Algebra and Discrete Mathematics
Number 2. **(2008).** pp. 1 – 41

RESEARCH ARTICLE

Planar trees, free nonassociative algebras, invariants, and elliptic integrals

Vesselin Drensky and Ralf Holtkamp

Communicated by V. A. Artamonov

Abstract. We consider absolutely free algebras with (maybe infinitely) many multilinear operations. Such multioperator algebras were introduced by Kurosh in 1960. Multioperator algebras satisfy the Nielsen-Schreier property and subalgebras of free algebras are also free. Free multioperator algebras are described in terms of labeled reduced planar rooted trees. This allows to apply combinatorial techniques to study their Hilbert series and the asymptotics of their coefficients. Then, over a field of characteristic 0, we investigate the subalgebras of invariants under the action of a linear group, their sets of free generators and their Hilbert series. It has turned out that, except in the trivial cases, the algebra of invariants is never finitely generated. In important partial cases the Hilbert series of the algebras of invariants and the generating functions of their sets of free generators are expressed in terms of elliptic integrals.

Introduction

Let K be an arbitrary field of any characteristic. Although probably most of the K-algebras considered in the literature are K-algebras equipped with just one binary operation, some classical objects are equipped with more than one binary operations, or even some non-binary operations. For example, quite often Jordan algebras are considered with the usual

The work of the first author was partially supported by Grant MI-1503/2005 of the Bulgarian National Science Fund.

2000 Mathematics Subject Classification: *17A50, 17A36, 17A42, 15A72, 33E05.*

multiplication $u \circ v$ and one more ternary operation, the triple product $\{uvw\}$. Poisson algebras have two binary operations – the Poisson bracket and the commutative and associative multiplication. Apart from the classical examples, there are of course also more recently introduced very important algebra types that extend this list, such as dendriform dialgebras and trialgebras. Then there are algebras with infinitely many operations, such as homotopy algebras. The study of primitive elements in free objects leads quite naturally to algebras with infinitely many operations, cf. [Lo, HLR].

In contrast to the case of the free associative algebra, where the primitive elements form the free Lie algebra, Akivis algebras (with up to ternary operations) were introduced in 1976 and later used to model the primitives of non-associative algebras. Then Shestakov and Umirbaev [SU] showed that there exist primitive elements in the universal enveloping algebras of free Akivis algebras which are not Akivis elements and gave a description of the primitive elements. They arrived at algebras with infinitely many operations, which they called hyperalgebras.

In 1960 Kurosh [K2] introduced multioperator algebras (or Ω-algebras) as a generalization of multioperator groups introduced by Higgins [12] in 1956. In [K2] Kurosh established that free Ω-algebras enjoy many of the combinatorial properties of free nonassociative algebras (with one binary operation) and suggested their simultaneous study.

Recall that a variety of universal algebras $\mathfrak{M}$ and its free algebras satisfy the Schreier property if any subalgebra of a free algebra of $\mathfrak{M}$ is free in $\mathfrak{M}$ again. For example, free groups are Schreier. A result of Kurosh [K1] from 1947 states that absolutely free binary algebras are also Schreier. In [K2] Kurosh proved that free multioperator algebras are Schreier again. The variety $\mathfrak{M}$ satisfies the Nielsen property if for any system of generators of a free subalgebra S of a free algebra of $\mathfrak{M}$ there exists an effective procedure (a sequence of elementary transformations similar to the Nielsen transformations in free groups) for obtaining a free set of generators of S. In many important cases a variety satisfies the Schreier property if and only if it satisfies the Nielsen property, see Lewin [L]. We shall say that such varieties and their free algebras satisfy the Nielsen-Schreier property. See for example the book by Mikhalev, Shpilrain and Yu [MSY] for different aspects of Nielsen-Schreier varieties. Schreier varieties of multioperator algebras have been considered by Burgin and Artamonov in [BA]. In particular, they showed that the Nielsen and Schreier properties are equivalent for varieties of multioperator algebras defined by homogeneous polynomial identities. A survey on the results before 1969 is given by Kurosh in [K3]. This article is also introductory for several other papers published in the same issue of Uspehi

Mat. Nauk (Russ. Math. Surv.) and devoted to different aspects of free and close to free Ω-algebras.

All above mentioned algebras are algebras defined over operads. In this paper, we consider Ω-algebras in the sense of Kurosh. Here Ω simply is a set of multilinear operations, which can be quite arbitrary. The only restriction is that we assume that $\Omega = \Omega_2 \cup \Omega_3 \cup \cdots$ is a union of finite sets of n-ary operations Ω_n, $n \geq 2$, otherweise our quantitative results have no sense. Then we consider the absolutely free nonunitary Ω-algebra $K\{X\}_\Omega$ freely generated by a set X. The "Ω-monomials" form the free Ω-magma $\{X\}_\Omega = \mathcal{M}ag_\Omega(X)$ which is a basis of the vector space $K\{X\}_\Omega$ and can be described in terms of labeled reduced planar rooted trees. In particular, if $\Omega = \Omega_2$ consists of a single binary operation, then $K\{X\}_\Omega = K\{X\}$ is the free nonassociative algebra and $\{X\} = \mathcal{M}ag_\Omega(X) = \mathcal{M}ag(X)$ is the usual free magma (the set of monomials in noncommuting nonassociative variables). If $X = \{x\}$ consists of one element, then $\mathcal{M}ag(X)$ is canonically identified with the set of planar rooted binary trees. Another special case is when Ω_n consists of one operation for each $n \geq 2$. Then we obtain the algebra $K\{X\}_\omega$ and for $X = \{x\}$ we may identify $\{X\}_\omega = \mathcal{M}ag_\omega(X)$ with the set of all reduced planar rooted trees.

Labeled reduced planar rooted trees have interesting combinatorics. This allows to apply classical enumeration techniques from graph theory and to study the Hilbert series of $K\{X\}_\Omega$ and the asymptotics of their coefficients.

Since free Ω-algebras have bases which are easily constructed in algorithmic terms, it is natural to develop a theory of Gröbner (or Gröbner-Shirshov) bases. It was surprising for us that the theory of Gröbner bases of free Ω-algebras is much simpler than the theory of Gröbner bases of free associative algebras. For example, if an ideal of $K\{X\}_\Omega$ is finitely generated then its Gröbner basis is finite. If J is a homogeneous ideal of $K\{X\}_\Omega$, we express the Hilbert series of the factor algebra $K\{X\}_\Omega/J$ in terms of the generating functions of Ω, X and the Gröbner basis of J.

Further, we assume that the base field K is of characteristic 0 and study subalgebras of the invariants $K\{X\}_\Omega^G$ under the action of a linear group G on the free Ω-algebra $K\{X\}_\Omega$ for a finite set of free generators X, in the spirit of classical algebraic invariant theory and its generalization to free and relatively free associative algebras, see the surveys [Dr2, F1, KS]. We show that the algebra of invariants $K\{X\}_\Omega^G$ is never finitely generated, except in the obvious cases, when all invariants (if any) are expressed by G-invariant free generators. The proof uses ideas of a similar result for relatively free Lie algebras, see Bryant [Br] and Drensky [Dr1]. Results of Formanek [F1] and Almkvist, Dicks and Formanek

[ADF], allow to express the Hilbert series of the algebra $K\{X\}_{\Omega}^{G}$ and the generating function of the set of its free generators in terms of the Hilbert series of $K\{X\}_{\Omega}$, which is an analogue of the Molien and Molien-Weyl formulas in commutative invariant theory. In the important partial cases of a unipotent action of the infinite cyclic group G and an action of the special linear group $SL_2(K)$ we give explicit expressions for the Hilbert series of the algebras of invariants and the generating functions of their sets of free generators. Applying these formulas to the free binary algebra $K\{X\}$ we express the results in terms of elliptic integrals. We give similar formulas when Ω has exactly one n-ary operation for each $n \geq 2$.

1. Preliminaries

We fix a field K of any characteristic and a set of variables X. In most of the considerations we assume that the set X is finite and $X = \{x_1, \ldots, x_d\}$. One of the main objects in our paper is the absolutely free nonassociative and noncommutative K-algebra $K\{X\}$ freely generated by the set X. As a vector space it has a basis consisting of all non-associative words in the alphabet X. For example, we make a difference between $(x_1x_2)x_3$ and $x_1(x_2x_3)$ and even between $(xx)x$ and $x(xx)$. Words of length n correspond to planar binary trees with n labeled leaves, see e.g. [GH, Ha]. We omit the parentheses when the products are left normed. For example, $uvw = (uv)w$ and $x^n = (x^{n-1})x$.

More generally, following Kurosh [K2], we consider a set Ω of multilinear operations. We assume that Ω contains n-ary operations for $n \geq 2$ only and for each n the number of n-ary operations is finite. We fix the notation

$$\Omega_n = \{\nu_{ni} \mid i = 1, \ldots, p_n\}, \quad n = 2, 3, \ldots,$$

for the set of n-ary operations. The free Ω-magma $\{X\}_{\Omega} = \mathcal{M}ag_{\Omega}(X)$ consists of all "Ω-monomials" and is obtained by recursion, starting with $\{X\}_{\Omega} := X$ and then continuing the process by

$$\{X\}_{\Omega} := \{X\}_{\Omega} \cup \{\nu_{ni}(u_1, \ldots, u_n) \mid \nu_{ni} \in \Omega, u_1, \ldots, u_n \in \{X\}_{\Omega}\}.$$

The set $\{X\}_{\Omega}$ is a K-basis of the free Ω-algebra $K\{X\}_{\Omega}$, and the operations of $K\{X\}_{\Omega}$ are defined using the multilinearity of the operations in Ω. We call the elements of $K\{X\}_{\Omega}$ Ω-polynomials.

The free Ω-magma $\{X\}_{\Omega}$ can be described also in terms of labeled reduced planar rooted trees. Recall that a finite connected graph $\emptyset \neq T = (\mathrm{Ve}(T), \mathrm{Ed}(T))$, with a distinguished vertex ρ_T, is called a rooted tree with root ρ_T, if for every vertex $\lambda \in \mathrm{Ve}(T)$ there is exactly one path connecting λ and ρ_T. Thinking of the edges as oriented towards the root, at each vertex there are incoming edges and (except for the root) one outgoing edge. The leaves of T have no incoming edges and the root has no outgoing edges. The tree is reduced if there are no edges with one incoming edge. A rooted tree T with a chosen order of incoming edges at each vertex is called a planar rooted tree. We label the vertices λ of the reduced planar rooted tree T in the following way. If λ is not a leaf and has n incoming edges, then we label it with an n-ary operation ν_{ni}. We call such trees Ω-trees. If λ is a leaf of the Ω-tree T, we label it with a variable $x_j \in X$. We refer to such trees as Ω-trees with labeled leaves. There is a one-to-one correspondence between the Ω-monomials and the Ω-trees with labeled leaves. For example, the monomial

$$\nu_{31}(\nu_{23}(x_1, x_1), x_3, \nu_{32}(x_2, x_1, x_4))$$

corresponds to the following tree:

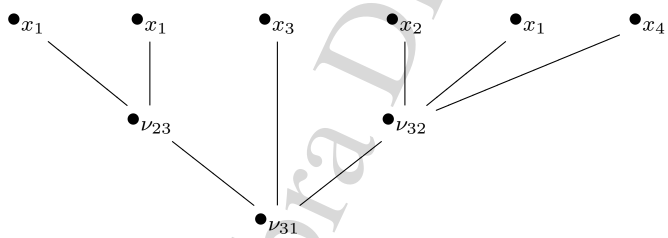

Fig. 1

We consider nonunitary algebras only. If we want to deal with unitary algebras, we need certain coherence conditions, because we have to express the monomials of the form $\nu_{ni}(u_1, \dots, 1, \dots, u_n)$, $u_j \in \{X\}_\Omega$, as linear combinations of elements of $\{X\}_\Omega$, see e.g. [H].

The algebra $K\{X\}_\Omega$ has a natural grading, defined by $\deg(x_j) = 1$, $x_j \in X$, and then extended on the Ω-monomials inductively by

$$\deg(\nu_{ni}(u_1, \dots, u_n)) = \sum_{j=1}^{n} \deg(u_j), \quad u_j \in \{X\}_\Omega.$$

Similarly, if $|X| = d$, then $K\{X\}_\Omega$ has a $\mathbb{Z}^d$-grading, or a multigrading, counting the degree $\deg_j(u)$ of any Ω-word u in each free generator $x_j \in X$. For a graded vector subspace V of $K\{X\}_\Omega$ we consider the homogeneous component $V^{(k)}$ of degree k. In the multigraded case the (multi)homogeneous component of V of degree $(k_1, \dots, k_d)$ is denoted by $V^{(k_1,\dots,k_d)}$. The formal power series with nonnegative integer coefficients

$$H(V, t) = \sum_{k \geq 1} \dim(V^{(k)}) t^k,$$

$$H(V,t_1,\dots,t_d)=\sum_{k_j\geq 0}\dim(V^{(k_1,\dots,k_d)})t_1^{k_1}\cdots t_d^{k_d},$$

are called the Hilbert series of V in the graded and multigraded cases, respectively. Similarly, if W is a set of (multi)homogeneous elements in $K\{X\}_\Omega$, the generating function of W is

$$G(W,t)=\sum_{k\geq 1}\#(W^{(k)})t^k,$$

$$G(W,t_1,\dots,t_d)=\sum_{k_j\geq 0}\#(W^{(k_1,\dots,k_d)})t_1^{k_1}\cdots t_d^{k_d},$$

where $\#(W^{(k)})$ and $\#(W^{(k_1,\dots,k_d)})$ are the numbers of homogeneous elements of the corresponding degree.

The above (multi)gradings work if the set X of free generators consists of d elements. We may consider more general situation of $\mathbb{Z}^d$-grading, when the set X is arbitrary (but still countable). We assign to each $x_j\in X$ a degree

$$\deg(x_j)=(a_{j1},\dots,a_{jd}),\quad a_{jk}\geq 0,$$

and assume that for each $(a_1,\dots,a_d)\in\mathbb{Z}^d$ there is a finite number of generators of this degree. Again, the algebra $K\{X\}_\Omega$ is $\mathbb{Z}^d$-graded and we may speak about the Hilbert series of its graded vector subspaces and generating functions of its subsets consisting of homogeneous elements.

2. Hilbert series and their asymptotics

The next result is standard and relates the Hilbert series of $K\{X\}_\Omega$ and the generating functions of its operations and generators. Compare with the cases $K\{X\}$ and $K\{X\}_\omega$. (For the relation between the Hilbert series of $K\{X\}$ with any $\mathbb{Z}^d$-grading and $K\{x\}$ see e.g. Gerritzen [G] and Rajaee [R]. For general references on enumeration techniques for graphs see the book by Harary and Palmer [HP].)

Proposition 2.1. *Let*

$$G(\Omega,t)=\sum_{n\geq 2}\#(\Omega_n)t^n=\sum_{n\geq 2}p_nt^n$$

be the generating function of the set Ω.

(i) *The Hilbert series*

$$H(K\{x\}_\Omega,t)=\sum_{k\geq 1}\dim(K\{x\}_\Omega^{(k)})t^k$$

satisfies the functional equation

$$G(\Omega, H(K\{x\}_\Omega, t)) - H(K\{x\}_\Omega, t) + t = 0. \qquad (1)$$

The equation (1) *and the condition* $H(K\{x\}_\Omega, 0) = 0$ *determine the Hilbert series* $H(K\{x\}_\Omega, t)$ *uniquely.*

(ii) *If the set* X *is* $\mathbb{Z}^d$*-graded in an arbitrary way, then the Hilbert series of* $K\{X\}_\Omega$ *is*

$$H(K\{X\}_\Omega, t_1, \dots, t_d) = H(K\{x\}_\Omega, G(X, t_1, \dots, t_d)).$$

Proof. (i) The Hilbert series of $K\{x\}_\Omega$ coincides with the generating function of $\{x\}_\Omega$. The set of all elements $\nu_{ni}(u_1, \dots, u_n) \neq x$ from $\{x\}_\Omega$, $u_j \in \{x\}_\Omega$, is in one-to-one correspondence with the set

$$\{(\nu_{ni}, u_1, \dots, u_n) \mid \nu_{nj} \in \Omega, u_j \in \{x\}_\Omega\}.$$

For example, if we apply this correspondence to the Ω-monomial (Ω-tree, respectively) given by the monomial $\nu_{31}(\nu_{23}(x, x), x, \nu_{32}(x, x, x))$, then $n = 3, \nu_{ni} = \nu_{31}$, and u_1, u_2, u_3 are given by the following Ω-trees:

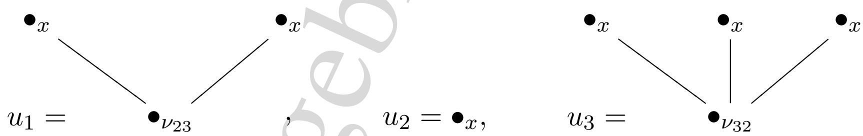

Fig. 2

Hence

$$H(K\{x\}_\Omega, t) - t = G(\{x\}_\Omega, t) - t = \sum_{k\geq 2} \#(\{x\}_\Omega^{(k)})t^k$$

$$= \sum_{n\geq 2} \#(\Omega_n) G(\{x\}_\Omega, t)^n = G(\Omega, G(\{x\}_\Omega, t)) = G(\Omega, H(K\{x\}_\Omega, t)).$$

The condition $H(K\{x\}_\Omega, 0) = 0$ means that the formal power series has no constant term, i.e., the algebra $K\{x\}_\Omega$ is nonunitary. Since Ω does not contain unary operations, its generating function does not have constant and linear terms. This easily implies that the k-th coefficient $\dim(K\{x\}_\Omega^{(k)})$ of $H(K\{x\}_\Omega, t)$ is determined by the first $k-1$ coefficients $\dim(K\{x\}_\Omega^{(k)})$, $m = 1, \dots, k-1$, and the Hilbert series $H(K\{x\}_\Omega, t)$ is determined in a unique way.

(ii) Let us fix an Ω-tree T with k leaves, and consider the set of all possible ways to label the leaves of T with elements of X. Clearly, the generating function of this set (with respect to the $\mathbb{Z}^d$-grading on X) is $G(X, t_1, \ldots, t_d)^k$. Hence

$$\begin{aligned} H(K\{X\}_\Omega, t_1, \ldots, t_d) &= \sum_{k\geq 1} \#(\Omega\text{-trees with } k \text{ leaves}) G(X, t_1, \ldots, t_d)^k \\ &= H(K\{x\}_\Omega, G(X, t_1, \ldots, t_d)). \end{aligned}$$

□

Remark 2.2. If $G(\Omega, t)$ is the generating function of the set Ω, and $u(t_1, \ldots, t_d), v(t_1, \ldots, t_d) \in \mathbb{C}[[t_1, \ldots, t_d]]$ are formal power series such that

$$G(\Omega, v(t_1, \ldots, t_d)) - v(t_1, \ldots, t_d) + u(t_1, \ldots, t_d) = 0 \tag{2}$$

and satisfying

$$u(0, \ldots, 0) = v(0, \ldots, 0) = 0 \tag{3}$$

then Proposition 2.1 (i) gives that

$$v(t_1, \ldots, t_d) = H(K\{x\}_\Omega, u(t_1, \ldots, t_d)).$$

Hence the functional equation (2) and the condition (3) determine uniquely the series $v(t_1, \ldots, t_d)$ as a function of $u(t_1, \ldots, t_d)$.

Example 2.3. (i) If Ω consists of one binary operation only, i.e., $K\{x\}_\Omega = K\{x\}$, then $G(\Omega, t) = t^2$ and Proposition 2.1 (i) gives

$$H(K\{x\}, t)^2 - H(K\{x\}, t) + t = 0.$$

This equation has two solutions

$$H(K\{x\}, t) = \frac{1 \pm \sqrt{1-4t}}{2}$$

and the condition $H(K\{x\}, 0) = 0$ implies that we have to choose the negative sign. Hence

$$H(K\{x\}, t) = \frac{1 - \sqrt{1-4t}}{2} \tag{4}$$

is the well known generating function of the Catalan numbers.

(ii) If Ω_n consists of one operation $\nu_n := \nu_{n1}$ for each $n \geq 2$, i.e., $K\{x\}_\Omega = K\{x\}_\omega$, then

$$G(\Omega, t) = t^2 + t^3 + \cdots = \frac{t^2}{1-t}.$$

Hence $H(K\{x\}_\omega,t)$ satisfies the equation

$$\frac{H(K\{x\}_\omega,t)^2}{1-H(K\{x\}_\omega,t)}-H(K\{x\}_\omega,t)+t=0,$$

$$2H(K\{x\}_\omega,t)^2-(1+t)H(K\{x\}_\omega,t)+t=0$$

and the solution satisfying the condition $H(K\{x\}_\omega,0)=0$ is

$$H(K\{x\}_\omega,t)=\frac{1+t-\sqrt{1-6t+t^2}}{4}. \tag{5}$$

This is the generating function of the super-Catalan numbers (cf. [Sl] A001003).

(iii) Let $\Omega=\underline{n}:=\{\nu_n\}$ consist of one n-ary operation only, i.e., $G(\Omega,t)=G(\underline{n},t)=t^n$. Then the Hilbert series of $K\{x\}_{\underline{n}}$ satisfies the algebraic equation of degree n

$$H(K\{x\}_{\underline{n}},t)^n-H(K\{x\}_{\underline{n}},t)+t=0$$

and is equal to the generating function of the planar rooted n-ary trees.

Remark 2.4. If $f(z)$ is an analytic function in a neighbourhood of 0, $f(0)\neq 0$, and

$$t=zf(z),$$

then the Lagrange inversion formula gives that

$$z=\sum_{k\geq 1}a_kt^k,\quad a_k=\frac{1}{k!}\frac{d^{k-1}}{d\zeta^{k-1}}\left(\frac{1}{f(\zeta)}\right)^k\Bigg|_{\zeta=0}.$$

The same holds if $f(z)$ is a formal power series with complex coefficients and $f(0)\neq 0$. Hence we may apply the formula for $zf(z)=G(\Omega,z)$ and express $H(K\{x\}_\Omega,t)$ in terms of $G(\Omega,t)$.

Example 2.5. To obtain the coefficients of the Hilbert series $H(K\{x\}_{\underline{n}},t)$ of Example 2.3 (iii) we apply Remark 2.4. (For an approach using Koszul duals of operads, compare also [BH]). We obtain $f(z)=1-z^{n-1}$,

$$\left(\frac{1}{f(\zeta)}\right)^k=\frac{1}{(1-\zeta^{n-1})^k}$$

$$=1+\binom{k}{1}\zeta^{n-1}+\binom{k+1}{2}\zeta^{2(n-1)}+\binom{k+2}{3}\zeta^{3(n-1)}+\cdots,$$

$$a_k=\frac{1}{k!}\frac{d^{k-1}}{d\zeta^{k-1}}\frac{1}{(1-\zeta^{n-1})^k}\Bigg|_{\zeta=0},\quad k\geq 1.$$

Direct calculations show that

$$a_k = \frac{1}{m(n-1)+1}\binom{mn}{m}, \qquad \text{for } k = m(n-1)+1, m \geq 0,$$

and $a_k = 0$ otherwise. Hence

$$H(K\{x\}_{\underline{n}}, t) = \sum_{m\geq 0} \binom{mn}{m} \frac{t^{m(n-1)+1}}{m(n-1)+1}$$

$$= t + t^n + nt^{2n-1} + \frac{n(3n-1)}{2} t^{3n-2} + \cdots .$$

For small m this can be seen also directly by counting the planar rooted n-ary trees with the corresponding number of leaves.

The set of n-ary trees with n leaves consists of exactly one tree, the so-called n-corolla. The set of n-ary trees with $2n-1$ leaves consists of n elements. The set of n-ary trees with $3n-2$ leaves consists of $\binom{n}{2} + n^2$ elements, typical examples (for $n = 3$) are depicted in Fig. 3.

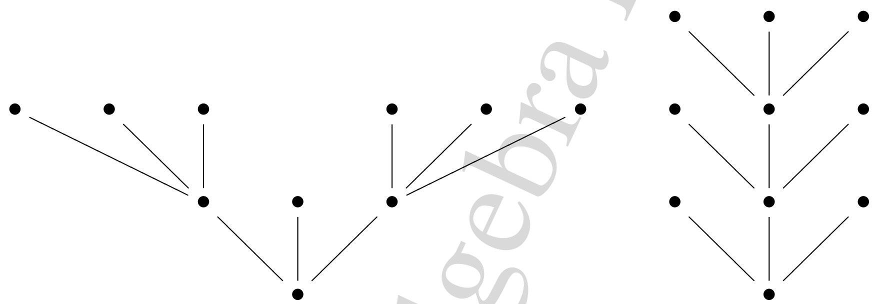

Fig. 3

For $n = 2$ we obtain the explicit formula for the Catalan numbers

$$c_k = \frac{1}{k}\binom{2k-2}{k-1}, \quad k = 1, 2, \ldots .$$

Example 2.6. For $\Omega = \omega$, as in Example 2.3 (ii), Remark 2.4 gives

$$2z^2 - (1+t)z + t = 0, \quad t = \frac{z(1-2z)}{1-z}, \quad f(z) = \frac{1-2z}{1-z}.$$

$$\frac{1}{f^k(\zeta)} = \left(\frac{1-\zeta}{1-2\zeta}\right)^k = \left(1 + \frac{\zeta}{1-2\zeta}\right)^k$$

$$= 1 + \binom{k}{1}\frac{\zeta}{1-2\zeta} + \binom{k}{2}\frac{\zeta^2}{(1-2\zeta)^2} + \cdots$$

$$+\binom{k}{k-1}\frac{\zeta^{k-1}}{(1-2\zeta)^{k-1}}+\binom{k}{k}\frac{\zeta^k}{(1-2\zeta)^k}$$

$$=1+\binom{k}{1}\zeta(1+2\zeta+2^2\zeta^2+2^3\zeta^3+\cdots)$$

$$+\binom{k}{2}\zeta^2\left(1+\binom{2}{1}2\zeta+\binom{3}{1}2^2\zeta^2+\binom{4}{1}2^3\zeta^3+\cdots\right)$$

$$+\binom{k}{3}\zeta^3\left(1+\binom{3}{2}2\zeta+\binom{4}{2}2^2\zeta^2+\binom{5}{2}2^3\zeta^3+\cdots\right)+\cdots$$

$$+\binom{k}{k-1}\zeta^{k-1}\left(1+\binom{k-1}{k-2}2\zeta+\binom{k}{k-2}2^2\zeta^2+\binom{k+1}{k-2}2^3\zeta^3+\cdots\right)$$

$$+\binom{k}{k}\zeta^k\left(1+\binom{k}{k-11}2\zeta+\binom{k+1}{k-1}2^2\zeta^2+\binom{k+2}{k-1}2^3\zeta^3+\cdots\right),$$

$$a_k=\frac{1}{k!}\frac{d^{k-1}}{d\zeta^{k-1}}\left(\frac{1}{f(\zeta)}\right)^k\Bigg\vert_{\zeta=0}=\frac{1}{k}\left(\binom{k}{1}\binom{k-2}{0}+\binom{k}{2}\binom{k-2}{1}2\right.$$

$$\left.+\binom{k}{3}\binom{k-2}{2}2^2+\cdots+\binom{k}{k-1}\binom{k-2}{k-2}2^{k-2}\right)$$

$$=\frac{1}{2k}\left(\sum_{j=1}^{k-1}\binom{k}{j}\binom{k-2}{j-1}2^j\right),\quad k=1,2,\dots\ .$$

Hence a_k is the constant term of the Laurent polynomial

$$\frac{1}{k}\zeta\left(1+\frac{1}{\zeta}\right)^k(1+2\zeta)^{k-2}.$$

One of the important characteristics of a formal power series $a(t)=\sum_{k\geq 0}a_kt^k$ is its radius of convergency

$$r(a(t))=\frac{1}{\limsup_{k\to\infty}\sqrt[k]{a_k}}.$$

By analogy with the (multilinear) codimension sequence for associative PI-algebras, see Giambruno and Zaicev [GZ], we introduce the exponent of free Ω-algebras.

Definition 2.7. Let $|X|=d<\infty$ and let

$$H(K\{X\}_\Omega,t)=\sum_{k\geq 1}a_kt^k$$

be the Hilbert series of the free Ω-algebra $K\{X\}_\Omega$. We define the exponent of $K\{X\}_\Omega$ by

$$\exp(K\{X\}_\Omega) = \limsup_{k\to\infty} \sqrt[k]{a_k}.$$

It is easy to see that

$$\exp(K\{X\}_\Omega) = d \cdot \exp(K\{x\}_\Omega),$$

i.e., it is sufficient to know the exponent of one-generated free Ω-algebras.

Example 2.8. Let $\Omega = \underline{n} := \{\nu_n\}$ consist of one n-ary operation only. Applying the Stirling formula

$$k! = \sqrt{2\pi k}\frac{k^k e^{\vartheta(k)}}{e^k}, \quad |\vartheta(k)| < \frac{1}{12k},$$

to Example 2.5, we obtain

$$\exp(K\{x\}_{\underline{n}}) = \lim_{m\to\infty} \sqrt[m(n-1)+1]{a_{m(n-1)+1}}$$

$$= \lim_{m\to\infty} \sqrt[m(n-1)+1]{\frac{1}{m(n-1)+1}\binom{mn}{m}} = \lim_{m\to\infty} \sqrt[m(n-1)+1]{\binom{mn}{m}}$$

$$= \lim_{m\to\infty} \sqrt[m(n-1)+1]{\frac{n^{nm}}{(n-1)^{(n-1)m}}} = \frac{n}{n-1}\sqrt[n-1]{n}.$$

Hence

$$\lim_{n\to\infty} \exp(K\{x\}_{\underline{n}}) = 1.$$

Example 2.9. For $\Omega = \omega$, as in Example 2.3 (ii), in order to find the coefficient a_k of the Hilbert series of $K\{x\}_\omega$, we may expand the function (5) as a power series. Let

$$\tau_{1,2} = 3 \pm 2\sqrt{2}$$

be the zeros of $1 - 6t + t^2$. The function

$$g_i = \sqrt{1 - \tau_i t}, \quad i = 1, 2,$$

is analytic in the open disc $|t| < 1/\tau_i$ and its radius of convergence is $1/\tau_i$. Since $\sqrt{1 - 6t + t^2} = g_1(t)g_2(t)$ and the radius of convergence of the product of two analytic functions is not less than the radius of convergence of each of the factors, we conclude that $r(H(K\{x\}_\omega, t)) \geq 1/\tau_1 = \tau_2$. More precisely, $r(H(K\{x\}_\omega, t)) = \tau_2$ because the derivatives of $H(K\{x\}_\omega, t)$ have singularities for $t = \tau_2$. Hence

$$\exp(K\{x\}_\omega) = \frac{1}{r(H(K\{x\}_\omega, t))} = \tau_1 \approx 5.8284.$$

Problem 2.10. *How does the exponent of $K\{x\}_\Omega$ depend on the analytic properties of the generating function of Ω? For $|\Omega| < \infty$, express* $\exp(K\{x\}_\Omega)$ *in terms of the coefficients and the zeros of the polynomial* $f(z) = (z - G(\Omega, z))/z$. *What happens if the number of operations of degree n is bounded by the same constant $a > 0$ for all n (or by an^k or by ak^n for a fixed positive integer k)?*

3. Nielsen-Schreier property and Gröbner bases

We assume that the free Ω-magma $\{X\}_\Omega$ is equipped with an admissible ordering $\prec$. This means that the set $(\{X\}_\Omega, \prec)$ is well ordered and if $u \prec v$ in $\{X\}_\Omega$, then

$$\nu_{ni}(w_1, \ldots, w_{j-1}, u, w_{j+1}, \ldots, w_n) \prec \nu_{ni}(w_1, \ldots, w_{j-1}, v, w_{j+1}, \ldots, w_n)$$

for any $\nu_{ni} \in \Omega$ and $w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n \in \{X\}_\Omega$. If

$$f = \sum_{i=1}^{m} \alpha_i u_i \in K\{X\}_\Omega, \quad 0 \neq \alpha_i \in K, u_i \in \{X\}_\Omega, u_1 \succ \cdots \succ u_m,$$

then $\overline{f} = u_1$ is the leading term of u.

Example 3.1. If $X = \{x_1, x_2, \ldots\}$ is countable, we order it by $x_1 \prec x_2 \prec \cdots$. If $u, v \in \{X\}_\Omega$ and $\deg(u) < \deg(v)$, we assume that $u \prec v$. If $\deg(u) = \deg(v) > 1$,

$$u = \nu_{n_1 i_1}(u_1, \ldots, u_{n_1}), \quad v = \nu_{n_2 i_2}(v_1, \ldots, v_{n_2}),$$

we fix $u \prec v$ if $n_1 < n_2$, or $n_1 = n_2$, $i_1 < i_2$ or, if $n_1 = n_2$, $i_1 = i_2$, and $(u_1, \ldots, u_{n_1}) \prec (v_1, \ldots, v_{n_1})$ lexicographically (i.e., $u_1 = v_1, \ldots, u_{k-1} = v_{k-1}$, $u_k \prec v_k$ for some k).

By a result of Kurosh [K2] every subalgebra of the free Ω-algebra $K\{X\}_\Omega$ is free. His proof provides an algorithm which easily produces a system of free generators of the subalgebra. We present this algorithm and some of its consequences for self-containess of our exposition from the point of view of admissible orders.

Algorithm 3.2. Let S be a subalgebra of $K\{X\}_\Omega$. Assuming that the base field K is constructive and starting with any system U of generators of the subalgebra S, we want to find a system of free generators of S.

Given $f_1, \ldots, f_m \in U$ which are algebraically dependent (i.e., the homomorphism $K\{x_1, \ldots, x_m\}_\Omega \to S$ defined by $x_j \to f_j$, $j = 1, \ldots, m$, has a nontrivial kernel), the procedure suggests in each step an elementary transformation which decreases one of the generators with respect to the admissible ordering.

We may assume here that the leading coefficient of each f_j is equal to 1, i.e., $f_j = \overline{f_j} + \cdots$, where $\cdots$ denotes a linear combination of lower Ω-monomials.

In the following we sketch how to find one generator f_j such that the Ω-monomial $\overline{f_j}$ belongs to the subalgebra generated by the other $\overline{f_i}, i \neq j$. (Then clearly we can replace in the generating set f_j by lower elements.)

Let $h(f_1, \ldots, f_m) = 0$ for some $0 \neq h(x_1, \ldots, x_m) \in K\{x_1, \ldots, x_m\}_\Omega$. For every Ω-monomial $h_i(x_1, \ldots, x_m) \in \{x_1, \ldots, x_m\}_\Omega$,

$$\overline{h_i(f_1, \ldots, f_m)} = h_i(\overline{f_1}, \ldots, \overline{f_m})$$

(because $h_i(\overline{f_1}, \ldots, \overline{f_m}) \neq 0$). Hence, there exist two different $h_1, h_2 \in \{x_1, \ldots, x_m\}_\Omega$ such that

$$h_1(\overline{f_1}, \ldots, \overline{f_m}) = h_2(\overline{f_1}, \ldots, \overline{f_m}). \tag{6}$$

If $h_1 = x_j$ then $\overline{f_j} = h_2(\overline{f_1}, \ldots, \overline{f_m})$. Since $h_2 \neq x_j$, comparing the degrees of $\overline{f_j}$ and $h_2(\overline{f_1}, \ldots, \overline{f_m})$, we conclude that $h_2(x_1, \ldots, x_m) = h_2(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)$ does not depend on x_j. The Ω-polynomials

$$f_1, \ldots, f_{j-1}, f_j^* = f_j - h_2(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_m), f_{j+1}, \ldots, f_m$$

generate the same algebra as $f_1, \ldots, f_m$. If $f_j^* = 0$, then we may remove f_j from the system of generators of S. Otherwise, $\overline{f_j^*} \prec \overline{f_j}$ and the new set $\{f_1, \ldots, f_j^*, \ldots, f_m\}$ is lower in the lexicographic ordering than $\{f_1, \ldots, f_m\}$.

The case where $h_2 = x_k$ is similar. In the remaining case, where neither $h_1(x_1, \ldots, x_m)$ nor $h_2(x_1, \ldots, x_m)$ is equal to a single variable x_j, may be treated recursively in view of equation (6).

The algorithm immediately gives the following well-known fact, see [K2, K3, BA].

Corollary 3.3. *Every graded subalgebra A of the $\mathbb{Z}^d$-graded Ω-algebra $K\{X\}_\Omega$ has a homogeneous system of free generators.*

Let $X = \{x_1, \ldots, x_d\}$. By analogy with the case of algebras with one binary operation, we call the automorphism φ of the free algebra $K\{X\}_\Omega$ tame if it belongs to the subgroup of $\mathrm{Aut}(K\{X\}_\Omega)$ generated by the linear and triangular automorphisms, defined, respectively, by

$$\varphi(x_j) = \sum_{i=1}^{d} \alpha_{ij} x_i, \quad \alpha_{ij} \in K, j = 1, \ldots, d,$$

where the matrix (α_{ij}) is invertible, and

$$\varphi(x_j) = \alpha_j x_j + f_j(x_{j+1}, \dots, x_d), \quad j = 1, \dots, d,$$

where $\alpha_j \in K^* = K \setminus \{0\}$ and the Ω-polynomials $f_j(x_{j+1}, \dots, x_d)$ do not depend on the variables $x_1, \dots, x_j$. The discussion of Algorithm 3.2 shows also that all automorphisms of $K\{X\}_\Omega$ are tame, which is a result of Burgin and Artamonov [BA]. If the base field K is constructive, it provides an algorithm which decomposes the given automorphism into a product of linear and triangular automorphisms.

Corollary 3.4 (Burgin and Artamonov [BA]). *If $|X| < \infty$, then every automorphism of $K\{X\}_\Omega$ is tame.*

The following gives a relation between the Hilbert series of graded subalgebras and generating functions of their systems of free generators.

Corollary 3.5. *If A is a graded subalgebra of the $\mathbb{Z}^d$-graded Ω-algebra $K\{X\}_\Omega$ with Hilbert series $H(A, t_1, \dots, t_d)$, then the generating function of every homogeneous free generating set Y of A is*

$$G(Y, t_1, \dots, t_d) = H(A, t_1, \dots, t_d) - G(\Omega, H(A, t_1, \dots, t_d)). \tag{7}$$

Proof. It is sufficient to use Corollary 3.3 and to apply (2) in Remark 2.2. □

Existence of admissible orderings allows to develop the theory of Gröbner (or Gröbner-Shirshov) bases of Ω-ideals J in $K\{X\}_\Omega$. The obvious definition of Ω-subwords of an Ω-word (or an Ω-monomial) $u = \nu_{ni}(u_1, \dots, u_n) \in \{X\}_\Omega$ is by induction. The subwords of u are the words u_j and the subwords of the u_j. Now we fix an admissible ordering on $\{X\}_\Omega$. The subset $B = B(J)$ of the ideal J of $K\{X\}_\Omega$ is a Gröbner basis of J if, for every nonzero $f \in J$, there is an element $g \in B$ such that the leading term $\overline{g}$ of g is a subword of the leading term $\overline{f}$ of f. Most of the standard properties of Gröbner bases for free noncommutative algebras hold also for free Ω-algebras. In particular, the set of normal words, i.e., the Ω-words which do not contain as subwords $\overline{g}$, $g \in B(J)$, form a basis of the factor algebra $K\{X\}_\Omega/J$. The set

$$I = I(J) = \{\overline{f} \mid 0 \neq f \in J\} \subseteq \{X\}_\Omega$$

is an ideal of $\{X\}_\Omega$ generated by the set $\{\overline{g} \mid g \in B(J)\}$. We call I the initial ideal of J.

There is an algorithm to compute the Gröbner basis of an ideal J of the free associative algebras $K\langle X\rangle$ which is an analogue of the Buchberger algorithm for ideals of polynomial algebras. (Compare with the approach based on the diamond lemma in the paper by Bergman [Be]. See also the survey article by Ufnarovski [U].) If the ideal J of $K\langle X\rangle$ is generated by a set $\{f_k\}$, we fix an admissible ordering and start the construction of the Gröbner basis $B(J)$ defining $B(J) := \{f_k\}$. If the leading terms of $f_1, f_2 \in B(J)$ are $\overline{f_1}, \overline{f_2}$, respectively, and $u_1\overline{f_1}v_1 = u_2\overline{f_2}v_2$ for some monomials u_k, v_k, $k = 1, 2$, then, for suitable nonzero $\alpha_1, \alpha_2 \in K$, the S-polynomial $f_{12} = \alpha_1 u_1 f_1 v_1 - \alpha_2 u_2 f_2 v_2$, if not 0, is lower than $u_1 f_1 v_1$ and $u_2 f_2 v_2$ in the admissible ordering of $K\langle X\rangle$. If $f_{12} \neq 0$, we have an "ambiguity" and, in order to solve it, we add f_{12} to $B(J)$. We have two kinds of S-polynomials. In the first case, $\overline{f_1}$ and $\overline{f_2}$ overlap, i.e., $u_1\overline{f_1} = \overline{f_2}v_2$. In the second case $\overline{f_2}$ is a subword of $\overline{f_1}$, i.e., $\overline{f_1} = u_2\overline{f_2}v_2$ (or $\overline{f_1}$ is a subword of $\overline{f_2}$). In the case of Ω-algebras there are no overlaps and it is sufficient to consider S-polynomials obtained when one of the leading terms is an Ω-subword of the other. Hence the Buchberger algorithm has the following form. (Of course, we fix an admissible ordering of $K\{X\}_\Omega$ and assume that the base field K is constructive.)

Algorithm 3.6. Let the Ω-ideal J of $K\{X\}_\Omega$ be generated by the set $\{f_k\}$. We may assume that the coefficients of the leading terms $\overline{f_k}$ are all equal to 1. We define $B(J) := \{f_k\}$. Let $\overline{f_2}$ be an Ω-subword of $\overline{f_1}$ for some $f_1, f_2 \in B(J)$, e.g. $\overline{f_1} = \nu_{ni}(u_1, \ldots, u_n)$, where $u_1, \ldots, u_n \in \{X\}_\Omega$ and $\overline{f_2}$ is a subword of some u_j. Then we replace in $B(J)$ the Ω-polynomial f_1 by

$$\widetilde{f_1} = f_1 - \nu_{ni}(u_1, \ldots, \widetilde{u_j}, \ldots, u_n),$$

where the Ω-polynomial $\widetilde{u_j}$ is obtained from the Ω-monomial u_j by replacing $\overline{f_2}$ by f_2. If $\widetilde{f_1} \neq 0$, we norm it (making the leading coefficient equal to 1). If $\widetilde{f_1} = 0$, we remove it from $B(J)$. We continue the process as long as possible.

Proposition 3.7. *Finitely generated ideals of $K\{X\}_\Omega$ have finite Gröbner bases with respect to any admissible ordering.*

Proof. Let the ideal J be generated by $f_1, \ldots, f_m$. Following Algorithm 3.6, we start with $B(J) := \{f_1, \ldots, f_m\}$. In each step we either remove one of the elements of $B(J)$ or replace it with a new polynomial, without adding more elements to $B(J)$. In a finite number of steps the procedure will stop and we obtain a finite Gröbner basis of J. □

Remark 3.8. Proposition 3.7 implies the solvability of the word problem for Ω-algebras. This means that if A is a finitely presented Ω-algebra, there is an algorithm which decides whether an element $f \in A$ is equal to 0. In other words, if $A \cong K\{x_1, \ldots, x_d\}_\Omega/J$ for a finitely generated Ω-ideal J, and the generators $f_1, \ldots, f_m$ of J are explicitly given, then we can decide whether $f \in K\{x_1, \ldots, x_d\}_\Omega$ belongs to J. One should pay attention to the fact that the solvability of decision problems cannot be transferred to factor algebras. There exist finitely generated ideals J_0 of the free associative algebra $K\langle x_1, \ldots, x_d\rangle$, such that the word problem has no solution in $A \cong K\langle x_1, \ldots, x_d\rangle/J_0$. Of course, in this case $A \cong K\{x_1, \ldots, x_d\}/J$ for some ideal J of $K\{x_1, \ldots, x_d\}$ and J is not finitely generated.

We conclude this section with an Ω-analogue of a theorem of Rajaee [R] for $K\{X\}$. We assume that $K\{X\}_\Omega$ is $\mathbb{Z}^d$-graded. Then, as in [R], the reduced Gröbner basis of a (multi)homogeneous Ω-ideal J of $K\{X\}_\Omega$ consists of (multi)homogeneous elements.

Theorem 3.9. *Let J be a homogeneous Ω-ideal of $K\{X\}_\Omega$ with respect to any $\mathbb{Z}^d$-grading of $K\{X\}_\Omega$. Then the Hilbert series of the factor algebra $A \cong K\{X\}_\Omega/J$ and the generating functions of the set of generators X and of the reduced Gröbner basis $B(J)$ of J with respect to any admissible ordering are related by*

$$H(A, t_1, \ldots, t_d) = H(K\{x\}_\Omega, G(X, t_1, \ldots, t_d) - G(B(J), t_1, \ldots, t_d)). \quad (8)$$

Proof. We follow the idea of the proof in [R]. For convenience, we denote the Hilbert series $H(P, t_1, \ldots, t_d)$ or the generating function $G(P, t_1, \ldots, t_d)$ of the graded object P by $H(P)$ and $G(P)$, respectively. Clearly, the isomorphism $A \cong K\{X\}_\Omega/J$ implies $H(A) = H(K\{X\}_\Omega) - H(J)$. Also, $H(K\{X\}_\Omega) = G(\{X\}_\Omega)$ and $H(J) = G(I)$, where $I = I(J) \triangleleft \{X\}_\Omega$ is the initial ideal of J. Finally, $G(B(J)) = G(B(I))$, where $B(I)$ is the minimal generating set of the Ω-ideal I. Hence (8) is equivalent to

$$G(\{X\}_\Omega) - G(I) = H(K\{x\}_\Omega, G(X) - G(B(I))).$$

The elements of I which do not belong to the minimal set of generators $B(I)$ are characterized by the property that they are of the form

$$u = \nu_{ni}(v_1, \ldots, v_{j-1}, w_j, v_{j+1}, \ldots, v_n), \quad v_k \in \{X\}_\Omega, \quad w_j \in I.$$

Hence

$$I = \left(\bigcup_{\nu_{ni} \in \Omega} \bigcup_{j=1}^{n} \nu_{ni}(\underbrace{\{X\}_\Omega, \ldots, \{X\}_\Omega}_{j-1 \text{ times}}, I, \underbrace{\{X\}_\Omega, \ldots, \{X\}_\Omega}_{n-j \text{ times}}) \right) \bigcup B(I),$$

$$G(I)=\sum_{\nu_{ni}\in\Omega} G\left(\bigcup_{j=1}^{n}\nu_{ni}(\underbrace{\{X\}_\Omega,\dots,\{X\}_\Omega}_{j-1\text{ times}},I,\underbrace{\{X\}_\Omega,\dots,\{X\}_\Omega}_{n-j\text{ times}})\right)+G(B(I)).$$

By the principle of inclusion and exclusion,

$$\begin{aligned}
&G\left(\bigcup_{j=1}^{n}\nu_{ni}(\underbrace{\{X\}_\Omega,\dots,\{X\}_\Omega}_{j-1\text{ times}},I,\underbrace{\{X\}_\Omega,\dots,\{X\}_\Omega}_{n-j\text{ times}})\right)\\
&\qquad=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}G^{n-k}(\{X\}_\Omega)G^k(I)\\
&\qquad=G^n(\{X\}_\Omega)-(G(\{X\}_\Omega)-G(I))^n.
\end{aligned}$$

This implies

$$\begin{aligned}
G(I)&=\sum_{\nu_{ni}\in\Omega}(G^n(\{X\}_\Omega)-(G(\{X\}_\Omega)-G(I))^n)+G(B(I))\\
&=G(\Omega,G(\{X\}_\Omega))-G(\Omega,G(\{X\}_\Omega)-G(I))+G(B(I)).
\end{aligned}$$

Applying (1) and Proposition 2.1 (ii) we obtain

$$G(I)=G(\{X\}_\Omega)-G(X)-G(\Omega,G(\{X\}_\Omega)-G(I))+G(B(I)),$$

$$G(\Omega,G(\{X\}_\Omega)-G(I))-(G(\{X\}_\Omega)-G(I))+(G(X)-G(B(I)))=0.$$

By (2) in Remark 2.2 we conclude that

$$H(A)=G(\{X\}_\Omega)-G(I)=H(K\{x\}_\Omega,G(X)-G(B(I)))$$

and this completes the proof. □

Corollary 3.10. *Let J be a homogeneous Ω-ideal of $K\{X\}_\Omega$ with respect to any $\mathbb{Z}^d$-grading of $K\{X\}_\Omega$. Let $B(J)$ be the reduced Gröbner basis of J with respect to any admissible ordering. Then*

$$\begin{aligned}
G(B(J),t_1,\dots,t_d)&=G(\Omega,H(K\{X\}_\Omega/J,t_1,\dots,t_d))\\
&\quad-H(K\{x\}_\Omega/J,t_1,\dots,t_d)+G(X,t_1,\dots,t_d).
\end{aligned}$$

Proof. Applying (2) in Remark 2.2 to (8) we obtain for $A\cong K\{X\}_\Omega/J$ that

$$G(\Omega,H(A))-H(A)+G(X)-G(B(J))=0,$$

which gives the expression for the generating function of the reduced Gröbner basis $B(J)$ of J. □

Example 3.11. Let $\Omega = \underline{n} = \{\nu_n\}$ consist of one n-ary operation only, as in Example 2.3 (iii), and let A be the free commutative and associative n-ary algebra in one variable, i.e., A is the homomorphic image of $K\{x\}_{\underline{n}}$ modulo the ideal generated by all

$$\nu_n(u_1,\dots,u_n) - \nu_n(u_{\sigma(1)},\dots,u_{\sigma(n)}), \quad \sigma \in S_n,$$

$$\nu_n(u_1,\dots,\nu_n(u_j,v_2,\dots,v_n),\dots,u_n)$$
$$-\nu_n(\nu_n(u_1,v_2,\dots,v_n),\dots,u_j,\dots,u_n), \quad j = 2,\dots,n,$$

where S_n is the symmetric group and $u_j, v_k \in K\{x\}_{\underline{n}}$. Hence the homogeneous component $A^{(k)}$ is one-dimensional for $k = (n-1)m+1$ and is equal to zero for all other k. We may assume that $A^{((n-1)m+1)}$ is spanned by

$$x^{(n-1)m+1} = \nu_n(x^{(n-1)(m-1)+1}, x, \dots, x), \quad m \geq 1.$$

Since $G(\Omega, t) = t^n$ and

$$H(A,t) = \sum_{m\geq 0} t^{(n-1)m+1} = \frac{t}{1-t^{n-1}},$$

Corollary 3.10 gives that the generating function of the reduced Gröbner basis with respect to any admissible ordering is

$$G(B(J),t) = \left(\frac{t}{1-t^{n-1}}\right)^n - \frac{t}{1-t^{n-1}} + t$$
$$= t^n \sum_{m\geq 1} \left(\binom{m+n-1}{n-1} - 1 \right) t^{(n-1)m}.$$

We fix the admissible ordering on $\{x\}_{\underline{n}}$ which compares the monomials first by degree and then by inverse lexicographic ordering: If $u = \nu_n(u_1,\dots,u_n)$, $v = \nu_n(v_1,\dots,v_n)$, then $u \prec v$ if either $\deg(u) < \deg(v)$ or $\deg(u) = \deg(v)$ and $u_k \prec v_k$, $u_{k+1} = v_{k+1},\dots,u_n = v_n$. In this way $x^{(n-1)m+1}$ is the smallest monomial of degree $(n-1)m+1$. Then the reduced Gröbner basis of J consists of all

$$\nu_n(x^{(n-1)m_1+1},\dots,x^{(n-1)m_n+1}) - x^{m(n-1)+1},$$

$m_1 + \cdots + m_n = m$, $(m_1,\dots,m_n) \neq (m,0,\dots,0)$. The number of such polynomials of degree $(n-1)m+1$ is equal to the number of all monomials $z_1^{m_1}\cdots z_n^{m_n} \neq z_1^m$ of total degree m in n variables $z_1,\dots,z_n$.

4. Invariant theory

Till the end of the paper we fix a field K of characteristic 0. We assume that the set $X = \{x_1, \ldots, x_d\}$ is finite and consists of d elements. The general linear group $GL_d(K)$ acts canonically on the d-dimensional vector space KX with basis X and we identify it with the group of invertible $d \times d$ matrices. If

$$g = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1d} \\ \vdots & \ddots & \vdots \\ \alpha_{d1} & \cdots & \alpha_{dd} \end{pmatrix} \in GL_d(K), \quad \alpha_{pq} \in K,$$

then the action on KX is defined by

$$g(x_j) = \alpha_{1j}x_1 + \cdots + \alpha_{dj}x_d, \quad j = 1, \ldots, d.$$

This action is extended diagonally on $K\{X\}_\Omega$ by

$$g(u(x_1, \ldots, x_d)) = u(g(x_1), \ldots, g(x_d)), \quad u(x_1, \ldots, x_d) \in K\{X\}_\Omega.$$

If G is a subgroup of $GL_d(K)$, then the algebra of G-invariants is defined in the obvious way as

$$K\{X\}_\Omega^G = \{f \in K\{X\}_\Omega \mid g(f) = f, \quad g \in G\}.$$

One of the main problems in classical invariant theory is the problem for finite generation of the algebra of invariants which is a partial case of the 14th Hilbert problem. The same problem has been intensively studied in noncommutative invariant theory. See the surveys [F1] and [Dr2] for free and relatively free associative algebras and the papers [Br] and [Dr1] for free and relatively free Lie algebras. It has turned out that in the noncommutative case the algebra of invariants is finitely generated in very special cases only.

For example, if G is a finite linear group acting on the free associative algebra $K\langle X\rangle$, a theorem established independently by Dicks and Formanek [DF] and Kharchenko [Kh2] states that $K\langle X\rangle^G$ is finitely generated if and only if G is cyclic and acts on KX by scalar multiplication. A simplified version of the proof of Kharchenko is given by Dicks in [C2]. Koryukin [Ko1] considered the case of the action of any linear group G on $K\langle X\rangle$. Let d_1 be the minimal integer with the property that there exist linearly independent $y_1, \ldots, y_{d_1}$ in the vector space KX such that $K\langle X\rangle^G \subset K\langle y_1, \ldots, y_{d_1}\rangle$. Changing linearly the system of free generators $X = \{x_1, \ldots, x_d\}$, we may assume that $K\langle X\rangle^G \subset K\langle x_1, \ldots, x_{d_1}\rangle$. The theorem of Koryukin [Ko1] gives that $K\langle X\rangle^G$ is finitely generated if and only if G acts on $Kx_1 \oplus \cdots \oplus Kx_{d_1}$ as a finite cyclic group of scalar multiplications.

In the case of the free Lie algebra $L(X)$ Bryant [Br] showed that $L(X)^G$ is not finitely generated if $G \neq \langle e \rangle$ is any finite group. The same result holds from [Dr1].

It is natural to expect that something similar holds for free Ω-algebras. If J is an Ω-ideal of $K\{X\}_\Omega$ which is $GL_d(K)$-invariant, i.e., $GL_d(K)(J) = J$, then the action of $GL_d(K)$ on $K\{X\}_\Omega$ induces an action on the factor algebra $K\{X\}_\Omega/J$. Hence the G-invariants of $K\{X\}_\Omega$ go to G-invariants of $K\{X\}_\Omega/J$. If the group G is finite or, more generally, acts as a reductive group on KX, then every invariant of $K\{X\}_\Omega/J$ can be lifted to an invariant of $K\{X\}_\Omega$. Hence, in this case we may study G-invariants of factor algebras and then lift the obtained results to the algebra $K\{X\}_\Omega^G$ itself.

In the case of ordinary polynomial algebras, if the group G is finite, the algebra of invariants $K[X]^G$ is always nontrivial and even of the same transcendence degree d as $K[X]$. Lifting the invariants, we obtain that the algebras $K\langle X\rangle^G$ and $K\{X\}^G$ are also nontrivial. In the case of (non-binary) free Ω-algebras, the picture is completely different:

Example 4.1. Let $G = \{e, -e\}$, where e is the identity $d \times d$ matrix and let $\Omega = \underline{3} = \{\nu_3\}$ consist of a single ternary operation. Since

$$(-e)(u(x_1, \ldots, u_d)) = (-1)^k u(x_1, \ldots, u_d),$$

$$k = \deg(u), u(x_1, \ldots, u_d) \in \{X\}_{\underline{3}},$$

$K\{X\}_{\underline{3}}^G$ is spanned by all homogeneous monomials of even degree. By Example 2.5 (or by easy induction), $K\{X\}_{\underline{3}}$ is spanned by monomials of odd degree only. Hence $K\{X\}_{\underline{3}}^G = \{0\}$.

Let T be an Ω-tree with N leaves. It is a reduced tree such that every internal vertex (i.e., vertex which is not a leaf) is labeled by an element of Ω_n, where n is the number of incoming edges of the vertex. We denote by ν_T the corresponding composition of operations from Ω. If we label the leaves of T by $x_1, \ldots, x_N$ and denote the corresponding Ω-monomial by $\nu_T(x_1, \ldots, x_N)$, then the labeling of the leaves of T by $x_{j_1}, \ldots, x_{j_N}$ gives rise to the monomial $\nu_T(x_{j_1}, \ldots, x_{j_N})$. For example, if T is the Ω-tree in Fig. 4, then

$$\nu_T(x_1, \ldots, x_6) = \nu_{31}(\nu_{23}(x_1, x_2), x_3, \nu_{32}(x_4, x_5, x_6)),$$

and $\nu_T(x_1, x_1, x_3, x_2, x_1, x_4)$ corresponds to the Ω-tree with labeled leaves given in Fig. 1.

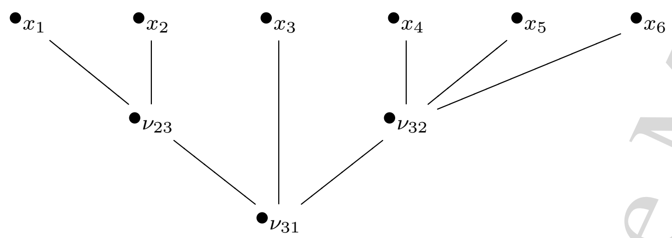

Fig. 4

Clearly, the $GL_d(K)$-module $\nu_T(KX,\dots,KX)$ is isomorphic to the N-th tensor power $(KX)^{\otimes N}$ by the isomorphism which deletes the operations

$$\pi_T : \nu_T(x_{j_1},\dots,x_{j_N}) \to x_{j_1}\otimes\dots\otimes x_{j_N}. \tag{9}$$

As in the classical case, if G is a subgroup of $GL_d(K)$, then the algebra of invariants $K\{X\}_\Omega^G$ is graded with respect to the usual grading defined by $\deg(x_j)=1$, $j=1,\dots,d$. Even more holds in $K\{X\}_\Omega$. The following proposition easily implies that we may use results on the G-invariants $K\langle X\rangle^G$ in the free associative algebra $K\langle X\rangle$ to describe the G-invariants $K\{X\}_\Omega^G$ for an arbitrary subgroup G of $GL_d(K)$.

Proposition 4.2. (i) *Let*

$$f(x_1,\dots,x_d)=\sum f_T(x_1,\dots,x_d)\in K\{X\}_\Omega,$$

where $f_T(x_1,\dots,x_d)\in\nu_T(KX,\dots,KX)$. Then $f(x_1,\dots,x_d)$ is G-invariant if and only if $\pi_T(f_T(x_1,\dots,x_d))\in K\langle X\rangle^G$ for all Ω-trees T;

(ii) *The Hilbert series of $K\langle X\rangle^G$, $K\{x\}_\Omega$ and $K\{X\}_\Omega^G$ are related as follows. If*

$$H(K\langle X\rangle^G,t)=\sum_{m\geq 1}a_mt^m,\quad H(\{x\}_\Omega,t)=\sum_{m\geq 1}b_mt^m,$$

then

$$H(\{X\}_\Omega^G,t)=\sum_{m\geq 1}a_mb_mt^m.$$

Proof. Since $GL_d(K)$ sends $\nu_T(x_{j_1},\dots,x_{j_N})$ to a linear combination of monomials of the same kind, we obtain immediately that each G-invariant is a linear combination of G-invariants $f_T\in\nu_T(KX,\dots,KX)$, which establishes (i). The proof of (ii) follows from the equality

$$K\{X\}_\Omega^G=\bigoplus\nu_T(KX,\dots,KX)^G,$$

where the direct sum of vector spaces is on all Ω-trees T. $\square$

Let G be an arbitrary subgroup of $GL_d(K)$ and let us consider the action of G on KX. Since every basis of the vector space KX is a system of free generators of $K\{X\}_\Omega$, we fix a basis of the subspace of G-invariants $(KX)^G$ and assume that $\{x_1,\ldots,x_{d_0}\}\subset X$ is a basis of $(KX)^G$. Then we complete this system to a basis of the whole KX by $x_{d_0+1},\ldots,x_d$. Obviously, every Ω-polynomial $f(x_1,\ldots,x_{d_0})$ is a G-invariant. We call such polynomials *obvious invariants*.

Example 4.3. Let $d=4$, let G be the cyclic subgroup of $GL_4(K)$ generated by the matrix

$$g=\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

and let $\Omega=\underline{3}=\{\nu_3\}$ consist of one ternary operation only. The subspace of G-invariants of KX is two-dimensional and is spanned by x_1,x_2. The polynomial

$$f(x_1,x_2,x_3,x_4)=\nu_3(x_3,x_1,x_2)-\nu_3(x_1,x_1,x_4)$$

is a nonobvious G-invariant because $g(f)=f$ and f depends on variables different from x_1,x_2.

The next result shows that the algebra of invariants is not finitely generated in all nontrivial cases.

Theorem 4.4. *Let G be any subgroup of $GL_d(K)$ and let $K\{X\}_\Omega^G$ contain nonobvious invariants. Then the algebra $K\{X\}_\Omega^G$ is not finitely generated.*

Proof. We choose a homogeneous nonobvious invariant of minimal degree. We may assume that it is of the form

$$f=\sum_{j=1}^m\alpha_jw_j,\quad \alpha_j\in K,w_j=\nu_T(x_{j_1},\ldots,x_{j_N})\in\{X\}_\Omega,\tag{10}$$

where all w_j correspond to the same Ω-tree T and there is no w_j which depends only on the G-invariant variables $x_1,\ldots,x_{d_0}$. Let, for some $k=1,\ldots,N$, and some w_j, the k-th coordinate x_{j_k} in $w_j=\nu_T(x_{j_1},\ldots,x_{j_N})$ be a noninvariant variable, i.e., $j_k>d_0$. We order the operations from Ω first by degree and then in an arbitrary way. We fix the admissible ordering on $\{X\}_\Omega$ which compares the Ω-monomials $w=\nu_{ni}(u_1,\ldots,u_n)$ first by total degree $\deg(w)$, then by the degree $\deg_{\text{noninv}}(w)$ with respect to the noninvariant variables $x_{d_0+1},\ldots,x_d$, then by the first (outer) operation ν_{ni}, and then lexicographically. In the special case of $w = \nu_T(x_{p_1}, \dots, x_{p_N})$, where the Ω-tree T is as in (10), we start the lexicographic ordering with the k-th position. Hence,

$$w' = \nu_{n_1 i_1}(u_1, \dots, u_{n_1}) \prec \nu_{n_2 i_2}(v_1, \dots, v_{n_2}) = w''$$

means that
1) $\deg(w') < \deg(w'')$;
2) or $\deg(w') = \deg(w'')$, $\deg_{\mathrm{noninv}}(w') < \deg_{\mathrm{noninv}}(w'')$;
3) or $\deg(w') = \deg(w'')$, $\deg_{\mathrm{noninv}}(w') = \deg_{\mathrm{noninv}}(w'')$, $n_1 < n_2$ or $n_1 = n_2$, $i_1 < i_2$;
4) $\deg(w') = \deg(w'')$, $\deg_{\mathrm{noninv}}(w') = \deg_{\mathrm{noninv}}(w'')$, $\nu_{n_1 i_1} = \nu_{n_2 i_2}$ and $u_1 = v_1, \dots, u_{c-1} = v_{c-1}$, $u_c \prec v_c$.

If in 4) both w' and w'' are of the same type $w' = \nu_T(x_{p_1}, \dots, x_{p_N})$ and $w'' = \nu_T(x_{q_1}, \dots, x_{q_N})$ for T from (10), we assume that first $u_k \prec v_k$ and if $u_k = v_k$, then $u_1 = v_1, \dots, u_{c-1} = v_{c-1}$, $u_c \prec v_c$. So, without loss of generality we may assume that $k = 1$, i.e., the first coordinate x_{j_1} of some w_j is noninvariant.

We construct a sequence $f_1, f_2, \dots$ of G-invariants, starting with $f_1 = f$. If in (10) each w_j has the form

$$w_j = \nu_T(x_{j_1}, \dots, x_{j_N}) = \nu_{n1}(u_{jr_1}, \dots, u_{jr_n}), \quad u_{jr_s} \in \{X\}_\Omega,$$

we define

$$f_{k+1} = \sum_{j=1}^{m} \alpha_j \nu_{n1}(u_{jr_1}, \dots, u_{jr_{n-1}}, \nu_{n1}(u_{jr_n}, \underbrace{f_k, \dots, f_k}_{n-1})).$$

In order to prove that f_{k+1} is G-invariant, we use the $GL_d(K)$-module isomorphism (9) which is also a G-module isomorphism and define the G-module isomorphism

$$\varphi_{k+1} : \nu_T(\underbrace{KX, \dots, KX}_{n-1}, \nu_{n1}(KX, \underbrace{f_k, \dots, f_k}_{n-1})) \to (KX)^{\otimes n} \otimes (Kf_k)^{\otimes(n-1)}$$

by

$$\varphi_{k+1} : \nu_T(x_{j_1}, \dots, x_{j_{N-1}}, \nu_{n1}(x_{j_N}, f_k, \dots, f_k)) \to x_{j_1} \otimes \dots \otimes x_{j_N} \otimes f_k^{\otimes(n-1)}.$$

Then

$$\varphi_{k+1}(f_{k+1}) = \sum_{j=1}^{m} \alpha_j \pi_T(f) \otimes f_{k+1}^{\otimes(n-1)},$$

which is G-invariant.

If the leading term of f with respect to the introduced admissible ordering is

$$\overline{f} = \nu_{n1}(u_1^0, \dots, u_{n-1}^0, u_n^0), \quad u_1^0, \dots, u_{n-1}^0, u_n^0 \in \{X\}_\Omega,$$

then the Ω-monomial u_1^0 depends also on a noninvariant variable. Also, it is easy to see that the leading term of f_{k+1} is

$$\nu_{n1}(u_1^0, \dots, u_{n-1}^0, \nu_{n1}(u_n^0, \underbrace{\overline{f_k}, \dots, \overline{f_k}}_{n-1})).$$

Now, let the algebra $K\{X\}_\Omega^G$ of G-invariants be finitely generated by some $h_1, \dots, h_m$. We may assume that the generators are homogeneous. We choose a sufficiently large k such that $\deg(f_{k+1}) > \deg(h_s)$, $s = 1, \dots, m$. Since f_{k+1} belongs to the Ω-subalgebra of $K\{X\}_\Omega$ generated by $h_1, \dots, h_m$, the leading term $\overline{f_{k+1}}$ of f_{k+1} can be expressed as an Ω-monomial of the leading terms $\overline{h_s}$ and is different from them. Hence

$$\overline{f_{k+1}} = \nu_{n1}(u_1^0, \dots, u_{n-1}^0, \nu_{n1}(u_n^0, \underbrace{\overline{f_k}, \dots, \overline{f_k}}_{n-1})) = \nu_{n_1 i_1}(v_1, \dots, v_{n_1}),$$

where each v_p is the leading term of an element of the Ω-subalgebra generated by $h_1, \dots, h_m$. This implies that $\nu_{n1} = \nu_{n_1 i_1}$ and $u_1^0 = v_1, \dots, u_{n-1}^0 = v_{n-1}$, $\nu_{n1}(u_n^0, \overline{f_k}, \dots, \overline{f_k}) = v_n$. Hence u_1^0 is a leading term of a G-invariant element. This is impossible because $\deg(f) > \deg(u_1^0)$ and $\deg(f)$ is the minimal degree of an invariant depending not only on G-invariant variables. $\square$

As an illustration of the proof, continuing Example 4.3, we can now start with

$$f_1 = f = f(x_1, x_2, x_3, x_4) = \nu_3(x_3, x_1, x_2) - \nu_3(x_1, x_1, x_4)$$

and construct

$$f_{k+1} = \nu_3(x_3, x_1, \nu_3(x_2, f_k, f_k)) - \nu_3(x_1, x_1, \nu_3(x_4, f_k, f_k)).$$

Remark 4.5. The main steps of the description of Koryukin [Ko1] of the finitely generated algebras of invariants $K\langle X\rangle^G$ can be applied also to the case of the free binary algebra $K\{X\}$. Tracing his proof we obtain that if $K\{X\}^G$ is finitely generated and $K\{X\}^G \subset K\{x_1, \dots, x_{d_1}\}$, where d_1 is minimal with this property (with respect to all linear changes of the system of generators of $K\{X\}$), then G acts on $Kx_1 \oplus \dots \oplus Kx_{d_1}$ as a finite cyclic group by scalar multiplications. If the order of this cyclic group is equal to r, then the final arguments of our proof of Theorem 4.4 show that the elements $x_1^{r-1}(x_1x_1^{rk})$, $k=1,2,\dots$, do not belong to any finitely generated subalgebra of $K\{X\}^G$.

Corollary 4.6. *Let $\Omega\neq\emptyset$, i.e., $K\{X\}_\Omega$ is not the vector space KX with trivial multiplication and let $G\neq\langle e\rangle$ be a finite subgroup of $GL_d(K)$. If the algebra of invariants $K\{X\}_\Omega^G$ is nonzero, then it is not finitely generated.*

Proof. If $K\{X\}_\Omega^G$ contains a nonobvious invariant, then we apply directly Theorem 4.4. Let us assume that all invariants are obvious and depend on the G-invariant variables $x_1,\dots,x_{d_0}$. Since $K\{X\}_\Omega^G\neq 0$ and $G\neq\langle e\rangle$, we derive that $0<d_0<d$. We use the well known fact that the tensor powers of any faithful representation of a finite group contain all irreducible representations of the group, including the trivial representation. Applying the theorem of Maschke, we choose the variables $x_{d_0+1},\dots,x_d$ to span a G-invariant complement of $Kx_1\oplus\dots\oplus Kx_{d_0}$. The representation of G in $\text{span}\{x_{d_0+1},\dots,x_d\}$ is faithful. Hence there exists a G-invariant $h\in(\text{span}\{x_{d_0+1},\dots,x_d\})^{\otimes p}$ of the form

$$h(x_{d_0+1},\dots,x_d)=\sum_{j=1}^m\alpha_jx_{j_1}\otimes\dots\otimes x_{j_p},\quad \alpha_j\in K,\quad j_1,\dots,j_p>d_0.$$

We choose an Ω-tree T with N leaves, $N>p$, and consider the associated operation ν_T. Now we consider the isomorphism π_T from (9). The variable x_1 is G-invariant and the Ω-polynomial

$$\pi_T^{-1}\left(\left(\sum_{j=1}^m\alpha_jx_{j_1}\otimes\dots\otimes x_{j_p}\right)\otimes x_1^{\otimes(N-p)}\right)$$
$$=\sum_{j=1}^m\alpha_j\nu_T(x_{j_1},\dots,x_{j_p},\underbrace{x_1,\dots,x_1}_{N-p})$$

is a nonobvious G-invariant. This contradiction completes the proof. □

Let us add in Example 4.1 one more variable x_0 which is fixed by G. Then G is generated by the $(d+1)\times(d+1)$ matrix

$$g=\begin{pmatrix}1&0&\cdots&0\\0&-1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&-1\end{pmatrix},$$

and $f = \nu_3(x_1, x_1, x_0)$ is a nonobvious G-invariant because

$$g(f) = \nu_3(g(x_1), g(x_1), g(x_0)) = \nu_3(-x_1, -x_1, x_0) = \nu_3(x_1, x_1, x_0) = g.$$

The proof of Dicks and Formanek [DF] that the finite cyclic groups G which act by scalar multiplication are the only groups such that the algebra $K\langle X\rangle^G$ is finitely generated uses ideas very different from the proof of Theorem 4.4 given above. Dicks and Formanek proved that for any finite group G the Hilbert series of $K\langle X\rangle^G$ satisfies

$$H(K\langle X\rangle^G, t) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{1-\mathrm{tr}_{KX}(g)t}, \tag{11}$$

where $\mathrm{tr}_{KX}(g)$ is the trace of the linear operator g in the d-dimensional vector space KX. This is an analogue of the classical Molien formula

$$H(K[X]^G, t) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(1-gt)}. \tag{12}$$

Later, Formanek [F1] generalized this result to the case of any factor algebra $K\langle X\rangle/J$ of $K\langle X\rangle$ modulo a $GL_d(K)$-invariant ideal J. The algebra $K\langle X\rangle/J$ inherits the multigrading of $K\langle X\rangle$. Let G be any finite subgroup of $GL_d(K)$ and let $\xi_1(g), \dots, \xi_d(g)$ be the eigenvalues of the $d\times d$ matrix $g \in G$. Then

$$H((K\langle X\rangle/J)^G, t) = \frac{1}{|G|}\sum_{g\in G} H(K\langle X\rangle/J, \xi_1(g)t, \dots, \xi_d(g)t). \tag{13}$$

Since the Hilbert series of $K[X]$ and $K\langle X\rangle$ are, respectively,

$$H(K[X], t_1, \dots, t_d) = \prod_{j=1}^{d}\frac{1}{1-t_j},$$

$$H(K\langle X\rangle, t_1, \dots, t_d) = \frac{1}{1-(t_1+\dots+t_d)},$$

and since

$$\prod_{j=1}^{d}(1-\xi_j(g)t) = \det(1-gt),$$

$$\sum_{j=1}^{d}\xi_j(g)t = \mathrm{tr}(gt),$$

the formula (13) gives immediately (12) and (11). Applied to invariants of free Ω-algebras, (13) implies the following.

Proposition 4.7. *Let G be a finite subgroup of $GL_d(K)$. Then*

$$H(K\{X\}_\Omega^G, t) = \frac{1}{|G|}\sum_{g\in G} H(K\{X\}_\Omega, \xi_1(g)t, \dots, \xi_d(g)t), \tag{14}$$

where $\xi_1(g), \dots, \xi_d(g)$ are the eigenvalues of the $d \times d$ matrix $g \in G$.

Proof. We recall the main steps of the proof of [F1], see also Section 6.3 of [Dr3], where the proof of (13) is given as a sequence of exercises. The first fact we need is that for a finite dimensional vector space W and a finite subgroup G of $GL(W)$, the vector space of G-invariants in W, i.e., $W^G = \{w \in W \mid g(w) = w, \quad g \in G\}$ coincides with the image of the Reynolds operator $\phi : W \to W$ defined by

$$\phi(w) = \frac{1}{|G|}\sum_{g\in G} g(w)$$

and $\dim(W^G) = \mathrm{tr}_W(\phi)$, the trace of ϕ acting on W. Further, if g is a diagonalizable matrix acting on KX (which is true when g is of finite order), and W is a $GL_d(K)$-invariant multihomogeneous finite dimensional vector subspace of $K\langle X\rangle$, then the Hilbert series $H(W, t_1, \dots, t_d)$ of W is a symmetric polynomial in $t_1, \dots, t_d$ and the trace of g acting on W is

$$\mathrm{tr}_W(g) = H(W, \xi_1(g), \dots, \xi_d(g)). \tag{15}$$

In particular, for the homogeneous components $W^{(k)}$ of degree k

$$\mathrm{tr}_{W^{(k)}}(g)t^k = H(W^{(k)}, \xi_1(g)t, \dots, \xi_d(g)t). \tag{16}$$

Finally, the equations (15) and (16) imply for the graded subspace of the G-invariants in W

$$H(W^G, t) = \sum_{k\geq 0} \dim((W^G)^{(k)})t^k = \sum_{k\geq 0} \mathrm{tr}_{W^{(k)}}(\phi)t^k$$

$$= \frac{1}{|G|}\sum_{g\in G}\sum_{k\geq 0} \mathrm{tr}_{W^{(k)}}(g)t^k = \frac{1}{|G|}\sum_{g\in G} H(W, \xi_1(g)t, \dots, \xi_d(g)t).$$

Since each homogeneous component $K\{X\}_\Omega^{(k)}$ of $K\{X\}_\Omega$ is isomorphic as a $GL_d(K)$-module to a direct sum of several copies of $K\langle X\rangle^{(k)}$, the equations (15) and (16) hold also for the finite dimensional $GL_d(K)$-submodules W of $K\{X\}_\Omega$. This implies the formula (14). □

Example 4.8. Let $G = \{e, -e\}$ act on the free nonassociative binary algebra $K\{x\}$, where $-e$ changes the sign of x. As in Example 4.1, $K\{x\}^G$ is spanned by all homogeneous monomials of even degree. Proposition 4.7 gives that

$$H(K\{x\}^G, t) = \frac{1}{2}\left(H(K\{x\}, t) + H(K\{x\}, -t)\right)$$

(which can be seen also directly). One can replace $H(K\{x\}, t)$ with its explicit form (4), as the generating function of the Catalan numbers, and obtain

$$H(K\{x\}^G, t) = \frac{1}{4}\left(2 - \sqrt{1-4t} - \sqrt{1+4t}\right).$$

Using the formula (7) we derive for the generating function of any homogeneous set Y of free generators of $K\{x\}^G$

$$G(Y, t) = \frac{1 - \sqrt{1-16t^2}}{8} = \frac{1}{4}H(K\{x\}, 4t^2). \tag{17}$$

Instead, we may proceed in the following way, with possible generalizations. Let $H = H(K\{x\}, t) = H_0 + H_1$, where

$$H_0 = \frac{1}{2}(H(t) + H(-t)), \quad H_1 = \frac{1}{2}(H(t) - H(-t))$$

are, respectively, the even and the odd components of the series H. Since

$$0 = H^2 - H + t = (H_0^2 + H_1^2 - H_0) + (2H_0H_1 - H_1 + t),$$

we separate the even and odd parts of this equation and obtain

$$H_0^2 + H_1^2 - H_0 = 2H_0H_1 - H_1 + t = 0, \quad H_1 = \frac{t}{1 - 2H_0},$$

$$H_0^2 + \frac{t^2}{(1-2H_0)^2} - H_0 = 0, \quad (H_0^2 - H_0)(1 + 4(H_0^2 - H_0)) + t^2 = 0.$$

Using that $H_0^2 - H_0 + G(Y, t) = 0$, we derive

$$-G(Y,t)(1 - 4G(Y,t)) + t^2 = 0, \quad (4G(Y,t))^2 - (4G(Y,t)) + 4t^2 = 0.$$

Hence, by (2) we obtain $4G(Y, t) = H(4t^2)$, i.e., (17) and

$$g_{2k} = \frac{1}{4^{k-1}}c_k, \quad k \geq 1, \tag{18}$$

where g_{2k} are the nonzero coefficients of the generating function $G(Y, t)$.

Since $K\{x\}^G$ is spanned by all nonassociative monomials of even degree, the monomials of the form uv, where both u, v are of odd degree form a free generating set of $K\{x\}^G$. Applying the Stirling formula to c_{2k} and g_{2k} we obtain

$$c_{2k} = \frac{1}{2k}\binom{4k-2}{2k-1} = \frac{(2(2k-1))!}{2k((2k-1)!)^2} \approx \frac{4^{2k}}{8k\sqrt{\pi(2k-1)}},$$

$$g_{2k} = \frac{4^{k-1}(2(k-1))!}{k((k-1)!)^2} \approx \frac{4^{2k}}{16k\sqrt{\pi(k-1)}},$$

$$\frac{g_{2k}}{c_{2k}} \approx \frac{1}{2}\sqrt{\frac{2k-1}{k-1}}, \quad \lim_{k\to\infty}\frac{g_{2k}}{c_{2k}} = \frac{\sqrt{2}}{2}.$$

Hence the quotient between the number of nonassociative monomials which are products of two monomials of odd degree to the number of all monomials of the same degree tends to $\sqrt{2}/2$. For example,

$$c_2 = 1, \quad g_2 = 1, \quad \frac{g_2}{c_2} = 1,$$

$$c_4 = 5, \quad g_4 = 4, \quad \frac{g_4}{c_4} = 0.8,$$

$$c_6 = 42, \quad g_6 = 32, \quad \frac{g_6}{c_6} \approx 0.761905,$$

$$c_8 = 429, \quad g_8 = 320, \quad \frac{g_8}{c_8} \approx 0.745921,$$

$$c_{10} = 4862, \quad g_{10} = 3584, \quad \frac{g_{10}}{c_{10}} \approx 0.737145,$$

$$c_{12} = 58786, \quad g_{12} = 43008, \quad \frac{g_{12}}{c_{12}} \approx 0.731603,$$

$$c_{14} = 742900, \quad g_{14} = 540672, \quad \frac{g_{14}}{c_{14}} \approx 0.727786,$$

$$c_{16} = 9694845, \quad g_{16} = 7028736, \quad \frac{g_{16}}{c_{16}} \approx 0.724997,$$

$$\frac{g_{20}}{c_{20}} \approx 0.721197, \quad \frac{g_{30}}{c_{30}} \approx 0.716308, \quad \frac{g_{40}}{c_{40}} \approx 0.713938,$$

$$\frac{g_{50}}{c_{50}} \approx 0.712539, \quad \frac{g_{100}}{c_{100}} \approx 0.709790,$$

which is very close to

$$\frac{\sqrt{2}}{2} \approx 0.707105.$$

The monomials which are products of two monomials of odd degree can be labeled by planar binary rooted trees with two branches with an odd number of leaves for each branch. Hence there are much more of such trees (about

$$\frac{\sqrt{2}/2}{1-\sqrt{2}/2} \approx 2.41420$$

times) than of planar binary rooted trees with the same number of leaves which have two branches with an even number of leaves.

In the case of infinite groups G the Molien formula (12) has no formal sense. Nevertheless, if G is compact, one can define Haar measure on G, replace the sum with an integral and obtain the Molien-Weyl formula for the Hilbert series of the algebra of invariants, see [We1, We2]. The analogue of (11) for G infinite is given by Almkvist, Dicks and Formanek [ADF]. For other applications of the Molien-Weyl formula for objects related with noncommutative algebra see also [F1], [Dr2] and the books [F2], [DrF]. We shall consider such applications in the next section.

Without presenting a comprehensive survey, we shall mention several results of action of other objects, different from groups, in the spirit of invariant theory. Instead of invariants of linear groups one may consider constants of derivations. A theorem of Jooste [J] and Kharchenko [Kh1] gives that for a Lie algebra D of linear derivations the algebra of constants

$$K\langle X\rangle^D = \{f \in K\langle X\rangle \mid \delta(f) = 0, \delta \in D\}$$

is free again. See also Koryukin [Ko2] and Ferreira and Murakami [FM1] for the problem of finite generation of $K\langle X\rangle^D$. One may study also invariants of Hopf algebras acting on $K\langle X\rangle$. We shall mention only [FMP] and [FM2] which show that the algebra of invariants of $K\langle X\rangle$ under the linear action of a Hopf algebra H is free and, under some mild restrictions on H, the algebra $K\langle X\rangle^H$ is finitely generated only if the action of H is scalar. It is a natural problem to study algebras of constants of derivations and invariants of Hopf algebras also in the case of free Ω-algebras.

5. Weitzenböck derivations, special linear groups and elliptic integrals

As in the previous section, we assume that K is a field of characteristic 0 and $X = \{x_1, \ldots, x_d\}$. Almkvist, Dicks and Formanek [ADF] studied invariants of the special linear group $SL_p(K)$ and the unitriangular group $UT_p(K)$ acting on the free associative algebra $K\langle X\rangle$. They expressed the Hilbert series of the algebra of invariants in terms of multiple integrals. We shall transfer these results to the case of $K\{X\}_\Omega$. We shall consider in detail the case $p = 2$ only. Also, instead for the unitriangular group $UT_2(K)$ we shall state the results for Weitzenböck derivations and unipotent actions of the infinite cyclic group.

For details on representation theory of general linear groups see e.g. the books by Macdonald [Mc] and Weyl [We2]. Let $W(\lambda) = W(\lambda_1, \lambda_2)$ be the irreducible $GL_2(K)$-module corresponding to the partition $\lambda = (\lambda_1, \lambda_2)$. The role of the character of $W(\lambda)$ is played by the Schur function $s_\lambda(u_1, u_2)$. This means that if the matrix $g \in GL_2(K)$ has eigenvalues ξ_1, ξ_2, then g acts as a linear operator on $W(\lambda)$ with trace $\mathrm{tr}_{W(\lambda)}(g) = s_\lambda(\xi_1, \xi_2)$. In the case of two variables $s_\lambda(u_1, u_2)$ has the simple form

$$s_\lambda(u_1, u_2) = u_1^{\lambda_1} u_2^{\lambda_2} + u_1^{\lambda_1 - 1} u_2^{\lambda_2 + 1} + \cdots + u_1^{\lambda_2 + 1} u_2^{\lambda_1 - 1} + u_1^{\lambda_2} u_2^{\lambda_1}$$

$$= (u_1 u_2)^{\lambda_2} \frac{u_1^{\lambda_1 - \lambda_2 + 1} - u_2^{\lambda_1 - \lambda_2 + 1}}{u_1 - u_2}.$$

The Schur functions form a basis of the vector space of symmetric polynomials in $K[u_1, u_2]$.

Recall that a linear operator δ of the K-algebra R is called a derivation if $\delta(uv) = \delta(u)v + u\delta(v)$ for all $u, v \in R$. Similarly, the linear operator δ is a derivation of the Ω-algebra R if

$$\delta(\nu_{ni}(v_1, \ldots, v_n)) = \sum_{j=1}^{n} \nu_{ni}(v_1, \ldots, \delta(v_j), \ldots, v_n) \tag{19}$$

for all $v_j \in R$ and all $\nu_{ni} \in \Omega$. The derivation is locally nilpotent if for any $v \in R$ there is a k such that $\delta^k(v) = 0$. If δ is a locally nilpotent derivation, then the exponential of δ

$$\exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots$$

is well defined and is an automorphism of R. The kernel $\ker(\delta) = R^\delta$ of δ is called the algebra of constants of δ. It coincides with the algebra $R^{\exp(\delta)}$ of the fixed points of the automorphism $\exp(\delta)$. Every mapping $\delta : X \to K\{X\}_\Omega$ can be extended to a derivation of $K\{X\}_\Omega$: If $v_1, \ldots, v_n$ are monomials in $\{X\}_\Omega$, then we define $\delta(\nu_{ni}(v_1, \ldots, v_n))$ inductively by (19).

Let $g \in GL_d(K)$ be a unipotent linear operator acting on the vector space $KX = Kx_1 \oplus \cdots \oplus Kx_d$. By the classical theorem of Weitzenböck [W], the (commutative and associative) algebra of invariants

$$K[X]^g = \{f \in K[X] \mid f(g(x_1), \ldots, g(x_d)) = f(x_1, \ldots, x_d)\}$$

is finitely generated. All eigenvalues of g are equal to 1 and, up to a change of the basis, g is determined uniquely by its Jordan normal form. Hence, for each fixed d we may consider only a finite number of linear unipotent operators. Equivalently, we may consider the linear locally nilpotent derivation $\delta = \log g$ called a Weitzenböck derivation. Clearly, the algebra of invariants $K[X]^g$ coincides with the algebra of constants $K[X]^\delta$ ($=\ker(\delta)$). See the book of Nowicki [N] for concrete generators of $K[X]^\delta$ for small d and the book by Freudenburg [Fr] for general information on locally nilpotent derivations of polynomial algebras. The paper by Drensky and Gupta [DrG] deals with Weitzenböck derivations for free and relatively free associative and Lie algebras. We present a short account on the properties of the algebra of constants $K\{X\}_\Omega^\delta$ and show how to calculate the Hilbert series of $K\{X\}_\Omega^\delta$. The proofs are based on the description of the invariants of the group of unitriangular matrices given by De Concini, Eisenbud and Procesi [DEP] and the work of Almkvist, Dicks and Formanek [ADF]. We assume that δ is a linear locally nilpotent derivation. It acts on KX as a nilpotent linear operator, with Jordan normal form consisting of k cells of size $n_1+1,\dots,n_k+1$, respectively, and X is a Jordan basis of δ. Hence either $\delta(x_j) = x_{j-1}$ or $\delta(x_j) = 0$, $j = 1,\dots,d$.

We equip KX with a $GL_2(K)$-module structure in the following way. If $X_r = \{x_{j_0}, x_{j_0+1},\dots,x_{j_0+n_r}\}$ is the part of the basis X corresponding to the r-th Jordan cell of δ, we assume that $GL_2(K)$ acts on KX_r as on the $GL_2(K)$-module of commutative and associative polynomials, homogeneous of degree n_r, in two variables x, y: If $g \in GL_2(K)$ and

$$g(x^{n_r-m}y^m) = \sum_{q=0}^{n_r} \alpha_{qm} x^{n_r-q} y^q$$

for some $\alpha_{qm} \in K$, then

$$g(x_{j_0+m}) = \sum_{q=0}^{n_r} \alpha_{qm} x_{j_0+q}.$$

Hence KX is isomorphic to the direct sum $W(n_1,0)\oplus\cdots\oplus W(n_k,0)$ as $GL_2(K)$-module. We extend the action of $GL_2(K)$ diagonally to $K\{X\}_\Omega$. Then, see [DEP] and [ADF], each irreducible $GL_2(K)$-submodule $W(\lambda_1,\lambda_2)$ of $K\{X\}_\Omega$ contains a one-dimensional δ-constant subspace and the algebra $K\{X\}_\Omega^\delta$ is spanned by these subspaces. We define on $K\{X\}_\Omega$ a $\mathbb{Z}^3$-grading assuming that the degree of x_{j_0+m} from X_r is equal to $(n_r - m, m, 1)$ and consider the Hilbert series

$$H_\delta(K\{X\}_\Omega, u_1, u_2, z) = H(K\{x\}_\Omega, z \sum_{q=1}^{k} s_{(n_r,0)}(u_1, u_2)). \qquad (20)$$

It is obtained from the Hilbert series $H(K\{X\}_\Omega, t_1, \dots, t_d)$ by replacing the variables $t_1, t_2, \dots, t_d$ respectively by

$$u_1^{n_1} z, u_1^{n_1-1} u_2 z, \dots, u_1 u_2^{n_1-1} z, u_2^{n_1} z, \dots,$$
$$u_1^{n_k} z, u_1^{n_k-1} u_2 z, \dots, u_1 u_2^{n_k-1} z, u_2^{n_k} z.$$

The variables u_1, u_2 take into account the bigrading related to the $GL_2(K)$-action and the extra variable z counts the usual grading.

The function $H_\delta(K\{X\}_\Omega, u_1, u_2, z)$ is symmetric in u_1, u_2. The coefficient of its linear in z component is equal to

$$\sum_{i=1}^{k} s_{(n_i,0)}(u_1, u_2)$$

which is the character of the $GL_2(K)$-module KX.

Hence $H_\delta(K\{X\}_\Omega, u_1, u_2, z)$ is the character of the $GL_2(K)$-module $K\{X\}_\Omega$. By [ADF], this means that if

$$H_\delta(K\{X\}_\Omega, u_1, u_2, z) = \sum_{q \geq 1} \left(\sum_{\lambda \vdash q} m_\lambda s_\lambda(u_1, u_2) \right) z^q,$$

then the homogeneous component of degree q decomposes as

$$K\{X\}_\Omega^{(q)} = \bigoplus_{\lambda \vdash q} m_\lambda W(\lambda_1, \lambda_2). \qquad (21)$$

Theorem 5.1. *Let δ be a linear locally nilpotent derivation of $K\{X\}_\Omega$, $X = \{x_1, \dots, x_d\}$, which, when acting on KX, has a Jordan normal form consisting of k cells of size $n_1 + 1, \dots, n_k + 1$, respectively. Then the Hilbert series of the algebra of constants $K\{X\}_\Omega^\delta$ is given by*

$$H(K\{X\}_\Omega^\delta, z) = 2 \int_0^1 \cos^2(\pi u) H_\delta(K\{X\}_\Omega, e^{2\pi i u}, e^{-2\pi i u}, z) du, \qquad (22)$$

where $H_\delta(K\{X\}_\Omega, u_1, u_2, z)$ is defined in (20). *Equivalently,*

$$H(K\{X\}_\Omega^\delta, z) = 2 \int_0^1 \cos^2(\pi u) H \left(K\{x\}_\Omega, z \sum_{i=1}^{k} \frac{\sin(2\pi(n_i+1)u)}{\sin(2\pi u)} \right) du. \qquad (23)$$

Proof. We follow the proof of Almkvist, Dicks, and Formanek [ADF] for the Hilbert series of $K\langle X\rangle^\delta$. If $K\{X\}_\Omega^{(q)}$ decomposes as in (21), then the Hilbert series of the algebra of δ-constants is

$$H(K\{X\}_\Omega^\delta, z) = \sum_{q\geq 1}\left(\sum_{\lambda\vdash q} m_\lambda\right) z^q.$$

Hence, for the proof of (22) it is sufficient to show that

$$2\int_0^1 \cos^2(\pi u) s_\lambda(e^{2\pi i u}, e^{-2\pi i u})du = 1,$$

which was already used in the proof of [ADF] (or may be verified directly). The expression (23) follows from the formula

$$s_{(n,0)}(e^{2\pi i u}, e^{-2\pi i u}) = \frac{e^{2\pi i(n+1)u} - e^{-2\pi i(n+1)u}}{e^{2\pi i u} - e^{-2\pi i u}} = \frac{\sin(2\pi(n+1)u)}{\sin(2\pi u)}.$$

□

Example 5.2. Let $K\{X\}_\Omega = K\{X\}$ be the free nonassociative algebra, i.e., Ω consist of one binary operation only.

(i) Let $d = 2$ and $\delta(x_1) = 0$, $\delta(x_2) = x_1$, i.e.,

$$\delta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{24}$$

By [DrG] the Hilbert series of $K\langle X\rangle^\delta$ (for the nonunitary algebra $K\langle X\rangle$) is

$$H(K\langle X\rangle^\delta, t) = \sum_{p\geq 0}\binom{2p+1}{p}t^{2p+1} + \sum_{p\geq 1}\binom{2p}{p}t^{2p}.$$

The algebra $K\langle X\rangle^\delta$ is a free associative algebra and has a homogeneous set of free generators Y with generating function

$$G(Y,t) = t + \sum_{p\geq 1} c_{p+1}t^{2p}. \tag{25}$$

Hence Proposition 4.2 gives

$$H(K\langle X\rangle^\delta, t) = \sum_{p\geq 0}\binom{2p+1}{p}c_{2p+1}t^{2p+1} + \sum_{p\geq 1}\binom{2p}{p}c_{2p}t^{2p},$$

where c_n are the Catalan numbers.

Applying Theorem 5.1 we obtain

$$H(K\{x\},t) = \frac{1-\sqrt{1-4t}}{2},$$

$$H_\delta(K\{X\},u_1,u_2,z) = \frac{1-\sqrt{1-4(u_1+u_2)z}}{2},$$

$$H(K\{X\}^\delta,z) = \int_0^1 \cos^2(\pi u)\left(1-\sqrt{1-8z\cos(2\pi u)}\right)du,$$

which is an elliptic integral. If Y is a homogeneous set of free generators of $K\{X\}^\delta$, then its generating function is

$$G(Y,z) = H(K\{X\}^\delta,z) - H(K\{X\}^\delta,z)^2.$$

The beginning of the expansion of $G(Y,z)$ as a formal power series is

$$G(Y,z) = z + z^2 + 2z^3 + 14z^4 + 56z^5 + 404z^6 + 2020z^7 + \cdots.$$

(Compare with the expansion of the generating function (25) of the free generators of $K\langle X\rangle^\delta$.) For the free generators of degree ≤ 3 of $K\{X\}^\delta$ one may choose

$$x_1, \quad x_1x_2 - x_2x_1, \quad (x_1x_1)x_2 - (x_2x_1)x_1, \quad x_1(x_1x_2) - x_2(x_1x_1).$$

(ii) Let $d = 3$ and

$$\delta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

Then (22) and (23) give

$$H(K\{X\}^\delta,z) = \int_0^1 \cos^2(\pi u)\left(1-\sqrt{1-4z(1+2\cos(4\pi u))}\right)du.$$

The beginning of the generating expansion for the free generators is

$$z + 2z^2 + 8z^3 + 58z^4 + 440z^5 + 3728z^6 + 33088z^7 + \cdots.$$

Example 5.3. Let $K\{X\}_\Omega = K\{X\}_\omega$, i.e., Ω_n consists of one operation for each $n \geq 2$. Applying Theorem 5.1 to (5) in Example 2.3 (ii) we obtain for $d = 2$ and δ from (24)

$$H(K\{X\}_\omega^\delta,z) = \frac{1}{2}\int_0^1 \cos^2(\pi u) f(2z\cos(2\pi u))du,$$

where

$$f(t) = 1 + t - \sqrt{1 - 6t + t^2}.$$

For $d = 3$ and δ from (26), we obtain again elliptic integrals:

$$H(K\{X\}^{\delta}_{\omega}, z) = \frac{1}{2}\int_0^1 \cos^2(\pi u) f(z(1 + 2\cos(4\pi u)))du,$$

which is, in low degrees,

$$z + 3z^2 + 21z^3 + 209z^4 + 2295z^5 + 27777z^6 + 354879z^7 + \cdots$$

Following Almkvist, Dicks and Formanek [ADF], we consider a polynomial representation of $GL_2(K)$ in KX, i.e., we assume that KX has the $GL_2(K)$-module structure

$$KX \cong W(\lambda_1^{(1)}, \lambda_2^{(1)}) \oplus \cdots \oplus W(\lambda_1^{(k)}, \lambda_2^{(k)}). \quad (27)$$

This induces also a representation of $SL_2(K) \subset GL_2(K)$. We translate the results of [ADF] for the Hilbert series of $K\langle X\rangle^{SL_2(K)}$ to the case of $K\{X\}^{SL_2(K)}_{\Omega}$.

Theorem 5.4. *Let the $GL_2(K)$-module structure of KX, $X = \{x_1, \ldots, x_d\}$, be given by (27). Then the Hilbert series of the algebra of $SL_2(K)$-invariants in $K\{X\}_{\Omega}$ is*

$$\begin{aligned} H(K\{X\}^{SL_2(K)}_{\Omega}, z) &= \\ &= 2\int_0^1 \sin^2(2\pi u) H\left(K\{x\}_{\Omega}, z\sum_{i=1}^{k} s_{(\lambda_1^{(i)}, \lambda_2^{(i)})}(e^{2\pi i u}, e^{-2\pi i u})\right) du = \\ &= 2\int_0^1 \sin^2(2\pi u) H\left(K\{x\}_{\Omega}, z\sum_{i=1}^{k} \frac{\sin(2\pi(\lambda_1^{(i)} - \lambda_2^{(i)} + 1)u)}{\sin(2\pi u)}\right) du. \end{aligned}$$

Proof. As in the proof of Theorem 5.1, it is sufficient to take into account that $W(\lambda_1, \lambda_2)$ contains a one-dimensional $SL_2(K)$-invariant if $\lambda_1 = \lambda_2$ and does not contain $SL_2(K)$-invariants otherwise. Then we need to show that

$$2\int_0^1 \sin^2(2\pi u) s_{\lambda}(e^{2\pi i u}, e^{-2\pi i u}) du = \delta_{\lambda_1\lambda_2},$$

where δ_{pq} is the Kronecker delta. This can be checked directly and was already used in [ADF]. □

Example 5.5. If $d = 2$ and $GL_2(K)$ acts naturally on KX, i.e. $KX \cong W(1,0)$, then the results in [DrG] for the Hilbert series $H(K\langle X\rangle^{SL_2(K)}, t)$ of the $SL_2(K)$-invariants of the nonunitary free associative algebra give

$$H(K\langle X\rangle^{SL_2(K)}, t) = \sum_{p\geq 1} c_{p+1}t^{2p} = \frac{1 - 2t^2 - \sqrt{1-4t^2}}{2t^2}.$$

By Proposition 4.2 (ii) we obtain that

$$H(\{X\}^{SL_2(K)}, z) = \sum_{p\geq 1} c_{2p}c_{p+1}z^{2p}.$$

On the other hand, Theorem 5.4 gives

$$H(K\{X\}^{SL_2(K)}, z) = \int_0^1 \sin^2(2\pi u)\left(1 - \sqrt{1 - 8z\sin(2\pi u)}\right) du$$

which is again an elliptic integral.

Remark 5.6. In order to obtain Ω-analogues of the results in [ADF] for the invariants of $SL_r(K)$ and $UT_d(K)$ we equip KX with the structure of a $GL_r(K)$-module with character (the trace of $g \in GL_r(K)$ acting on KX)

$$T_{KX}(u_1, \ldots, u_r) = \sum_{i=1}^{k} s_{\lambda(i)}(u_1, \ldots, u_r).$$

Then we replace in [ADF] the GL_r-character

$$H(K\langle X\rangle, u_1, \ldots, u_r) = \frac{1}{1 - zT_{KX}(u_1, \ldots, u_r)}$$

with $H(K\{x\}_\Omega, z\sum_{i=1}^{k} zT_{KX}(u_1, \ldots, u_r))$ and obtain integral expressions for $H(K\{X\}_\Omega^{SL_r(K)}, z)$ and $H(K\{X\}_\Omega^{UT_r(K)}, z)$.

Acknowledgements

The first author wants to thank for the hospitality at the Department of Mathematics of the Ruhr University in Bochum during his visit there. Both authors thank Lothar Gerritzen for stimulating discussions about the subject. They are especially indebted to the anonymous referee for the historical comments which led to the improvement of the exposition and the extension of the list of references.

References

[ADF] G. Almkvist, W. Dicks, E. Formanek, *Hilbert series of fixed free algebras and noncommutative classical invariant theory*, J. Algebra **93** (1985), 189-214.

[Be] G.M. Bergman, *The diamond lemma for ring theory*, Adv. Math. **29** (1978), 178-218.

[Br] R.M. Bryant, *On the fixed points of a finite group acting on a free Lie algebra*, J. London Math. Soc. (2) **43** (1991), 215-224.

[BA] M.S. Burgin, V.A. Artamonov, *Certain properties of subalgebras in varieties of linear Ω-algebras* (Russian), Mat. Sbornik **87** (1972), No. 1, 67-82. Translation: Math. USSR, Sb. **16** (1972), 69-85.

[BH] E. Burgunder, R. Holtkamp, *Partial magmatic bialgebras*, Preprint 2007.

[C1] P.M. Cohn, *Subalgebras of free associative algebras*, Proc. Lond. Math. Soc., III. Ser. 14 (1964), 618-632.

[C2] P.M. Cohn, *Free Rings and Their Relations*, Second Edition, London Mathematical Society Monographs, **19**, Academic Press, London, 1985.

[DEP] C. De Concini, D. Eisenbud, C. Procesi, *Young diagrams and determinantal varieties*, Invent. Math. **56** (1980), 129-165.

[DF] W. Dicks, E. Formanek, *Poincaré series and a problem of S. Montgomery*, Lin. Multilin. Algebra **12** (1982), 21-30.

[Dr1] V. Drensky, *Fixed algebras of residually nilpotent Lie algebras*, Proc. Amer. Math. Soc. **120** (1994), 1021-1028.

[Dr2] V. Drensky, *Commutative and noncommutative invariant theory*, Math. and Education in Math., Proc. of the 24-th Spring Conf. of the Union of Bulgar. Mathematicians, Svishtov, April 4-7, 1995, Sofia, 1995, 14-50.

[Dr3] V. Drensky, *Free Algebras and PI-Algebras*, Springer-Verlag, Singapore, 1999.

[DrF] V. Drensky, E. Formanek, *Polynomial Identity Rings*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser, Basel – Boston, 2004.

[DrG] V. Drensky, C.K. Gupta, *Constants of Weitzenböck derivations and invariants of unipotent transformations acting on relatively free algebras*, J. Algebra **292** (2005), 393-428.

[FM1] V.O. Ferreira, L.S.I. Murakami, *Finitely generated constants of free algebras*, Groups, Rings and Group Rings, 149-153, Lect. Notes Pure Appl. Math., **248**, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[FM2] V.O. Ferreira, L.S.I. Murakami, *Finitely generated invariants of Hopf algebras on free associative algebras*, Linear Algebra Appl. **420** (2007), 70-78.

[FMP] V.O. Ferreira, L.S.I. Murakami, A. Paques, *A Hopf-Galois correspondence for free algebras*, J. Algebra **276** (2004), 407-416.

[F1] E. Formanek, *Noncommutative invariant theory*, Contemp. Math. **43** (1985), 87-119.

[F2] E. Formanek, *The Polynomial Identities and Invariants of $n \times n$ Matrices*, CBMS Regional Conf. Series in Math. **78**, Published for the Confer. Board of the Math. Sci. Washington DC, AMS, Providence RI, 1991.

[Fr] G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, Encyclopaedia of Mathematical Sciences, **136**, Invariant Theory and Algebraic Transformation Groups, VII. Springer-Verlag, Berlin, 2006.

[G] L. Gerritzen, *Hilbert series and non-associative Gröbner bases*, Manuscr. Math. **103** (2000), 161-167.

[GH] L. Gerritzen, R. Holtkamp, *Hopf co-addition for free magma algebras and the non-associative Hausdorff series*, J. Algebra **265** (2003), 264-284.

[GZ] A. Giambruno, M. Zaicev, *On codimension growth of finitely generated associative algebras*, Adv. Math. **140** (1998), 145-155.

[Ha] M. Hall Jr., *Combinatorial Theory*, Reprint of the Second Edition, Wiley Classics Library, Chichester, John Wiley & Sons, 1998.

[HP] F. Harary, E.M. Palmer, *Graphical Enumeration*, Academic Press, New York – London, 1973.

[Hi] P.J. Higgins, *Groups with multiple operators*, Proc. London Math. Soc. (3) **6** (1956), 366-416.

[H] R. Holtkamp, *On Hopf algebra structures over free operads*, Adv. Math. **207** (2006), 544-565.

[HLR] R. Holtkamp, J.-L. Loday, M. Ronco, *Coassociative magmatic bialgebras and the Fine numbers*, J. Algebraic Combin. (to appear).

[J] T.W. Jooste, *Primitive derivations in free associative algebras*, Math. Z. **164** (1978), 15-23.

[Kh1] V.K. Kharchenko, *Constants of derivations of primary rings* (Russian) Izv. Akad. Nauk SSSR, Ser. Mat. **45** (1981), 435-462. Translation: Math. USSR, Izv. **18** (1982), 381-401.

[Kh2] V.K. Kharchenko, *Noncommutative invariants of finite groups and Noetherian varieties*, J. Pure Appl. Algebra **31** (1984), 83-90.

[KS] O.G. Kharlampovich, M.V. Sapir, *Algorithmic problems in varieties*, Intern. J. Algebra and Computation **5** (1995), 379-602.

[Ko1] A.N. Koryukin, *Noncommutative invariants of reductive groups* (Russian), Algebra i Logika **23** (1984), 419-429. Translation: Algebra Logic **23** (1984), 290-296.

[Ko2] A.N. Koryukin, *Noncommutative invariants of bialgebras* (Russian), Algebra i Logika **33** (1994), 654-680. Translation: Algebra Logic **33** (1994), 366-380.

[K1] A.G. Kurosh, *Non-associative free algebras and free products of algebras* (Russian), Mat. Sbornik, **20** (1947), 239-260 (English summary, 260-262).

[K2] A.G. Kurosh, *Free sums of multiple operator algebras* (Russian), Siberian Math. J., **1** (1960), 62-70.

[K3] A.G. Kurosh, *A cycle of papers on multioperator rings and algebras* (Russian), Uspehi Mat. Nauk **24** (1969), No. 1 (145), 3-15. Translation: Russ. Math. Surv. **24** (1969), No. 1, 1-13.

[L] J. Lewin, *On Schreier varieties of linear algebras*, Trans. Amer. Math. Soc. **132** (1968), 553-562.

[Lo] J.-L. Loday, *Generalized bialgebras and triples of operads*, Preprint 2006.

[Mc] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press (Clarendon), Oxford, 1979, Second Edition, 1995.

[MSY] A.A. Mikhalev, V. Shpilrain, J.-T. Yu, *Combinatorial Methods. Free Groups, Polynomials, and Free Algebras*, CMS Books in Mathematics **19**, Springer, New York, 2004.

[N] A. Nowicki, *Polynomial Derivations and Their Rings of Constants*, Uniwersytet Mikolaja Kopernika, Torun, 1994.

[R] S. Rajaee, *Non-associative Gröbner bases*, J. Symb. Comp. **41** (2006), 887-904.

[Sl] N. Sloane (Edt.), *The On-Line Encyclopedia of Integer Sequences*, 2004, http://www.research.att.com/~njas/sequences/index.html

[SU] I.P. Shestakov, U.U. Umirbaev, *Free Akivis algebras, primitive elements, and hyperalgebras*, J. Algebra **250** (2002), 533-548.

[U] V.A. Ufnarovski, *Combinatorial and asymptotic methods in algebra*, in A.I. Kostrikin, I.R. Shafarevich (Eds.), "Algebra VI", Encyclopedia of Math. Sciences **57**, Springer-Verlag, 1-196, 1995.

[W] R. Weitzenböck, *Über die Invarianten von linearen Gruppen*, Acta Math. **58** (1932), 231-293.

[We1] H. Weyl, *Zur Darstellungstheorie und Invariantenabzählung der projektiven, der Komplex- und der Drehungsgruppe*, Acta Math. **48** (1926), 255-278; reprinted in "Gesammelte Abhandlungen", Band III, Springer-Verlag, Berlin – Heidelberg – New York, 1968, 1-25.

[We2] H. Weyl, *The Classical Groups, Their Invariants and Representations*, Princeton Univ. Press, Princeton, N.J., 1946, New Edition, 1997.

Contact information

V. Drensky
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria
E-Mail: drensky@math.bas.bg

R. Holtkamp
Fakultät für Mathematik, Ruhr-Universität, 44780 Bochum, Germany
E-Mail: ralf.holtkamp@ruhr-uni-bochum.de

Received by the editors: 03.01.2008
and in final form 15.07.2008.