Algebra and Discrete Mathematics
Number 4. **(2009).** pp. 158 – 166

RESEARCH ARTICLE

Groups with many generalized FC-subgroup

Alessio Russo and Giovanni Vincenzi

Communicated by I. Ya. Subbotin

To Professor L. A. Kurdachenko on his sixties birthday

Abstract. Let FC^0 be the class of all finite groups, and for each non-negative integer m define by induction the group class FC^{m+1} consisting of all groups G such that the factor group $G/C_G(x^G)$ has the property FC^m for all elements x of G. Clearly, FC^1 is the class of FC-groups and every nilpotent group with class at most m belongs to FC^m. The class of FC^m-groups was introduced in [6]. In this article the structure of groups with finitely many normalizers of non-FC^m-subgroups (respectively, the structure of groups whose subgroups either are subnormal with bounded defect or have the property FC^m) is investigated.

Introduction

The structure of groups for which the set of non-normal subgroups (or more generally of non-subnormal subgroups) has prescribed properties has been investigated by several authors. The first step was of course the description of groups in which all subgroups are normal (*Dedekind groups*). It is well known that Dedekind groups either are abelian or can be decomposed as a direct product of Q_8 (the quaternion group of order 8) and a periodic abelian group with no elements of order 4. G. M. Romalis and N. S. Sesekin ([14], [15], [16]) considered locally soluble groups whose non-normal subgroups are abelian (*metahamiltonian groups*), proving in particular that commutator subgroup of these groups is finite. Moreover, S. N. Chernikov [2] studied the groups whose non-normal subgroups are

2000 Mathematics Subject Classification: *20F24.*
Key words and phrases: *Conjugacy class, FC-groups, normalizer subgroup, subnormal subgroup.*

finite, while recently L. A. Kurdachenko, J. Otal and the authors [7] have described (generalized) soluble groups whose non-normal subgroups are FC-groups. They prove, among other results, that such groups are locally (central-by-finite). Finally, B. Bruno and R. E. Phillips [1] are concerned with (generalized) soluble groups whose non-normal subgroups are locally nilpotent, showing in particular that the groups of this class have finite commutator subgroup if they are not locally nilpotent. In a famous paper of 1955, B. H. Neumann [10] proved that each subgroup of a group G has finitely many conjugates if and only if the factor group $G/Z(G)$ is finite, and the same conclusion holds if the restriction is imposed only to abelian subgroups (see [3]). Therefore central-by-finite groups are precisely those groups in which all the normalizers of (abelian) subgroups have finite index, and this result suggests that the behaviour of normalizers has a strong influence on the structure of the group. In this context, it is interesting to recall a result of Y. D. Polovickiĭ [12] stating that a group G has finitely many normalizers of abelian subgroups if and only if it is central-by-finite. These considerations show that it is natural to investigate the structure of groups with finitely many normalizers of subgroups with a given property. For instance, F. De Mari and F. de Giovanni recently have studied the classes of groups with finitely many normalizers of non-abelian (respectively, of non-(locally nilpotent)) subgroups (see [4], [5]) improving the quoted result of Romalis and Sesekin (respectively, of Bruno and Phillips).

The next step to combine the above ideas seems to be the consideration of classes of groups which are a good extension both of the nilpotency and of the property FC. To this purpose we will concerned with the class of FC^m*-groups* introduced in [6]. Recall that FC^0 is the class of all finite groups, and for each non-negative integer m define by induction the group class FC^{m+1} consisting of all groups G such that for every element $x \in G$ the factor group $G/C_G(x^G)$ has the property FC^m. Clearly, the FC^1-groups are precisely the FC-groups while the class FC^m contains all finite groups and all nilpotent groups with class at most m. The aim of this article is to investigate the structure of the classes of (generalized) soluble groups which either have finitely many normalizers of non-FC^m-subgroups or have the non-subnormal subgroups with bounded defect satisfying the property FC^m.

Most of our notation is standard and can for instance be found in [9] and [13].

1. Statementes and proofs

We will prove many of our results within the universe of W-groups. Recall that a group G is said to be a W*-group* if every finitely generated non-nilpotent subgroup of G has a finite, non-nilpotent, homomorphic image. It is well known that all hyper-(abelian or finite) groups and all linear groups have the property W (see [13], theorem 10.51 and [17]). Let k and m be non-negative integers. In what follows we denote by X_m and $X_{k,m}$ respectively, the class of groups with finitely many normalizers of non-FC^m-groups and the class of groups whose subgroups either are subnormal with defect at most k or satisfy the property FC^m.

Proposition 1. *Let G be a W-group whose non-subnormal subgroups are FC^m-groups for some non-negative integer m. Then every finitely generated subgroup of G is polycyclic-by-finite.*

Proof. Clearly we may suppose that G is a finitely generated non-nilpotent group. It follows that there exists a non-subnormal subgroup H of G with finite index, so that the core H_G of H in G is a finitely generated FC^m-group. Therefore the factor group $H_G/Z_m(H_G)$ is finite (see [6], proposition 3.6), and hence G contains a subgroup of finite index which is nilpotent with class at most m. □

In order to prove the corresponding result for the class X_m we need a lemma which has a formulation and proof similar to those obtained for other classes of groups with finitely many normalizer of subgroups of certain types (see for instance [4], [5], [8]).

Lemma 1. *Let G be an X_m-group. Then G contains a characteristic $X_{2,m}$-subgroup of finite index.*

Proof. Let H be a subgroup of G which has not the property FC^m. Clearly its normalizer $N_G(H)$ has finitely many conjugates in G, and hence the index $|G : N_G(N_G(H))|$ is finite. It follows that the characteristic subgroup

$$M(H) = \bigcap_{\alpha \in AutG} N_G(N_G(H))^\alpha$$

has finite index. Let L be the set of all non-FC^m-subgroups of G. If H and K are elements of L such that $N_G(H) = N_G(K)$, then $M(H) = M(K)$. Thus also

$$M = \bigcap_{H \in L} M(H)$$

is a characteristic subgroup of finite index of G. Now let H be any element of L contained in M. Then $M \leq M(H) \leq N_G(N_G(H))$, and hence H is subnormal in M with defect at most 2. □

Proposition 2. *Let G be a W-group satisfying the property X_m. Then every finitely generated subgroup of G is polycyclic-by-finite.*

Proof. The statement immediately follows from Proposition 1 and Lemma 1. □

Our next results are turned to prove that the W-groups in the classes X_m and $X_{k,m}$ have the torsion subgroup. To this end we need to investigate the behaviour of the finitely generated torsion-free abelian normal subgroups of these groups. We begin with an easy result.

Lemma 2. *Let A be a torsion-free abelian group, and let g be a non-trivial automorphism of G. If g has finite order, then the semidirect product $G = \langle g \rangle \ltimes A$ is not nilpotent.*

Proof. Assume, for a contradiction that G is nilpotent. Let $g^i\bar{a}$ be an element of $Z(G) \setminus A$, where $\bar{a} \in A$ and $|g|$ does not divide i. As $(g^i\bar{a})a = a(g^i\bar{a})$ for each $a \in A$, then $g^i \in Z(G)$, a contradiction because the centralizer $C_{\langle g \rangle}(A)$ is trivial. Therefore $Z(G) \leq A$, and MacLain theorem yields that G is torsion-free. □

Lemma 3. *Let G be is any $X_{k,m}$-group (any X_m-group, respectively), and let A be a finitely generated torsion-free abelian normal subgroup of G. If g is an element of finite order of G, then $[A, g] = \{1\}$.*

Proof. Assume, for a contradiction that $[A, g] \neq \{1\}$. Clearly, there exists a prime number p such that $Y_n = \langle g \rangle A^{p^n}$ is not abelian for all positive integers n. If Y_n is an FC^m-group, then $\gamma_{m+1}(Y_n)$ is periodic (see [6], corollary 3.3) and hence Y_n is nilpotent since A is torsion-free. Therefore, also the factor group $Y_n/C_{\langle g \rangle}(A^{p^n})$ is nilpotent, a contradiction by the Lemma 2. Thus Y_n is not an FC^m-group for all n.

First suppose that G is an $X_{k,m}$-group. Then Y_n is subnormal in G with defect at most k for all positive integers n, whence so is the subgroup

$$\langle g \rangle = \bigcap_{n \in N} Y_n.$$

It follows by Fitting's theorem that Y_n is nilpotent, and we reach a contradiction using the argument above. Finally, let G be an X_m-group, and assume that G is a counterexample with a minimal number t of proper normalizers of subgroups which are not FC^m-groups. Clearly, we may suppose that $t \geq 1$. Then every subgroup Y_n is normal in G, since otherwise its normalizer would be a counterexample with less than t proper normalizers of non-FC^m-subgroups. It follows that also the subgroup $\langle g \rangle$ is normal in G. Thus $[A, g] \leq A \cap \langle g \rangle = \{1\}$, and this last contradiction completes the proof of the lemma. □

Theorem 1. *Let G be a W-group satisfying the property $X_{k,m}$ (the property X_m, respectively). Then the elements of finite order of G form a (fully-invariant) subgroup of G.*

Proof. Let x and y be elements of finite orders of G. Clearly, we may assume that $G = \langle x, y\rangle$, and by Proposition 1 (by Proposition 2, respectively) G is polycyclic-by-finite. Moreover, as the largest periodic normal subgroup T of G is finite, replacing G by the factor group G/T, it can also be assumed without loss of generality that G has no periodic non-trivial normal subgroups. Let S be a soluble normal subgroup of G such that G/S is finite. For a contradiction, let G be infinite, and choose the soluble subgroup S with minimal derived length. If A is the smallest non-trivial term of the derived series of S, then A is torsion-free, so that by Lemma 3 A is contained in the centre $Z(G)$ of G. On the other hand, by the minimal choice of G, the factor group G/A is finite, and hence G' is likewise finite by Schur's theorem. It follows that also G is finite, and this contradiction proves the statement. □

Recall that if H is a subgroup of a group G, the *series of normal closures* of H in G is defined inductively by the positions $H^{G,0} = G$ and $H^{G,k+1} = H^{H^{G,k}}$ for each non-negative integer k. Thus H is subnormal in G with defect k if and only if $H^{G,k} = H$ and k is the smallest non-negative integer with this property.

Corollary 1. *Let G be a W-group satisfying the property $X_{k,m}$. Then the subgroup $\gamma_{f(k+m+1)+1}(G)$ is periodic, where f is the function of Roseblade's theorem.*

Proof. By Theorem 1 we can assume that G is torsion-free. It follows that every subgroup of G either is subnormal with defect at most k or is nilpotent with class at most m (see [6], corollary 3.4), and hence G is locally nilpotent (see [11], theorem A). Suppose that G is finitely generated, and let X be a subgroup of G which is not subnormal with defect at most k. Then $X < X^{G,k}$. Clearly, if M is a maximal subgroup of $X^{G,k}$ containing X, then M is nilpotent with class at most m, and hence X is subnormal in M with defect at most m. On the other hand, $M \lhd X^{G,k}$, so that X is subnormal in G with defect at most $m + k + 1$. Therefore the statement follows from the Roseblade's theorem. □

Let H be a subgroup of a group G. Denote by $I_G(H)$ the *isolator* of H in G, i.e., the set of all $x \in G$ such that $x^m \in H$ for some positive integer m. In general, $I_G(H)$ need not be a subgroup, as is shown by the isolator of the identity subgroup in D_∞, the infinite dihedral group. On the other hand, if G is locally nilpotent, then the isolator of every subgroup of G is likewise a subgroup. Actually locally nilpotent groups have a rich isolator theory. For details we refer to [9], Section 2.3. In particular, in what follows we will use the statement which ensures that if H is a subgroup of a torsion-free, locally nilpotent group G, then $Z_m(H) = Z_m(I_G(H)) \cap H$ for all non-negative integers m.

Corollary 2. *Let G be a W-group satisfying the property X_m. Then the subgroup $\gamma_{m+1}(G)$ is periodic.*

Proof. By Theorem 1 we can assume that G is torsion-free, and hence by corollary 3.4 of [6] it has finitely many normalizers of subgroups which are not nilpotent with class at most m. It follows that G is locally nilpotent (see [5], theorem A), and we may also suppose that G is finitely generated. Now we argue by induction on the number of proper normalizers of subgroups with class greater that m, and so we may reduce to the case that all non-normal subgroups of G have class at most m. Suppose that G does not contain a subgroup of finite index with class at most m. Let L be the set of all subgroups of finite index of G. Then each element K of L is normal in G and G/K is a Dedekind group. It follows that

$$\gamma_3(G) \leq \bigcap_{K \in L} K = \{1\}.$$

Thus G is metahamiltonian and so even abelian, a contradiction. Now let H be a normal subgroup of G with class at most m such that the factor group G/H is finite. Clearly, the isolator $I_G(H)$ of H in G is equal to G. It follows by the quoted remark that $H \leq Z_m(G)$, and hence G is nilpotent with class at most m as required. □

Corollary 3. *Let G be a W-group satisfying the property X_m. If H is a finitely generated subgroup of G, then the factor group $H/Z_m(H)$ is finite.*

Proof. By Proposition 2 and Corollary 2 the subgroup $\gamma_{m+1}(H)$ is finite, whence so is $H/Z_m(H)$. □

Note that it is easy to find $X_{1,1}$-groups with an infinite commutator subgroup which are not FC-groups. For, let p be a prime number, and let P be a group of type p^∞. Consider the automorphism x of $P \times P$ defined by the rule $x(a_1, a_2) = (a_1a_2, a_2)$ for every $(a_1, a_2) \in P \times P$. Then the semidirect product

$$G = \langle x \rangle \ltimes (P \times P)$$

is a non-periodic group whose commutator subgroup is $G' = Z(G) \simeq P$. Therefore, each non-normal subgroup of G has finite commutator subgroup. Moreover, G is not an FC-group since $G/Z(G)$ is not periodic.

In [7], lemma 2.7, it has been proved that an $X_{1,1}$-group has countable commutator subgroup if it is not an FC-group. Our last result extends this statement to the class X_m. For this purpose we need two lemmas concerning the FC^m-groups.

Lemma 4. *Let G be an FC^m-group. Then every non-trivial normal subgroup N of G contains a non-trivial FC-element of G.*

Proof. Clearly, we may suppose $m \geq 2$ and $[N, G] \neq \{1\}$. Denote by F the FC-centre of G. By theorem 3.2 of [6] the subgroup $\gamma_m(G)$ is contained in F so that $[N,_{m-1} G] \leq N \cap F$. It follows that there exists a positive integer $n \leq m - 1$ such that $\{1\} \neq [N,_n G] \leq N \cap F$. □

Recall that a group class X is said to be *countably recognizable* if a group G belongs to X provided that all its countable subgroups are X-groups. In what follows we denote by $FC^m(G)$ the subgroup of a group G consisting of all elements x such that $G/C_G(\langle x^G \rangle)$ is an FC^{m-1}-group. In particular, $FC^1(G)$ is the FC-centre of G.

Lemma 5. *Let G be a group, and let H be a countable subgroup of G. If every countable subgroup of G containing H is an FC^m-group, then $H \leq FC^m(G)$. In particular, the class FC^m is countably recognizable for all non-negative integers m.*

Proof. Put $N = FC^m(G) \cap H^G$, and let K/N be any countable subgroup of G/N containing HN/N. Then $K = LN$, where L is a countable subgroup of G, and H is contained in the countable FC^m-subgroup $\langle H, L \rangle$. It follows that $K/N \leq \langle H, L \rangle N/N$, and hence K/N is an FC^m-group. Therefore the hypotheses are inherited by the group G/N and its countable subgroup HN/N. Assume for a contradiction that the statement is false, so that H is not contained in N. Replacing G by G/N it can be assumed without loss of generality that the normal closure H^G contains no non-trivial elements having finitely many conjugates in G. Write

$$H = \{h_n \mid n \in N_0\}$$

where $h_0 = 1$, and put $X_0 = \{1\}$. Now suppose that for some non-negative integer n a countable subgroup X_n of G has been defined containing the elements $h_0, \ldots, h_n$. Clearly there exists a countable subset W_n of G such that every non-trivial element of $H^G \cap X_n$ has infinitely many conjugates under the action of W_n. Consider the countable subgroup $X_{n+1} = \langle X_n, h_{n+1}, W_n \rangle$, and put

$$X = \bigcup_{n \in N_0} X_n.$$

Thus X is a countable subgroup of G containing H, and hence X is an FC^m-group. It follows from Lemma 4 that there exists a non-trivial element u of $H^X \cap FC^1(X)$. If m is a positive integer such that $u \in X_m$, then u has infinitely many conjugates under the action of X_{m+1}, and this last contradiction proves the lemma. □

Theorem 2. *Let G be an X_m-group. If G is not an FC^m-group, then the commutator subgroup G' of G is countable.*

Proof. Let $N_G(X_1), \ldots, N_G(X_t)$ be the proper normalizers of non-FC^m-subgroups of G. First assume that the set $N_G(X_1) \cup \cdots \cup N_G(X_t)$ is properly contained in G. Let x be an element of

$$G \setminus (N_G(X_1) \cup \cdots \cup N_G(X_t)).$$

Then each subgroup of G containing x either is normal or an FC^m-group. Since G does not satisfy the property FC^m, by Lemma 5 G contains a countable subgroup H which is not an FC^m-group. Therefore the subgroup $\langle H, x\rangle$ is normal in G and $G/\langle H, x\rangle$ is a Dedekind group. Thus $|G'\langle H, x\rangle/\langle H, x\rangle| \leq 2$, and hence G' is countable.

Now suppose that

$$G = N_G(X_1) \cup \cdots \cup N_G(X_t).$$

It follows from a result of B. H. Neumann that it is possible to omit from the above union any subgroup of infinite index (see [13], lemma 4.17), and hence we can write

$$G = N_G(X_{i_1}) \cup \cdots \cup N_G(X_{i_s}),$$

where the index $|G : N_G(X_{i_j})|$ is finite for all $j = 1, \ldots, s$. Clearly, every $N_G(X_{i_j})$ has less than t proper normalizers of non-FC^m-subgroups, so that arguing by induction (on t), we may assume that $N_G(X_{i_j})'$ is countable. On the other hand, as the index $|G : N_G(N_G(X_{i_j})')|$ is finite, then $(N_G(X_{i_j})')^G$ is countable, whence so is the subgroup $N = \langle N_G(X_{i_1})', \ldots, N_G(X_{i_s})'\rangle^G$. Moreover, the factor group G/N has a finite covering consisting of abelian subgroups, and hence it is central-by-finite (see [13], theorem 4.16). It follows from Schur's theorem that $G'N/N$ is finite whence G' is countable also in this case. □

References

[1] B. Bruno, R.E. Phillips, *Groups with restricted non-normal subgroups*, Math. Z., N. **176**, 1981, pp.199–221.

[2] S. N. Chernikov, *The groups with prescribed properties of family of infinite subgroups*, Ukrain. Math. J., N. **19**, 1967, pp.111–131.

[3] I. I. Eremin, *Groups with finite classes of conjugate abelian subgroups*, Mat. Sb. N. **47**, 1959, pp.45–54.

[4] F. De Mari, F. de Giovanni, *Groups with finitely many normalizers of non-abelian subgroups*, Ricerche Mat. N. **55**, 2006, pp.311–317.

[5] F. De Mari, F. de Giovanni, *Groups with finitely many normalizers of non-nilpotent subgroups*, Math. Proc. Roy. Irish Acad., N. **105**A(2), 2007, pp.143–152.

[6] F. de Giovanni, A. Russo, G. Vincenzi, *Groups with restricted conjugacy classes*, Serdica Math. J., N. **28**, 2002, pp.241–254.

[7] L.A. Kurdachenko, J. Otal, A. Russo, G. Vincenzi, *Groups whose non-normal subgroups have finite conjugacy classes*, Math. Proc. Roy. Irish Acad. N. **104** A(2), 2004, pp.177–189.

[8] L.A. Kurdachenko, A. Russo, G. Vincenzi, *Groups without proper abnormal subgroups*, J. Group Theory N. **9**, 2006, pp.507–518.

[9] J. C. Lennox, D.J.S. Robinson, *The Theory of Infinite Soluble Groups*, Clarendon, Oxford, 2004.

[10] B. H. Neumann, *Groups with finite classes of conjugate subgroups*, Math. Z. N. **63**, 1955, pp.76–96.

[11] R.E. Phillips, J. S. Wilson, *On certain minimal conditions for infinite groups*, J. Algebra N. **51**, 1978, pp.41–68.

[12] Y. D. Polovickiĭ, *Groups with finite classes of conjugate infinite abelian subgroups*, Soviet Math. (Iz. VUZ) N. **24**, 1980, pp.52–59.

[13] D.J.S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Springer, Berlin 1972.

[14] G.M. Romalis, N.F. Sesekin, *Metahamiltonian Groups*, Ural Gos. Univ. Zap., N. **5**, 1966, pp.101–106.

[15] G.M. Romalis, N.F. Sesekin, *Metahamiltonian Groups II*, Ural Gos. Univ. Zap., N. **6**, 1968, pp.52–58.

[16] G.M. Romalis, N.F. Sesekin, *Metahamiltonian Groups II*, Ural Gos. Univ. Zap., N. **7**, 1969-1970, pp.195–199.

[17] B. A. F. Wehrfritz, *Frattini subgroups in finitely genereted linear groups*, J. London Math. Soc., N. **43**, 1968, pp.619–622.

CONTACT INFORMATION

A. Russo Dipartimento di Matematica, Seconda Università di Napoli, Via Vivaldi 43, I - 81100 Caserta (Italy)
E-Mail: `alessio.russo@unina2.it`

G. Vincenzi Dipartimento di Matematica e Informatica, Università di Salerno, Via Ponte Don Melillo, I - 84084 Fisciano, Salerno (Italy)
E-Mail: `gvincenzi@unisa.it`

Received by the editors: 11.07.2009
and in final form 11.07.2009.