup of A has a quasinormal complement in G, then G is nilpotent.*

Proof. Suppose that this is false and let G be a counterexample of minimal order. Then

(1) *A and every proper subgroup of G containing A are nilpotent.*

Let $A \leq M \leq G$ with $M \neq G$. Then $M = M \cap AB = A(M \cap B)$, where $M \cap B$ is nilpotent in G, A is a subnormal subgroup in M. Let A_p be a Sylow subgroup of A and T a subnormal complement for A_p in G. In view of Lemma 1(3), $M \cap T$ is subnormal in M, so $M = M \cap A_pT = A_p(M \cap T)$. Thus the hypothesis of the theorem is true for M. But $|M| < |G|$, contrary to the choice of G. Thus M is nilpotent. Clearly, A is nilpotent.

(2) *G is soluble.*

By the condition, A is subnormal in G. Then in view of (1) and Lemma 1(8), A contains in some soluble normal subgroup N of G. But $G/N \simeq B/B \cap N$ is nilpotent, so G is soluble.

(3) *G/P is nilpotent for every normal p-subgroup P of G, containing Sylow p-subgroup of A.*

We shall show that the hypothesis of the theorem is true for G/P. Clearly, that $(AP/P)(BP/P) = G/P$, where BP/P is nilpotent and

AP/P a subnormal in G/P. Let Q/P be a Sylow q-subgroup of $AP/P \simeq A/A \cap P$. Then $(q, |P|) = 1$ and $Q = A_qP$ for some Sylow q-subgroups A_q of A. In view of (1), A is nilpotent, so A_q is subnormal in G and $Q = A_q \times P$. Let T be a subnormal complement for A_q in G. Let $D = Q \cap TP = Q_1 \times P_1$, where Q_1 is a Sylow q-subgroup of D and $P_1 \leq P$. Clearly, $Q_1 \leq A_q$. Since $(q, |P|) = 1$, $Q_1 \leq T_q$ for any Sylow q-subgroups T_q of T and therefore $Q_1 \leq T \cap A_q = 1$. Thus $D = P_1$ and hence $TP/P \cap Q/P = 1$. It follows that TP/P is the subnormal complement for Q/P in G/P. At the choice of G we conclude that G/P is nilpotent.

(4) *$A \leq F(G)$ and $F(G)$ is a r-group for some prime r.*

Let P be a Sylow r-subgroup of A. Then in view of (1), P is subnormal in G. By Lemma 1(6), $P \leq O_r(G)$. According to (3), $G/O_r(G)$ is nilpotent. Since G is not nilpotent group, $A \leq F(G) = O_r(G)$.

(5) *$|G| = p^aq$ for some primes p and q and Sylow p-subgroup of G is normal.*

Let M be a normal subgroup of group G such that $A \leq M$ and G/M a simple group. In view of (2), $|G : M| = q$ is a prime. According to (1), M is nilpotent. As every Sylow subgroup P of M is characteristic in M, P is normal in G and in view of (4), $M = P$.

(6) *A is a p-group.*

It directly follows from (4) and (5).

Final contradiction.

Let T be a subnormal complement to a subgroup A in G. Then by Lemma 1(5), the Sylow q-subgroup Q of B contains in T. Let $D = AQ$. Then by Lemma 1(3), $T \cap D = Q(T \cap A) = Q$ is subnormal in D. Thus $D = A \times Q$, so $A \leq N_G(Q)$. Hence $B \leq N_G(Q)$. Then Q is normal in G. Hence in view of (5), G is nilpotent. The received contradiction finishes the proof of the lemma. □

Lemma 1.7. *If $G = AB$, where every Sylow subgroup of A is well p-embedded in G and B is a Hall nilpotent subgroup in G, then G is soluble.*

Proof. Suppose that this is not true and that G is a counterexample of minimal order. Then every minimal normal subgroup of G contained in A is not abelian. Indeed, if for some abelian the minimal normal subgroup L we have $L \leq A$, then by Lemma 4, the hypothesis of lemma is true for G/L. Consequently to the choice of group G, G/L is metanilpotent. It then follows that G is soluble, contrary to the choice of G.

Now assume that $A = G$ and let P be any Sylow subgroup in G. Let $D = P_{qG}$. By Lemma 2(3), the subgroup D is subnormal in G. By [13, II, Corollary 7.7.2], $D \le F(G)$. But G has not the abelian minimal normal subgroups and therefore $D = F(G) = 1$. According to the condition, a subgroup P is well p-embedded in G, so G has such permutable subgroup T that is the complement to P in G. It is clear that T is subnormal in G and consequently T is a normal subgroup in G. Thus every Sylow subgroup of G has normal complement in G. But then G is a nilpotent group, a contradiction. □

Lemma 1.8. *Suppose that $G = [P]M$ and P is a Sylow p-subgroup in G, M is a soluble group. If all maximal subgroups of P are well p-embedded in G, then G is p-supersoluble.*

Proof. Suppose that this is not true and that G is a counterexample of minimal order.

(1) *If N is a minimal normal subgroup of G, then G/N is a p-supersoluble group.*

Indeed, $G/N = [PN/N](MN/N)$, where PN/N is a Sylow p-subgroup in G/N, MN/N is a soluble group. Let K/N be any maximal subgroup of PN/N.

We shall show that a subgroup K/N is well p-embedded in G/N. Since P is a Sylow p-subgroup in G, so $K = K \cap PN = N(K \cap P)$. We shall show first that $K \cap P$ is a maximal subgroups of P. Note that $K \cap P \neq P$. Indeed, if $K \cap P = P$, then $P \subseteq K$ and $K/N = PN/N$, contrary to the choice of K/N. Now assume that exists a subgroup T such that $K \cap P \subset T \subset P$. Then $K = N(K \cap P) \subseteq TN \subseteq PN$. But K is a maximal subgroup of P, so either $K = TN$ or $TN = NP$. If $K = TN$, then $T \subseteq K \cap P \subset T$ that is impossible. Hence $TN = NP$, so $P = P \cap TN = T(P \cap N) \subseteq T(P \cap K) = T$. This gives a contradiction. So $K \cap P$ is a maximal subgroup of P.

By condition of lemma, $K \cap P_p$ is well p-embedded in G. Thus by Lemma 4(2), $(K \cap P_p)N/N$ is well p-embedded in GN/N, so K/N is a well p-embedded subgroup. Thus the hypothesis is still true for G/N. By the choice of G, G/N is a p-supersoluble group.

(2) *N is the only minimal normal subgroup of G and N is a p-group.*

Since the class of all p-supersoluble groups is the saturated formation (see [13, p. 35]), so N is the only minimal normal subgroup of G. Since G is p-supersoluble, so either N is a p'-group or N a p-group. If N is a p'-group, then G is p-supersoluble. Hence N is a p-group.

(3) $N = P$.

Since $N \not\le \Phi(G)$, there exists a subgroup L of G such that $G = [N]L$. We show that $N = O_p(G)$. Indeed, $O_p(G) = O_p(G) \cap NL = N(O_p(G) \cap L)$. Since $O_p(G) \le F(G) \le C_G(N)$, so $O_p(G) \cap L$ is normal in G. It follows that $O_p(G) \cap L = 1$. Hence $N = O_p(G) = P$.

Final contradiction.

Let K be a maximal subgroup of P. Then by hypothesis, G has a quasinormal subgroup T such that $KT = G$ and $T \cap K \leq K_{sG}$. Since $K \leq N$, so $NT = G$. If $N \cap T = 1$, then $KT \neq G$. Hence $N \cap T \leq N$. If $N \cap T < N$, then we have a contradiction to the minimality of N. Thus $N \cap T = N$, so $N \leq T$ and $T = G$. But K is well p-embedded in G, so $K \cap T = K \leq K_{sG}$. Hence K is s-permutable in G, a contradiction. □

2. Characterizations of finite soluble, supersoluble, metanilpotent and dispersive groups

Theorem 2.1. *G is soluble if and only if $G = AB$, where A, B are subgroups of G sutisfying every maximal subgroup of A and every maximal subgroup of B are well p-embedded in G.*

Proof. Necessity. Suppose that this is false and let G be a counterexample of minimal order.

(1) *If N is a minimal normal subgroup of G contained in $A \cap B$, then G/N is soluble* (it directly follows from Lemma 4(1)).

(2) $A \neq G \neq B$.

Indeed, let $A = G$. Let R be a minimal normal subgroup of G. Then the hypothesis of our theorem is true for $G/R = (G/R)(G/R)$. In view of (1), G/R is soluble. Thus R is the only minimal normal subgroup of G, $R \not\leq \Phi(G)$ and $R = A_1 \times \ldots \times A_t$, where $A_1 \simeq \ldots \simeq A_t$ is a simple non-abelian group. Let p be a prime divisor of the order $|R|$ and M a maximal subgroup of G containing $N = N_G(P)$, where P is a Sylow p-subgroup of R. Then by Frattini's Lemma, $G = RM$, so $M_G = 1$. Let T be a quasinormal subgroup in G such that $G = TM$ and $M \cap T \leq M_{sG}$. By Lemma 3(4), $M \cap T \leq M_{sG} = M_G = 1$. Hence T is a complement for M in G. Clearly, p does not divide $|G : M|$, so $(p, |T|) = 1$. It follows that $T \cap R = 1$. By [3, A, Lemma 14.3], $TR = T \times R$. Since R is the only minimal normal subgroup of G and R is not abelian, $T \leq C_G(R) = 1$. Hence $G = TM = M$. This is a contradiction.

(3) *A, B are solube* (it follows from (2) and a choice of group G).

Final contradiction.

Let R be a largest normal soluble subgroup of G. We shall show, that AR/R is nilpotent. If $A \leq R$ it is obvious. Let now $A \not\subseteq R$ and $R \cap A \leq M$, where M is the maximal subgroup of A. Let T be a quasinormal subgroup of G such that $G = MT$ and $M \cap T \leq M_{sG}$. Then $A = A \cap MT = M(A \cap T)$ and $A \cap T$ is a quasinormal subgroup in A. Since $T \cap M$ is a s-permutable subgroup in G, so by lemma 2(3), $T \cap M$ is a subnormal subgroup in G. In view of (3), $T \cap M$ is soluble. Hence

$T \cap M \leq R$. Then we have

$$(R \cap A)(T \cap A) \cap M = (R \cap A)(T \cap A \cap M) = (R \cap A)(T \cap M) \leq R \cap A.$$

Hence by Lemma 5, $A/R \cap A$ is nilpotent, so $AR/R \simeq A/R \cap A$ is nilpotent. It is similarly possible to show that BR/R is nilpotent. Hence by [7, Theorem 3], $G/R = (AR/R)(BR/R)$ is soluble. Thus G is soluble, a contradiction.

Sufficiency. Suppose G is soluble and let M be a maximal subgroup of group G. Then by [3, A, Theorem 15.6], M/M_G has a normal complement in G/M_G and therefore M/M_G is well p-embedded in G/M_G. Thus by Lemma 4(1), M is well p-embedded in G. □

Corollary 1. *G is soluble if and only if all maximal subgroups are well p-embedded in G.*

Theorem 2.2. *G is metanilpotent if and only if $G = AB$, where A is a subnormal subgroup in G, B is a Hall abelian subgroup in G and every Sylow subgroup of A is well p-embedded in G.*

Proof. Necessity. Suppose that this is false and let G be a counterexample of minimal order. By Lemma 7, G is soluble. Then following statements hold.

(1) *Let N be a minimal normal subgroup in G, being p-subgroup for some prime p. If either $N \leq A$ or $(p, |A|) = 1$, then a quatient G/N is metanilpotent.*

Clear, A/N is subnormal in G/N, $BN/N \simeq B/B \cap N$ is a Hall abelian subgroup in G/N and $G/N = (A/N)(BN/N)$. Let P/N be a Sylow q-subgroup in AN/N. Let Q be a Sylow subgroup in AN such that $P = QN$. By [13, III, Lemma 11.6], $Q = A_q N_q$ for some Sylow q-subgroups A_q of A and for Sylow q-subgroups N_q of N. Since group G is soluble, N is the abelian p-group for some prime p. And if either $N \leq A$ or $(p, |A|) = 1$, $A_q N/N$ is a Sylow q-subgroup in AN/N. By Lemma 4(1), $A_q N/N$ is well p-embedded in G/N. Thus the hypothesis of the theorem is true for G/N. Thus the quotient G/N is metanilpotent according to the choice of G.

(2) *$P_{sG} = P_G$ for any Sylow p-subgroup P of A* (it directly follows from Lemma 3(4)).

(3) $A_G \neq 1$.

Assume that $A_G = 1$. By hypothesis, B is the abelian group, so $(A \cap B)^G = (A \cap B)BA = (A \cap B)^A \leq A$ and $A \cap B = 1$. Since $G = AB$ and by [13, III, Lemma 11.6], for any prime p will be such Sylow p-subgroups A_p, B_p and G_p in A, B and G, respectively, that $G_p = A_p B_p$.

Since B is a Hall subgroup, it then follows from equality $A \cap B = 1$ that A is a Hall subgroup in G. By hypothesis, A is subnormal in G. In view of [13, II, Corollary 7.7.2 (1)], A is normal in G. The received contradiction finishes the proof of the statement (3).

(4) *In G there is the only minimal normal subgroup L contained in A and L is a p-group for some prime number p.*

Indeed, by (3), one of the minimal normal subgroups L of G contains in A. Since the class of all metanilpotent groups is the saturated formation (see [13, II, p. 36]), L is the only minimal normal subgroup of G contained in A. But G is soluble, so L is a p-group for some prime p.

(5) *Every Sylow q-subgroup of A has a quasinormal supplement in G with $q \neq p$.*

Let Q be a Sylow q-subgroup in A with $q \neq p$. By hypothesis of our theorem, G has a quasinormal subgroup T such that $G = QT$ and $Q \cap T \leq Q_{sG}$. In view of (2) and (4), $Q_{sG} = 1$. Thus T is a quasinormal supplement to Q in G.

Final contradiction.

Let A_p be a Sylow p-subgroup in A and $P = (A_p)_{sG} = A_G$. We shall consider a quotient group $G/P = (A/P)(BP/P)$. By hypothesis, G has a quasinormal subgroup T such that $TA_p = G$ and $T \cap A_p \leq P$. Then $(A_p/P)(TP/P) = G/P$ and $A_p/P \cap TP/P = P(A_p \cap T)/P = P/P$, so TP/P is a quasinormal supplement to A_p/P in G/P. On the other hand, if Q/N is a Sylow q-subgroup in A/N with $q \neq p$, then in view of (5), Q/P has a quasinormal supplement in G/P (see the proof of the statement (3) Lemmas 6). Thus by Lemma 6, G/P is nilpotent. Hence G is metanilpotent. The received contradiction finishes the proof of the metanilpotently of G.

Sufficiency. Suppose that G is metanilpotent. We shall show that every Sylow subgroup of G is well p-embedded in G. Suppose that is false and let G be a counterexample of minimal order. Then G has a Sylow subgroup P which is not well p-embedded in G. Let N be any minimal normal subgroup in G and F is a Fitting subgroup of G. Suppose that $N \leq P$. Then P/N is well p-embedded in G/N. By Lemma 4(1), P is well p-embedded in G, a contradiction.

Thus $P_G = 1$, so $F \cap P \leq P_{sG} = P_G = 1$. Since G is metanipotent and FP/F is a Sylow subgroup in G, we see that FP/F has a normal supplement T/F in G/F. But F and T/F are p'-groups, so T is a normal supplement to P in G. Hence P is well p-embedded in G. The received contradiction shows that every Sylow subgroup of G is well p-embedded in G. □

Corollary 2. *G is metanilpotent if and only if every Sylow subgroup is well p-embedded in G.*

Theorem 2.3. *Suppose that $G = AB$ and A is a quasinormal subgroup in G, B is a dispersive. If every maximal subgroup of any non-cyclic Sylow subgroup of A is well p-embedded in G, then G is dispersive.*

Proof. Suppose that this theorem is not true and let G be a counterexample of minimal order.

(1) *Every proper subgroup M of G containing A is dispersive.*

Let $A \leq M \leq G$ and $M \neq G$. Then $M = M \cap AB = A(M \cap B)$, where $M \cap B$ is dispersive and A is s-quasinormal in M. By Lemma 4(2), any maximal subgroup of every non-cyclic Sylow subgroup of A is well p-embedded in M and $|M| < |G|$, then by the choice of group G, we have (1).

(2) *Let H be not uniqueal normal subgroup in G being p-group for some prime p. Suppose either H contains a Sylow p-subgroup P of A or P is cyclic or $H \leq A$. Then G/H is dispersive.*

If $A \leq H$, then $G/H = BH/H \simeq B/B \cap H$ is dispersive. Let now $A \nsubseteq H$. Since $|G/H| < |G|$, we need to be shown that hypothesis of the theorem is true for G/H. Clearly, $G/H = (HA/H)(BH/H)$, where HA/H is s-quasinormal in G/H and BH/H is dispersive. Let Q/H be a Sylow q-subgroup of AH/H and M/H any maximal subgroup in Q/H. Let Q_1 be a Sylow q-subgroup of Q such that $Q = HQ_1$. Clearly, Q_1 is a Sylow q-subgroup of AH. Thus $Q = A_qH$ for some Sylow q-subgroup A_q of A. Assume that Q/H is not a cyclic subgroup. Then A_q is not cyclic. We shall show that M/H is well p-embedded in G/H. If $H \leq A$, it directly follows from Lemma 4. Admit that either Sylow p-subgroup P of A cyclic or $P \leq H$. Then $p \neq q$. We shall show $M \cap A_q$ is maximal in A_q. Since $M \neq Q$ and $A_qH = Q$, we see that $M \cap A_q \neq A_q$. Assume that for some subgroup T of G we have $M \cap A_q \leq T \leq A_q$, where $M \cap A_q \neq T \neq A_q$. Then $M = H(M \cap A_q) \leq HT \leq HA_q = Q$. Since M is maximal in Q, or $M = TH$ or $TH = HA_q$. If $M = TH$, then $T \leq M \cap A_q$, contrary to the choice of T. Thus $TH = HA_q$ and we have $A_q = A_q \cap TH = T(A_q \cap H) \leq T(M \cap A_q) = T$, a contradiction. Hence $M \cap A_q$ is a maximal subgroup in A_q. By hypothesis, $M \cap A_q$ is well p-embedded in G. Therefore $M/H = (M \cap A_q)H/H$ is well p-embedded in G/H. Hence the conditions of the theorem are true for G/H.

(3) *If p is a prime and $(p, |A|) = 1$, then $O_p(G) = 1$.*

Let $H = O_p(G) \neq 1$. Then in view of (2), G/H is dispersive. On the other hand, if π is a set of all prime divisors $|A|$, then in view of [10] and [13, II, Corollary 7.7.2], $A \leq E$, where E is a normal π-subgroup in G. Thus $G/E \simeq B/B \cap E$ is dispersive. But then $G \simeq G/H \cap E$ is dispersive, the contradiction.

(4) *G is soluble.*

By hypothesis, A is s-quasinormal in G. In view of [10] and [13, II, Corollary 7.7.2], A contains in some soluble normal subgroup E of G. Since $G/E \simeq B/B \cap E$ is dispersive, G is soluble.

(5) $A_G \neq 1$.

Suppose that $A_G = 1$. Then by [8], A is nilpotent. Let P be a Sylow p-subgroup of A. Since A is subnormal in G, so P is subnormal in G. Thus by [13, II, Corollary 7.7.2], $P \leq O_p(G)$. But in view of (2), $G/O_p(G)$ is dispersive. By the choice of G, $P = A$. Let q be a smallest prime divisor $|G/O_p(G)|$. Then G has a normal maximal subgroup M such that $P \leq M$ and $|G : M| = q$. Let r be a largest prime divisor $|G|$ and R be a Sylow r-subgroup of M. Then in view of (1), R is normal in M, so $R \triangleleft G$. If $r \neq q$, R is a Sylow r-subgroup of G and G/R dispersive. It follows that G is dispersive, a contradiction. Hence $r = q$. But then $G/O_p(G)$ is a r-group. Let B_r be a Sylow r-subgroup in B. Then B_r is a Sylow r-subgroup in G. Since AB_q is a subgroup of G and in view of (1), we have AB_q is dispersive and $B_q \triangleleft AB_q$. As B is dispersive, $B_q \triangleleft B$ and $B_q \triangleleft G$. Hence G is dispersive. The received contradiction proves (5).

Final contradiction.

Let H be a minimal normal subgroup of G containing in A. Let H be a p-group and P a Sylow p-subgroup of A. In view of (2), G/H is dispersive. Let q be a smallest prime divisor $|G/H|$. Then G has a normal maximal subgroup M such that $P \leq M$ and $|G : M| = q$. Let r be a largest prime divisor $|G|$, R be a Sylow r-subgroup of M. Then in view of (1), R is normal in M and so $R \triangleleft G$. As above we see $r = q$. Then G/H is a r-group. Thus $H = A$. By Theorem 1.4 in [15], G is dispersive, a contradiction. □

Theorem 2.4. *If $G = AB$, where A is a subnormal subgroup in G and B is a Hall subgroup in G, which all Sylow subgroups are cyclic groups and any maximal subgroup of every non-cyclic Sylow subgroup of A is well p-embedded in G, then G is supersoluble.*

Proof. Suppose that this is false and that G is a counterexample of minimal order.

(1) *Each proper subgroup M of G containing A is supersoluble.*

Let $A \leq M \leq G$ and $M \neq G$. Then $M = M \cap AB = A(M \cap B)$, where $M \cap B$ is nilpotent and A is a subnormal in M. By Lemma 4(2), any maximal subgroup of every non-cyclic Sylow subgroup of A is well p-embedded in M and $|M| < |G|$, then by the choice of group G, we have (1).

(2) *Let H be a non-uniqueal normal subgroup in G. Suppose that H is a p-group. Admit that H contains Sylow p-subgroup P of A or P is cyclic or $H \leq A$. Then G/H is supersoluble* (see the proof of the statement (2) Theorems 2.3).

(3) *One of the Sylow subgroup of A is not cyclic.*

Indeed, easily to see, that any Sylow subgroup of G contains or in some subgroup interfaced with A or in some subgroup interfaced with B. If all Sylow subgroups of A are the cyclic groups, then every Sylow subgroup of G is cyclic. But then by [5, VI, Theorem 10.3], G is supersoluble, contrary to the choice of G.

(4) *G is soluble.*

Assume that $A \neq G$. Then by view of (1), A is supersoluble. By [13, II, Corollary 7.7.2 (4)], A contains in some normal soluble subgroup R of G. But $G/R = RB/R \simeq B/B \cap R$ is supersoluble group, so G is soluble.

Now assume that $A = G$. If there is such prime p and such maximal subgroup M in some Sylow subgroup G_p of G that $M_{sG} \neq 1$, then $O_p(G) \neq 1$, this attracts resolvability of group G in view of (2). Thus we can assume that for any Sylow subgroup G_p of G and for its any maximal subgroup M we have $M_{sG} = 1$. Then M has a quasinormal supplement T in G and the order Sylow p-subgroup of T is equal p. By Lemma 4(2), condition of the theorem is true for T. Then by view of the choice of group G, T is supersoluble. But it again attracts resolvability of group G.

(5) *A is supersoluble.*

Let $A = G$ be a soluble group in which for any non-cyclic Sylow subgroup G_p all its maximal subgroups are well p-embedded in G. Since the class of all supersoluble groups is the saturated formation (see [13, p. 35]), there is the only minimal normal subgroup N. Thus $N = C_G(N) \nsubseteq \Phi(G)$. By [5, III, Lemma 3.3(a)], $N \nsubseteq \Phi(G_p)$. Since $N \nsubseteq \Phi(G)$, so $G = [N]E$ for some maximal subgroup E of G. Thus $M_{sG}E = EM_{sG}$. But $N \nsubseteq M$, so $M_{sG} \neq N$. If $M_{sG} \neq 1$, in view of maximality of a subgroup E, then $M_{sG} = G$, that attracts $N = N \cap M_{sG}E = M_{sG}(N \cap E) = M_{sG}$, a contradiction. Hence $M_{sG} = 1$ and M has a quasinormal supplement T in G.

It is clear that the order Sylow p-subgroup of T is equal p. Hence in view of Lemma 4(2), the condition of the theorem is true for T. By the choice of group G, T is a supersoluble group. Let q be a largest prime divisor of the order of T. And let T_q be a Sylow q-subgroup in T. We shall admit that $q \neq p$. Then T_q is a Sylow q-subgroup in G. Since T is subnormal in G, so $T_q \triangleleft G$. Then $T_q \leq C_G(N) = N$, a contradiction. Hence $q = p$ is the largest prime divisor of the order of G. In view of [13, I, Lemma 3.9], $O_p(G/C_G(N)) = O_p(G/N) = 1$. Hence by view of (2),

$N = G_p$, a contradiction.

(6) $A_G \neq 1$.

Let p be a largest prime divisor of the order of A and A_p be a Sylow p-subgroup in A. By (5), a group A is supersoluble and $A_p \lhd A$. By [13, II, Corollary 7.7.2 (1)], $A_p \leq O_p(G)$. In view of (2), $G/O_p(G)$ is a supersoluble group and $O_p(G)$ non-cyclic group by the choice of group G. It follows that $A_p \not\subseteq B^x$ for all $x \in G$. Therefore A_p is a Sylow subgroup in G, so $A_p = O_p(G)$.

(7) *Let N be a minimal normal subgroup of group G contained in A. Then $N = A_p = G_p$ is a Sylow subgroup in G, where p is the largest prime divisor of the order of A.*

Let N be a minimal normal subgroup of G contained in A. And let p be the largest prime divisor of A. If p divides $|B|$, $G_p \leq B$, where G_p is a Sylow p-subgroup of G. By the condition, G_p is a cyclic group. But $N \leq G_p$, so N is a cyclic group. In view of (2), G is supersoluble. The received contradiction with a choice of group G shows, that p does not divide $|B|$. Thus in view of (5), $O_p(G) = O_p(A) = A_p$, where A_p is a Sylow p-subgroup of A. Since $O_p(A) \subseteq C_G(N) = N$, we have $N = A_p$ is a Sylow subgroup in G.

(8) *G is p-supersoluble* (it directly follows from Lemma 8).

Final contradiction.

By (2), G/N is supersoluble. By (8), $|N| = p$. Hence G is supersoluble. The received contradiction finishes the proof of the theorem. □

References

[1] R. Baer, *Classes of finite groups and their propeties*, Illinois Math. Journal **1**, 1957, pp. 115-187.

[2] W.E. Deskins, *On quasinormal subgroups of finite groups*, Math. Z. **82**, 1963, pp. 125-132.

[3] K. Doerk and T. Hawkes, *Finite soluble groups*, Walter de Gruyter. Berlin - New York, 1992.

[4] D. Gorenstein, *Finite groups, 2nd edn.*, Chelsea. New York, 1980.

[5] B. Huppert, *Endliche Gruppen I.*, Springer. Berlin - Heidelberg - New York, 1967.

[6] N. Ito , J. Szép, *Uber die Quasinormalteiler von endlichen Gruppen*, Act. Sci. Math. **23**, 1962, pp. 168-170.

[7] O.H. Kegel, *Produkte nilpotenter Gruppen*, Arch. Math. **12**, 1961, pp. 90-93.

[8] O.H. Kegel, *Sylow-Gruppen und Subnormalteiler endlicher Gruppen*, Math. Z. **78**, 1962, pp. 205-221.

[9] R. Maier , P. Schmid, *The embedding of permutable subgroups in finite groups*, Z. Math. **131**, 1973, pp. 269-272.

[10] O. Ore, *Contributions in the theory of groups of finite order*, Duke Math. J. **5**, 1939, pp. 431-460.

[11] P. Schmid, *Subgroups permutable with all Sylow subgroups*, J. Algebra **207**, 1998, pp. 285-293.

[12] R. Schmidt, *Subgroup Lattices of Groups. Vol. 14: De Gruyter Expositions in Mathematics*, Walter de Gruyter. Berlin - New York, 1994.

[13] L.A. Shemetkov, *Formations of finite groups*, Nauka. Minsk, 1978.

[14] A.N. Skiba, *On weakly s-permutable subgroups of finite groups*, J. Algebra **315**, 2007, pp. 192-209.

[15] A.N. Skiba , O.V. Titov, editors *Finite groups with the weakly quasinormal subgroups* , GGU im. F. Skoriny. Gomel, 2006.

[16] S.E. Stonehewer, *Permutable subgroups in Infinite Groups*, Math. Z. **125**, 1972, pp. 1-16.

[17] J.G. Thompson, *An example of core-free quasinormal subgroups of p-groups*, Math. Z. **96**(2), 1967, pp. 226-227.

[18] Y. Wang, *c-normality of groups and its properties*, J. Algebra **180**, 1995, pp. 954-965.

[19] H. Wielandt, *Subnormal subgroups and permutation groups. Lectures given at the Ohio State University*, Columbus. Ohio, 1971.

Contact information

N. V. Hutsko Gomel State University of F.Skorina, Belarus, 246019, Gomel, Sovetskaya Str., 104
E-Mail: `GutskoN@tut.by`

A. N. Skiba Gomel State University of F.Skorina, Belarus, 246019, Gomel, Sovetskaya Str., 104
E-Mail: `Skiba@gsu.unibel.by`

Received by the editors: 30.08.2007
and in final form 30.08.2007.