Algebra and Discrete Mathematics
Number 4. **(2009).** pp. 135 – 157

SURVEY ARTICLE

Groups with small cocentralizers

Javier Otal and Nikolaj N. Semko

Communicated by I. Ya. Subbotin

Dedicated to Leonid with occasion of his 60^{th} birthday

Abstract. Let G be a group. If $S \subseteq G$ is a G–invariant subset of G, the factor-group $G/C_G(S)$ is called *the cocentralizer of S in G*. In this survey-paper we review some results dealing with the influence of several cocentralizers on the structure of the group, a direction of research to which Leonid A. Kurdachenko was an active contributor, as well as many mathematicians all around the world.

Introduction

Let G be a group. A subset S of G is called *G–invariant* if $a^g \in S$ for every element $a \in S$ and $g \in G$. If S is a G–invariant subset, then its centralizer $C_G(S)$ is a normal subgroup of G. The corresponding factor-group $Coc_G(S) = G/C_G(S)$ is called *the cocentralizer of the set S in the group G* (L.A. Kurdachenko [37]). Clearly, $Coc_G(S)$ is isomorphic to some subgroup of Aut $(\langle S\rangle)$. The influence of the cocentralizers of many objects related to the group on the structure of the group itself is a subject of study in many branches of Group Theory. For instance, in the Theory of Finite Groups many examples have been developed with the consideration of the cocentralizers of chief factors, which play there a significant role therein. For example, we can mention Formation Theory, because local formations are defined through the cocentralizers of the chief factors of the groups involved.

Supported by Proyecto MTM2007-60994 of DGI del MEC (Spain).
2000 Mathematics Subject Classification: *20F24, 20F17.*
Key words and phrases: *Cocentralizer in a group. Groups with prescribed conjugacy classes: FC–groups. CC–groups. PC–groups. MC–groups.*

In the Theory of Infinite Groups, many types of groups appear in studying cocentralizers of their conjugacy classes. We recall here the definitions. Let $\mathfrak{X}$ be a class of groups. We say that *a group G has $\mathfrak{X}$–conjugacy classes* (or G is an $\mathfrak{X}C$–group) if $Coc_G(g^G) \in \mathfrak{X}$ for each $g \in G$. If $\mathfrak{X} = \mathfrak{I}$ is the class of all identity groups, then the class of $\mathfrak{I}C$–groups is exactly the class $\mathfrak{A}$ of all abelian groups. Thus the class of $\mathfrak{X}C$–groups can be considered as a natural generalization of the class of abelian groups. If $\mathfrak{X} = \mathfrak{F}$ is the class of all finite groups, then the $\mathfrak{F}C$–groups is the class of groups with finite conjugacy classes (as in the literature, from now on we shall denote FC–groups to this class). This class is a suitable extension both of the classes $\mathfrak{F}$ and $\mathfrak{A}$ mentioned above, so that it inherited many properties of them.

Some natural subclasses of FC–groups were introduced quite long time ago. One of the important results on groups is the Schur's theorem [55]. I. Schur proved that if the center of a group G has finite index, then the derived subgroup $[G, G]$ is finite. Thus, in that paper, the classes of central–by–finite groups and finite–by–abelian groups were introduced. As it is well-know now, these classes play a very important role in the theory of FC–groups. The next important step were done by A. P. Ditsman in [9]. He proved that every finite G–invariant subset of a group G whose elements have finite order generates a finite normal subgroups. In the paper due to R. Baer [2], the class of FC–groups has been introduced. With this and consequential works by P. A. Goldberg [12] and S. N. Chernikov [5], the series of researches having as main goal the extension of main theorems from finite groups theory to periodic FC–groups has begun. In the paper by R. Baer [3], arbitrary FC–groups were considered at first, and the term FC–group was introduced here. This paper, papers by S. N.Chernikov [4, 6, 7] and B. H. Neumann [46, 47, 48] set the bases of the theory of FC–groups.

The papers by M. I. Kargapolov [24], S. N. Chernikov [8] and P. Hall [23] started the study of the conditions of embedding periodic FC–groups in direct products of finite groups. The influence of the class $\mathbf{SD}\mathfrak{F}$ of subgroups of direct products of finite groups on the structure of an FC–group was also considered there. We note that an interesting parallel with abelian groups raised there. A significant role here is played by the work by Yu.M. Gorchakov [16, 17, 18, 19, 20, 21] and M.J. Tomkinson [61, 62, 63]. Since these results are completely developed in the books [22, 64] and the survey-paper [65], we do not consider them here any longer. Instead we focus on some later results from this area.

1. Non-periodic FC–groups

Leonid Kurdachenko initiated this investigation. We note that all previous results on FC–groups developed in the mentioned books by Yu. M. Gorchakov and by M. J. Tomkinson concern with periodic FC–groups only. However non-periodic FC–groups have their own nature. From the results of the paper [3, 46], it follow that the set $Tor(G)$ of all elements of finite order of an FC–group G is a characteristic subgroup of G and the factor-group $G/Tor(G)$ is a torsion-free abelian group. From the results of [7], it follows that an FC–group G can be embedded in a direct product $T \times A$, where T is a periodic FC–group and A is a torsion-free abelian group. However it is worth to mention that not all properties of the periodic part $Tor(G)$ can be extended on the direct factor T. In particular, if $Tor(G)$ is a subgroup of a direct product of finite groups, the question on finding conditions under which G itself can be embedded in a direct product of finite groups and abelian torsion-free groups is naturally raised and it is not a simple one. This problem was considered by L. A. Kurdachenko. He begun the developing of the theory of non-periodic FC–groups. Even since some interesting parallel with the theory of abelian groups could be find here, we remark that the general situation is more complicated.

Let G be a group and let H be a normal subgroup of G. Given a class of groups $\mathfrak{X}$, we say that H *can be embedded in a direct product of* $\mathfrak{X}$*–groups relative to* G if H has a family of subgroups $\{H_lambda \mid \lambda \in \Lambda\}$ satisfying the following conditions

(i) H_λ is a normal subgroup of G;

(ii) $\bigcap_{\lambda\in\Lambda} H_\lambda = \langle 1\rangle$;

(iii) if $x \in H$, then there exists a subset $\Lambda_x \subseteq \Lambda$ such that $\Lambda \setminus \Lambda_x$ is finite and $x \in H_\lambda$ for all $\lambda \in \Lambda_x$; and

(iv) $H/H_\lambda \in \mathfrak{X}$ for all $\lambda \in \Lambda$.

If G is a subgroup of a direct product of finite groups and a abelian torsion-free group, then its periodic part can be embedded in a direct product of finite groups relative to the group G. Thus we see that a condition of a relative embedding are natural. In the paper [25, 26], FC–groups G whose periodic part can be embedded in a direct product of finite groups relative to G were considered.

Theorem 1 ([25]). *Let G be an FC–group whose periodic part can be embedded in a direct product of finite groups relative to G. Then the factor-group $G/\zeta(G)$ can be embedded in a direct product of finite groups.*

Theorem 2 ([25]). *Let G be an FC–group. Suppose that $\zeta(Tor(G)) = \langle 1 \rangle$. Then the group G can be embedded in a direct product of finite groups with identity centers and a torsion-free abelian group.*

Theorem 3 ([25]). *Let G be an FC–group. Suppose that $Tor(G)$ is a semisimple group. Then the group G can be embedded in a direct product of finite semisimple groups and a torsion-free abelian group.*

Theorem 4 ([25]). *Let G be an FC–group. Suppose that $Tor(G)$ is a bounded subgroup and $G/Tor(G)$ has finite 0–rank. If $Tor(G)$ can be embedded in a direct product of finite groups relative to a G, then the group G itself can be embedded in a direct product of finite groups and a torsion-free abelian group.*

In the paper [26], Kurdachenko considers FC–groups, whose periodic part can be embedded in a direct product of finite groups.

Theorem 5 ([26]). *Let G be an FC–group whose periodic part can be embedded in a direct product of finite groups. Then the factor-group $G/\zeta(Tor(G)$ can be embedded in a direct product of finite groups and a torsion-free abelian group.*

Corollary 1 ([26]). *Let G be an FC–group. Then $G/\zeta(Tor(G)$ can be embedded in a direct product of finite groups and a torsion-free abelian group in the following cases:*

(1) the Sylow p–subgroups of $Tor(G)$ are finite for all primes p.

(2) $Tor(G)$ is a countable residually finite group.

(3) $Tor(G)$ is a residually finite group and $Tor(G) = [G, G]$.

(4) $Tor(G)$ is a subgroup of direct product of finite groups whose order are bounded.

Corollary 2 ([26]). *Let G be an FC–group and let C be the third hypercenter of $Tor(G)$. Then G/C can be embedded in a direct product of finite groups and a torsion-free abelian group.*

The next theorem gives conditions that ensures than a usual embedding in a direct product of finite groups is equivalent to the embedding relative to a group G.

Theorem 6 ([26]). *Let G be an FC–group whose periodic part can be embedded in a direct product of finite groups. If the factor-group $G/Tor(G)$ has finite 0–rank, then $Tor(G)$ can be embedded in a direct product of finite groups relative to G.*

Theorem 7 ([26]). *Let G be an FC–group whose periodic part can be embedded in a direct product of finite groups. If the factor-group $G/Tor(G)$ has finite 0–rank, then $G/\zeta(G)$ can be embedded in a direct product of finite groups.*

Theorem 8 ([26]). *Let G be an FC–group whose periodic part can be embedded in a direct product of finite groups. If the factor-group $G/Tor(G)$ has finite 0–rank and $Tor(G)$ is bounded, then G can be embedded in a direct product of finite groups and a torsion-free abelian group.*

Some results from the paper [26] were generalized in [27]. We say that a group G is said to be *co-layer-finite* if G/G^n is finite for each positive integer n. Here $G^n = \langle g^n \mid g \in G\rangle$. Since it is well-known that *a layer-finite group* can be defined as a group G in which $G[n] = \langle g \mid g^n = 1\rangle$ is finite for each $n \geq 1$, the concept of a co-layer-finite group is clearly dual to the concept of a layer-finite group. We remark that the class of co-layer-finite groups is quite large.

Theorem 9 ([27]). *Let G be an FC–group whose periodic part can be embedded in a direct product of finite groups. If the factor-group $G/Tor(G)$ is co-layer-finite, then $Tor(G)$ can be embedded in a direct product of finite groups relative to a G.*

Theorem 10 ([27]). *Let G be an FC–group whose periodic part can be embedded in a direct product of finite groups. If $G/Tor(G)$ is a countable co-layer-finite group and the set $\Pi(Tor(G))$ is finite, then G can be embedded in a direct product of finite groups and a torsion-free abelian group.*

Corollary 3 ([27]). *Let G be an FC–group whose periodic part can be embedded in a direct product of finite groups. If the factor-group $G/Tor(G)$ has finite 0–rank and the set $\Pi(Tor(G))$ is finite, then G can be embedded in a direct product of finite groups and a torsion-free abelian group.*

The last two results were significantly generalized in the paper [29].

Let A be a torsion-free abelian group of finite 0–rank and let M be a maximal $\mathbb{Z}$–independent subset of A. Put $B = \langle M\rangle$ so that the factor-group A/B is periodic. Let

$$Sp(A) = \{p \mid \text{the Sylow } p\text{–subgroup of } A/B \text{ is infinite}\}.$$

It is not hard to prove that the set $Sp(A)$ is an invariant of the group A.

Theorem 11 ([29]). *Let G be an FC–group whose periodic part can be embedded in a direct product of finite groups. If $G/Tor(G)$ is a countable co-layer-finite group and the set $\Pi(\zeta(G))$ is finite, then G can be embedded in a direct product of finite groups and a torsion-free abelian group.*

Theorem 12 ([29]). *Let G be an FC–group whose periodic part can be embedded in a direct product of finite groups. If the factor-group $A = G/Tor(G)$ has finite 0–rank and the set $\Pi(\zeta(G)) \cap Sp(A)$ is finite, then G can be embedded in a direct product of finite groups and a torsion-free abelian group.*

Corollary 4 ([29]). *Let G be an FC–group whose periodic part can be embedded in a direct product of finite groups. Suppose that $A = G/Tor(G)$ has finite 0–rank. Then G can be embedded in a direct product of finite groups and a torsion-free abelian group if one of the following equivalent conditions is satisfied:*

(i) A is a free abelian group;

(ii) A is a minimax group, that is A has a finite subnormal series whose factors satisfy either the minimal or the maximal condition on subgroups;

(iii) the set $Sp(A)$ is finite;

(iv) the set $\Pi(G)$ is finite;

(v) the set $\Pi(\zeta(G))$ is finite; and

(vi) the set $\Pi(\zeta(Tor(G)))$ is finite.

As we can see from above results the problem of whether or not an FC–group G can be embedded in a direct product of finite groups and a torsion-free abelian group depends heavily not only on the structure of its torsion subgroup (the torsion subgroup must be embeddable in a direct product of finite groups; however, this is not sufficient), but also depends on the structure of the factor out the torsion subgroup. It is therefore important to study classes of locally normal groups and classes of torsion-free abelian groups that are "well-behaved" with respect to the type of embeddability that we are interested in.

Denote by $\mathfrak{A}^{tf}$ the class of all torsion-free abelian groups, by $\mathbf{SD}\mathfrak{F}$ the class of the subgroups of direct products of finite groups, and by $\mathbf{SD}(\mathfrak{F}\cup\mathfrak{A}^{tf})$ the class of the subgroups of direct products of finite groups and torsion-free abelian groups. Following [28] we say that a torsion-free abelian group A belongs to the class $\mathrm{A}(\mathbf{SD}\mathfrak{F})$ if every FC–group G such that $Tor(G) \in\mathbf{SD}\mathfrak{F}$ and $A \cong G/Tor(G)$, can be embedded in a direct product of finite groups and a torsion-free abelian group. And dually, we say that a periodic FC–group T belongs to the class $\mathrm{T}(\mathfrak{A}^{tf})$ if $T \in\mathbf{SD}\mathfrak{F}$ and every FC–group G such that $Tor(G) \cong T$ can be embedded in a direct product of finite groups and a torsion-free abelian group. Applying Theorem 2, it is not hard to obtain a following result.

Theorem 13. *The class T($\mathfrak{A}^{tf}$) is exactly the class of all periodic FC–groups having identity center.*

A complete description of the class A($\mathbf{SD}\mathfrak{F}$) was obtained in the paper [28]. This description depends on of the structure of the abelian co-layer-finite groups, which was carried out in the same paper.

Theorem 14 ([28])**.** *(1) A torsion-free abelian group A is co-layer-finite if and only if it has an ascending series of pure subgroups*

$$\langle 1\rangle = A_0 \leq A_1 \leq \cdots \leq A_n \leq A_{n+1} \leq \cdots A_\omega = \bigcup_{n\geq 0} A_n$$

with the following properties:

(a) The factor-groups A_n/A_{n-1} have all finite 0–rank equal to 1,

(b) given a prime q, there exists a number $t(q)$ such that $q \in Sp(A_{n+1}/A_n)$ for every $n > t(q)$, and

(c) A/A_ω is a divisible group.

(2) A periodic abelian group A is co-layer-finite if and only if $A = Dr_{p\in\Pi(A)} A_p$, where A_p is the Sylow p-subgroup of A, we have that $Ap = F_p \times D_p$ with F_p finite and D_p divisible.

(3) An abelian group A is co-layer-finite if and only if $Tor(A)$ and $A/Tor(A)$ are co-layer-finite.

The following theorem gives the description of the class A($\mathbf{SD}\mathfrak{F}$).

Theorem 15 ([28])**.** *A torsion-free abelian group A belong to the class A($\mathbf{SD}\mathfrak{F}$) if and only if it has an ascending series of pure subgroups*

$$\langle 1\rangle = A_0 \leq A_1 \leq \cdots \leq A_n \leq A_{n+1} \leq \cdots \bigcup_{n\geq 0} A_n = A$$

with the following properties:
(a) The factor-groups A_n/A_{n-1} have all finite 0–rank equal to 1,
(b) all the sets $Sp(A_n/A_{n-1})$ are all finite, and
(c) if q is a prime, there exists $t(q) \geq 1$ such that $q \in Sp(A_{n+1}/A_n)$ for every $n > t(q)$.

This setting can be adequately generalized as follows. Let $\mathfrak{X}$ be a subclass of $\mathbf{SD}\mathfrak{F}$. Following [29] we say that a torsion-free abelian group A belongs to the class A($\mathfrak{X}$) if every FC–group G such that $Tor(G) \in \mathfrak{X}$ and $A \cong G/Tor(G)$, can be embedded in a direct product of finite groups and a torsion-free abelian group. And dually, if $\mathfrak{Y}$ is a subclass of $\mathfrak{A}^{tf}$, then we say that a periodic FC–group T belongs to the class T($\mathfrak{Y}$)) if $T \in \mathbf{SD}\mathfrak{F}$

and every FC–group G such that $Tor(G) \cong T$ and $G/Tor(G) \in \mathfrak{Y}$ can be embedded in a direct product of finite groups and a torsion-free abelian group ([32]).

If the class $\mathfrak{X}$ is generated by a single group, say T (i.e. it consists of all isomorphic copies of T and all identity groups), then we will write $\mathrm{A}(T)$ instead of $\mathrm{A}(\mathfrak{X})$. Similarly, if the class $\mathfrak{Y}$ is generated by a single group A, then we will write $\mathrm{T}(A)$ instead of $\mathrm{T}(\mathfrak{Y})$.

The class $\mathrm{A}(T)$ is determined by the classes $\mathrm{A}(C)$, where C is a central subgroup of T; more precisely, $\mathrm{A}(T) \subseteq \mathrm{A}(C)$, and therefore two cases of study naturally arise whether or not the center $\zeta(T)$ of T is bounded. The first of these cases was considered in the paper [30]. Indeed if we choose $\mathrm{A}(T)$ in the more natural way, this leads us to the developing conditions of embedding FC–groups in direct products of finite groups and abelian torsion-free groups.

Theorem 16 ([30])**.** *Let G be an FC–group and put $T = Tor(G)$ and $A = G/T$. If T is a bounded subgroup of a direct product of finite groups and A/A^q is finite for each $q \in \Pi(\zeta(T))$, then G can be embedded in a direct product of finite groups and a torsion-free abelian group.*

Theorem 17 ([30])**.** *Let T be a bounded subgroup of a direct product of finite groups. Then the class $A(T)$ consists of all torsion-free abelian groups A such that A/A^q is finite for each $q \in \Pi(\zeta(T))$.*

Corollary 5 ([30])**.** *Let $\mathfrak{X}$ be a subclass of $\boldsymbol{SD\mathfrak{F}}$ and suppose that there is a positive integer n such that $G^n = \langle 1 \rangle$ for every $G \in \mathfrak{X}$. Then the class $A(\mathfrak{X})$ consists of all torsion-free abelian groups A such that A/A^n is finite.*

Corollary 6 ([30])**.** *Let $\mathfrak{X}$ be a subclass of $\boldsymbol{SD\mathfrak{F}}$ and suppose that each group G of this class is bounded. Then the class $A(\mathfrak{X})$ is exactly the class of all co-layer-finite torsion-free abelian groups.*

Another approach was considered in the paper [32].

Theorem 18 ([32])**.** *Let T be a subgroup of a direct product of finite groups and let $q \in \Pi(\zeta(T))$. If A is a torsion-free abelian group such that A/A^q is infinite, then $A \notin A(T)$.*

Theorem 19 ([32])**.** *Let G be an FC–group whose periodic part T is a subgroup of a direct product of finite groups. Put $A = G/Tor(G)$ and suppose that A is countable. Let*

$$\langle 1 \rangle = A_0 \leq A_1 \leq \cdots \leq A_n \leq A_{n+1} \leq \cdots \bigcup_{n \geq 0} A_n = A$$

be an ascending series of pure subgroups such that the factor-groups A_n/A_{n-1} *have all finite* 0*–rank equal to* 1*. If* A/A^q *is finite for every* $q \in \Pi(\zeta(T))$ *and the sets* $Sp(A_n/A_{n-1}) \cap \Pi(\zeta(T))$ *are all finite, then* G *can be embedded in a direct product of finite groups and a torsion-free abelian group.*

Theorem 20 ([32])**.** *Let* T *be a subgroup of a direct product of finite groups, and suppose that* $\Pi(\zeta(T))$ *is infinite. Then* $A \in A(T)$ *if and only if* A *has an ascending series of pure subgroups*

$$\langle 1\rangle = A_0 \le A_1 \le \cdots \le A_n \le A_{n+1} \le \cdots \bigcup_{n\ge 0} A_n = A$$

whose factors A_n/A_{n-1} *have all finite* 0*–rank equal to* 1*, such that*

(a) all the sets $Sp(A_n/A_{n-1})\Pi(\zeta(T))$ *are finite, and*

(b) if $q \in \Pi(\zeta(T))$*, there exists* $t(q) \ge 1$ *such that* $q \in Sp(A_{n+1}/A_n)$ *for every* $n > t(q)$*.*

Theorem 21 ([32])**.** *Let* T *be a subgroup of a direct product of finite groups and suppose that* $\Pi(\zeta(T))$ *is finite, but* T *is unbounded. Then* $A \in A(T)$ *if and only if* A *is countable and* A/A^q *is finite for all* $q \in \Pi(\zeta(T))$*.*

On the other hand the classes $\mathrm{T}(\mathfrak{Y})$ were studied in the paper [33].

Theorem 22 ([33])**.** *Let* A *be a torsion-free abelian group. Suppose that there exists some prime* q *such that* A/A^q *is infinite. If* T *is a subgroup of a direct product of finite groups and* $q \in \Pi(\zeta(T))$*, then* $T \in T(A)$*.*

Corollary 7 ([33])**.** *Let* $\mathfrak{Y} \subseteq \mathfrak{A}^{tf}$*. If* $\mathfrak{Y}$ *contains a free abelian group of infinite* 0*–rank, then* $T(\mathfrak{Y})$ *is exactly the class of all periodic* FC*–groups with identity center.*

Corollary 8 ([33])**.** $T(\mathfrak{A}^{tf})$ *is the class of all periodic* FC*–groups with identity center.*

Theorem 23 ([33])**.** *Let* G *be an* FC*–group whose periodic part* T *is a subgroup of a direct product of finite groups. Put* $A = G/Tor(G)$ *and suppose that* A *is countable. Let*

$$\langle 1\rangle = A_0 \le A_1 \le \cdots \le A_n \le A_{n+1} \le \cdots \bigcup_{n\ge 0} A_n = A$$

be an ascending series of pure subgroups whose factors A_n/A_{n-1} *have all finite* 0*–rank equal to* 1*. Put*

$$\Theta = \{q \mid q \text{ is prime and } A/A^q \text{ is infinite}\}.$$

If $\Theta \cap \Pi(\zeta(T)) = \emptyset$ and all the sets $Sp(A_n/A_{n-1})\Pi(\zeta(T))$ are finite, then G can be embedded in a direct product of finite groups and a torsion-free abelian group.

Theorem 24 ([33]). *Let A be a countable torsion-free abelian group and let*

$$\Theta = \{q \mid q \text{ is prime and } A/A^q \text{ is infinite}\}.$$

Suppose further that

$$\langle 1 \rangle = A_0 \leq A_1 \leq \cdots \leq A_n \leq A_{n+1} \leq \cdots \bigcup_{n \geq 0} A_n = A$$

is an ascending series of pure subgroups whose factors A_n/A_{n-1} have all finite 0–rank equal to 1. If T is a subgroup of a direct product of finite groups, then $T \in T(A)$ if and only if $\Theta \cap \Pi(\zeta(T)) = \emptyset$ and all the sets $Sp(A_n/A_{n-1})\Pi(\zeta(T))$ are finite.

2. Another topic related to the class of FC–groups

In the theory of FC–groups, nilpotent groups of class at most 2 play a very specific role, as they ensure the proof of some highlighting results of the theory. For example, Yu. M. Gorchakov showed that the factor group of an FC–group by the second hypercenter can be embedded in a direct product of finite groups (see [22, Corollary II.3.8]). Also, any FC–group contains an equipotent nilpotent subgroup of class 2 (T. Ya. Semenova; see [22, Theorem III.1.8]), although this fails even for abelian subgroups of FC–groups. Nilpotent FC–groups of class 2 were studied in the paper [34]. Following Yu. M. Gorchakov (see [22, Chapter I, Section 4]), a class of groups $\mathfrak{D}$ is called *a direct variety* if it is closed under taking of subgroups, factor groups, and direct products. If $\mathfrak{X}$ is an arbitrary class of groups, then *the direct variety generated by* $\mathfrak{X}$ is by definition the class $\mathbf{QSD}\mathfrak{X}$ (see [22, Lemma I.4.2]).

Theorem 25 ([34]). *Let $\mathfrak{X}$ be the class of all nilpotent FC–groups having nilpotency class at most 2. Then $\mathfrak{X}$ is generated as a direct variety by all FC–groups having quasicyclic derived subgroups.*

Let $\mathfrak{N}_2$ be the class of all nilpotent groups having nilpotency class at most 2, and let $\mathfrak{F}\mathfrak{A}$ the class of groups having finite derived subgroup (finite–by–abelian groups).

Theorem 26 ([34]). *If $G \in \mathfrak{N}_2 \cap \boldsymbol{QSD}(\mathfrak{F}\mathfrak{A})$, then $G/\zeta(G)$ can be embedded in a direct product of finite groups.*

This result has an important consequence. M. J. Tomkinson had shown that the derived subgroup of a periodic FC–group belonged to the class $\mathbf{QSD}\mathfrak{F}$ ([64, Theorem 3.6]). In a natural way the following arise: Do all periodic FC–groups belong to $\mathbf{QSD}(\mathfrak{F}\mathfrak{A})$?; see [64, Question 3.H]. Yu. M. Gorchakov had constructed an example of a p–group G such that $[G,G] = \zeta(G)$ was a quasicyclic subgroup and $G/\zeta(G) \notin \mathbf{SD}\mathfrak{F}$ ([22, Example II.2.11]). Theorem 24 shows that this group is not contained in $\mathbf{QSD}(\mathfrak{F}\mathfrak{A})$, so we obtain

Corollary 9 ([34]). *The class of periodic FC–groups is not contained in the direct variety generated by the groups with finite derived subgroups.*

Theorem 27 ([34]). *Let G be an FC–group and suppose that $G \in \mathfrak{N}_2$. If $[G,G]$ has no element of order 2 and $G/\zeta(G) \in \boldsymbol{SD}\mathfrak{F}$, then $G \in \boldsymbol{QSD}(\mathfrak{F}\mathfrak{A})$.*

Recall that a p–group G is called *extraspecial* if $\zeta(G) = [G,G]$ is a subgroup of order p and $G/\zeta(G)$ is an elementary abelian p–group, where p is a prime.

Theorem 28 ([34]). *Let G be an extraspecial p–group. If $G \in \boldsymbol{QSD}\mathfrak{F}$, then G can be embedded in a central product of non-abelian subgroups of order p^3.*

Theorem 28 gives a positive answer to [64, Question 3G].

A group G is called *a $\mathfrak{Z}$–group* if for each infinite cardinal $\mathfrak{m}$ and each subset S of G such that $|S| < \mathfrak{m}$ we have $|G : C_G(S)| < \mathfrak{m}$ ([64, Section 3]). This class of groups hah appeared in a paper of P. Hall [23]. The class of periodic $\mathfrak{Z}$–groups includes the direct variety generated by the class of finite groups ([23]). However it was unknown whether or not these mentioned classes coincide ([64, Question 3F]). M. J. Tomkinson constructed an example of an extraspecial p–group that could not be embedded in a central product of non-abelian groups of order p^3 ([64, Example 3.16]). Applying Theorem 28 above, this group cannot be contained in $\mathbf{QSD}\mathfrak{F}$, so we have the following

Corollary 10 ([34]). *The class $\boldsymbol{QSD}\mathfrak{F}$ is not equal to the class of periodic $\mathfrak{Z}$–groups.*

The study of the relationship between the classes $\mathbf{QSD}\mathfrak{F}$ and FC–groups was continued by L. A. Kurdachenko in the paper [35]. Since P. Hall [23] had constructed an example of n extraspecial p–group outside of the class $\mathbf{QSD}\mathfrak{F}$, it was known that the class $\mathbf{QSD}\mathfrak{F}$ and the class of periodic FC–groups are not coincide. Therefore it appeared in the most natural way to compare natural the direct variety of FC–groups and the class $\mathbf{QSD}\mathfrak{F}$, in particular, the minimal direct variety of FC–groups and $\mathbf{QSD}\mathfrak{F}$. This natural approach were already applied to some other important classes of groups, for instance, varieties and formations. At the beginning of the research, the natural question on the existence of such direct variety raised, that is there is a minimal direct variety outside $\mathbf{QSD}\mathfrak{F}$? In [35, Theorem 1], an strategy to go further on this was developed. Actually, P. Hall had constructed an extraspecial p–group, which have allowed to solve positively the question. In this setting, the following result generalizes Theorem 28.

Theorem 29 ([36])**.** *Let $G \in \mathfrak{N}_2 \cap \boldsymbol{QSD}\mathfrak{F}$. Then G can be embedded in a central product of finite $\mathfrak{N}_2$–groups with two generators.*

This theorem answered the following question of the theory of FC–groups. The results of P. Hall, Yu. M. Gorchakov and M. J. Tomkinson showed that the class $\mathfrak{V}$ of those FC–groups, whose factor-groups by the center can be embedded in a direct product of finite groups formed a distinguished class of FC–groups that included the class $\mathbf{QSD}\mathfrak{F}$ (see [22, Theorem I.4.6]). Hence the following question appeared: is the class $\mathfrak{V}$ a direct variety? The answer was negative, as the following result proved.

Theorem 30 ([36])**.** *There exists an FC–group G with the following properties:*

(1) G is a p–group, p prime;

(2) $G/\zeta(G) \in \boldsymbol{SD}\mathfrak{F}$; and

(3) G has an epimorphic image B such that $B/\zeta(B) \notin \boldsymbol{SD}\mathfrak{F}$.

In particular, the class $\mathfrak{V}$ is not a direct variety.

M. J. Tomkinson proved that every periodic residually finite FC–group G can be embedded in a residually finite group $L = TZ$, where T is a subgroup of a direct product of finite groups and $Z \leq \zeta(L)$ ([64, Theorem 2.24]). This Theorem shoved that the structure of a periodic FC–group heavily depends on the structure its central subgroups. Related to this, there are the following results.

Theorem 31 ([31])**.** *Let $G = TZ$, where T is a subgroup of a direct product of finite groups and $Z \leq \zeta(G)$. If Z is a bounded subgroup, then G can be embedded in a direct product of finite subgroups.*

Theorem 32 ([31])**.** *Let $G = TZ$, where T is a subgroup of a direct product of finite groups and $Z \leq \zeta(G)$. If $Z/(Z \cap T)$ is a layer-finite group and G is residually finite, then G can be embedded in a direct product of finite subgroups.*

Corollary 11 ([31]). *Let G be a residually finite periodic FC–group. If the factor-group $G/[G,G]$ is layer-finite, then G can be embedded in a direct product of finite subgroups.*

Theorem 33 ([31]). *Let G be a residually finite periodic FC–group. If the center $\zeta(G)$ is layer-finite, then G can be embedded in a direct product of finite subgroups.*

If G is a periodic FC–group, then $G/\zeta(G) \in \mathbf{SD}\mathfrak{F}$ (see [22, Corollary II.3.7]) and $[G,G] \in \mathbf{SD}\mathfrak{F}$ ([64, Corollary 2.27]). Hence the following question makes sense: If G is an FC–group and $[G,G] \in \mathbf{SD}\mathfrak{F}$, is it necessarily true that $G/\zeta(G) \in \mathbf{SD}\|\mathfrak{F}$? ([64, Question 2A]). Some results on this topic were obtained in [44].

Theorem 34 ([44]). *Suppose that G is an FC–group such that $[G,G] \in \boldsymbol{SD}\mathfrak{F}$ but $G/\zeta(G) \notin \boldsymbol{SD}\mathfrak{F}$, and let $\pi = \Pi(\zeta(G) \cap [G,G])$. Then there is a metabelian section $H = U/V$ of G satisfying the following conditions:*

(1) G/U is finite;

(2) H is a p–group for some prime p;

(3) $[H,H]$ is bounded abelian; and

(4) $H/\zeta(H) \notin \boldsymbol{SD}\mathfrak{F}$.

Theorem 35 ([44]). *Let G be an FC–group. If $[G,G] \in \boldsymbol{SD}\mathfrak{F}$ and the Sylow p–subgroups of $[G,G]$ are countable for every $p \in \Pi(\zeta(G)[G,G])$, then $G/\zeta(G) \in \boldsymbol{SD}\mathfrak{F}$.*

Theorem 36 ([44]). *Let G be an FC–group and let $A = G/[G,G]$. If $[G,G] \in \boldsymbol{SD}\mathfrak{F}$ and A/A^p is countable for every $p \in \Pi(\zeta(G) \cap [G,G])$, then $G/\zeta(G) \in \boldsymbol{SD}\mathfrak{F}$.*

Let G be a residually finite group. We say that G *is an strong residually finite group* if every factor-group of G is residually finite as well ([40]). Examples of strong residually finite groups are direct products of finite simple non-abelian groups, bounded abelian groups and periodic FC–groups with finite Sylow subgroups. If G is a residually finite group, then G can be given a topology in which the family of all subgroups of finite index in G forms the neighborhoods of the identity. In fact G becomes a topological group and the topology is called *the profinite topology* on G. Under this meaning, a strong residually finite group G is a residually finite group in which every normal subgroup is closed in the profinite topology on G. In the paper [40], strong residually finite FC–groups were characterized. Main results of this paper can be formulated as follows.

Theorem 37 ([40])**.** *Let G be an FC–group and let $T = Tor(G)$. Then G is strong residually finite if and only if the following conditions hold:*

(1) G/T has finite 0–rank;

(2) $Sp(G/T) = \emptyset$; and

(3) T is a strong residually finite group.

Theorem 38 ([40])**.** *Let G be a locally nilpotent FC–group. Then G is a strong residually finite group if and only if $\zeta(G)$ includes a finitely generated torsion-free subgroup V such that $G/V = Dr_{p\in\Pi(G)}Z_p$, where Z_p is a bounded central–by–finite group.*

Let M be a minimal normal subgroup of a group G. Since $[M, G]$ is a G–invariant subgroup of M either $[M, G] = \langle 1\rangle$ or $[M, G] = M$. In the first case M is central in G. In the other, $C_G(M) \not\leq G$, and so M is non-central in G. Consider *the socle of the group G*; this is the subgroup $\mathrm{Soc}(G)$ generated by all minimal normal subgroups of G. It is known that $\mathrm{Soc}(G) = \mathrm{Dr}_{\lambda\in\Lambda}M_\lambda$ is the direct product of some of them. Put

$$Z = \{\lambda \in \Lambda \mid M \text{ is central in } G\}$$

and set $E = \Lambda \setminus Z$. Then $\mathrm{Soc}(G) = S_1 \times S_2$, where $S_1 = \mathrm{Dr}_{\lambda\in Z}M_\lambda$ and $S_2 = \mathrm{Dr}_{\lambda\in E}M_\lambda$. It is not hard to see that S_1 and S_2 are independent of the choice of the decomposition of $\mathrm{Soc}(G)$. The subgroup $S1 = \mathrm{Socc}(G)$ is called *the central socle of* G and the subgroup $S_2 = \mathrm{Socnc}(G)$ is called *the non-central socle of* G. Starting from the non-central socle we may construct the upper noncentral socle series of G,

$$\langle 1\rangle = M_0 \leq M_1 \cdots \leq M_\alpha \leq M_{\alpha+1} \leq \cdots M_\gamma,$$

where $M_1 = \mathrm{Socnc}(G)$, $M_{\alpha+1}/M_\alpha = \mathrm{Socnc}(G/M_\alpha)$ for all $\alpha < \gamma$ and $\mathrm{Socnc}(G/M_\gamma) = \langle 1\rangle$. The last term M_γ of this series is called *the non-central hypersocle of* G, and will be denoted by $Z^*(G)$.

Theorem 39 ([40])**.** *Let G be an FC–group. Then G is a strong residually finite group if and only if $G/Z^*(G)$ is a strong residually finite group.*

Theorem 40 ([40])**.** *Let G be a periodic locally soluble FC–group. If the Sylow subgroups of $G/Z^*(G)$ are bounded central–by–finite groups, then G is residually finite.*

Corollary 12 ([40])**.** *Let G be a periodic locally soluble FC–group. If the Sylow subgroups of G are bounded central–by–finite groups, then G is residually finite.*

Corollary 13 ([40]). *Let G be a periodic locally soluble FC–group. If the Sylow subgroups of $G/Z^*(G)$ are bounded central–by–finite groups, then G is strong residually finite.*

Theorem 41 ([40]). *Let G be a (hypercentral–by–hypercentral) FC–group. Then G is strong residually finite group if and only if $G/Z^*(G) \leq T \leq A$, where*

(1) A is a torsion-free abelian group of finite rank and $Sp(A) = \emptyset$;

(2) T has a normal subgroup $L = Dr_{p\in\Pi(L)}L_p$, where L_p is a finite p–group;

(3) $T/L = Dr_{p\in\Pi(T/L)}Q_p$, where Q_p is a bounded central–by–finite p–group; and

(4) The Sylow p–subgroups of T are bounded central–by–finite groups.

Theorem 42 ([40]). *Let G be a periodic FC–group. If G is strong residually finite group, then G can be embedded in a direct product of finite groups.*

Corollary 14 ([40]). *Let G be a periodic FC–group. Then G is a strong residually finite group if and only if every factor-group of G can be embedded in a direct product of finite groups.*

3. Some generalizations of FC–groups

A Chernikov groups is the most classic generalization of a finite group. Thus the first natural generalization of the class of FC–groups is the class of CC–groups or *groups with Chernikov conjugacy classes.* Ya. D.Polovitsky [53] introduced this class and obtained some initial results. Although CC–groups are not investigated as far as FC–groups, they are the subject of many recent papers. As in FC–groups, the first step here is extending the fundamental results on finite groups to periodic CC–groups. This was done in papers due to J. Alcázar and J. Otal [1], J. Otal and J. M. Peña [49, 50, 51] and J. Otal, J. M. Peña, M. J. Tomkinson [52]. In the papers of M. González and J. Otal [13, 14, 15] the important results of embeddings due formerly to P. Hall, Yu. M. Gorchakov and M. J. Tomkinson on FC–groups could be extended to the class of periodic CC–groups. In the paper by S. Franciosi, F. de Giovanni and M. J. Tomkinson [10], the consideration of groups with polycyclic–by–finite conjugacy classes or PC–groups. In a series of papers, L. A. Kurdachenko considered another classes of $\mathfrak{X}C$–groups. A natural generalization of the classes of Chernikov groups and polycyclic–by–finite groups is the class of minimax groups. Recall that a group G is called *minimax* if G has a finite subnormal series whose factors are either polycyclic–by–finite or Chernikov groups. A group G is called an MC–group or *group with minimax conjugacy classes* if $Coc_G(g^G)$ is minimax for each $g \in G$. This class was introduced by L. A. Kurdachenko in the paper [37]. Whitin the paper the following results were obtained.

Theorem 43 ([37])**.** *Let G be an MC–group. Then for each element $g \in G$, the subgroup $\langle g\rangle^G$ is minimax.*

Corollary 15 ([37])**.** *Let G be an MC–group. Then for each element $g \in G$, the subgroup $[G, g]$ is minimax.*

Theorem 44 ([37])**.** *Let G be an MC–group. Then the following assertions are equivalent:*

(1) G is a Baer-nilpotent group;

(2) G is a locally nilpotent group;

(3) G is a $\mathfrak{Z}$–group;

(4) G is a hypercentral group;

(5) G has a ascending central series of length at most 2ω;

(6) G is a group with the normalizer condition; and

(7) G is an $\overline{N}$–group.

Theorem 45 ([37])**.** *Let G be an MC–group. Then the following assertions are equivalent:*

(1) G is an SN–group;

(2) G is an SN^*–group;

(3) G is an $\overline{SN}$–group;

(4) G is an SI–group;

(5) G is a hyperabelian group;

(6) G is a hypoabelian group;

(7) G is an $\overline{SI}$–group;

(8) G is a locally soluble group;

(9) G is a radical group;

(10) G has a ascending series of normal subgroups whose factors are abelian and whose length is at most ω;

(11) G has a descending series of normal subgroups whose factors are abelian and whose length is at most $\omega + 1$;

(12) G is a group with the normalizer condition; and

(13) G is an $\overline{N}$–group.

The study of MC–groups was continued in [39].

Theorem 46 ([39]). *Let G be an MC–group. Then we have*

(1) G has an ascending series of normal subgroups whose factors are finite groups or abelian minimax groups.

(2) the chief factors of G are finite.

(3) every maximal subgroup of G has finite index.

If G is a group, we denote by $D(G)$ the subgroup generated by all Chernikov divisible normal subgroups of G. It is not hard to see that $D(G)$ is a periodic divisible abelian normal subgroup of G. If G is an MC–group and $D(G) = \langle 1 \rangle$, then we say that G is *a reduced MC–group*. A group G is said to be an M_rC–group if $Coc_G(x^G)$ is a reduced minimax group for each element $x \in G$ (see [39]).

Theorem 47 ([39]). *Let G be an M_rC–group. Then for each element $g \in G$, the subgroup $\langle g \rangle^G$ is reduced minimax.*

Corollary 16 ([39]). *Let G be an MC–group. Then $G/D(G)$ is an M_rC–group.*

Corollary 17 ([39]). *Let G be an MC–group (respectively, an M_rC–group). If $G = Fratt(G)$, then G is $\mathfrak{F}$–perfect and nilpotent of class at most 2 (respectively, G is divisible abelian).*

Theorem 48 ([39]). *(1) Let G be an MC–group. If $G/Fratt(G)$ is minimax, then there exists a finite subset X such that $L = \langle X \rangle^G$ is minimax and G/L is an $\mathfrak{F}$–perfect nilpotent group of class at most 2.*

(2) Let G be an M_rC–group. If $G/Fratt(G)$ is minimax, then there exists a finite subset X such that $L = \langle X \rangle^G$ is reduced minimax and G/L is divisible abelian.

Theorem 49 ([39]). *Let G be an M_rC–group. Then*

(1) $Fratt(G)$ *is locally nilpotent.*

(2) *If* $G/Fratt(G)$ *is locally nilpotent, then so is* G.

We recall that an abelian group A is called an $\mathfrak{A}_0$–group if the Sylow p–subgroups of A are Chernikov and $A/Tor(A)$ has finite 0–rank. A soluble group G is said to be an $\mathfrak{S}_0$–group, if G has a finite series of normal subgroups whose factors are abelian $\mathfrak{A}_0$–groups. Related to this, we have the class $\mathfrak{S}_0\mathfrak{F}$ of finite extensions of $\mathfrak{S}_0$, which is a large extension of the class of minimax groups. Following A.I. Maltsev [45] we say that a group G *has finite special rank* r if every finitely generated subgroup of G has at most r generators and r is the minimum under such property. The class of abelian groups of finite special rank will denote by $\mathfrak{A}^{\triangle}$, whereas the class of soluble groups of finite special rank will denoted by $\mathfrak{S}^{\triangle}$. In the paper [43], $\mathfrak{A}_0C$–groups and $\mathfrak{A}^{\triangle}C$–groups were considered in some detail. If $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of groups, then *the central direct product* of the given groups is

$$\mathrm{Zr}_{\lambda\in\Lambda}G_\lambda = (\mathrm{Dr}_{\lambda\in\Lambda}G_\lambda)\zeta(\mathrm{Cr}_{\lambda\in\Lambda}G_\lambda).$$

Theorem 50 ([43]). *Let* G *be an* $\mathfrak{A}_0C$*–groups and let* $g \in G$. *Then* $\langle g\rangle^G \in \mathfrak{A}_0$.

Theorem 51 ([43]). *Let* G *be a countable* $\mathfrak{A}_0C$*–group. Then* G *is isomorphic to a subgroup of* $Zr_{n\geq 1}(A_n \times B_n)$, *where the groups* A_n *and* B_n *satisfy the following conditions:*

(1) $A_n,\ B_n \in \mathfrak{N}_2$;

(2) $[A_n, A_n]$ *is a periodic* $\mathfrak{A}_0$*–subgroup;*

(3) B_n *is a torsion-free group; and*

(4) $[B_n, B_n]$ *has finite* 0*–rank.*

Theorem 52 ([43]). *Let* G *be a countable* $\mathfrak{A}^{\triangle}C$*–group. Then* G *is isomorphic to a subgroup of* $Zr_{n\geq 1}G_n$, *where the groups* G_n *satisfy the following conditions:*

(1) $G_n \in \mathfrak{N}_2$;

(2) $[G_n, G_n]$ *has finite special rank equals to* 1.

Moreover, if $[G_n, G_n]$ *is torsion-free, then* G_n *is also torsion-free.*

Let $\mathfrak{H}$ be the class of all groups G such that $\zeta(G) \cong \mathbb{Q}$ and $G/\zeta(G) \cong \mathbb{Q} \times \mathbb{Q}$ (here $\mathbb{Q}$ stands the full rational group), and all identity groups.

Theorem 53 ([43]). *Let G be a countable $\mathfrak{A}^{\triangle}C$–group. Then $G \in \boldsymbol{QSD}\mathfrak{H}$.*

In the paper [38], locally nilpotent $\mathfrak{S}_0C$–groups were studied. We quote some of the main results of the paper.

Theorem 54 ([38]). *Let G be a locally nilpotent $\mathfrak{S}_0C$–groups and let $g \in G$. Then $\langle g\rangle^G \in \mathfrak{S}_0$.*

Theorem 55 ([38]). *Let G be an $\mathfrak{S}_0C$–group. Then the following assertions are equivalent:*

(1) G is a Baer-nilpotent group;

(2) G is a locally nilpotent group;

(3) G is a $\overline{\mathfrak{Z}}$–group;

(4) G is an Engel group;

(5) G has a ascending central series of length at most 3ω;

(6) G is a group with the normalizer condition; and

(7) G is an $\overline{N}$–group.

At the beginning of this survey-paper we mentioned a classical result due to I. Schur [55], namely that if the center of a group G has finite index, then the derived subgroup $[G, G]$ is finite. Following [11], we say that a class of groups $\mathfrak{X}$ is *a Schur class* if it satisfies that $G/\zeta(G) \in \mathfrak{X}$ implies that $[G, G] \in \mathfrak{X}$. It is not hard to show that the class of polycyclic–by–finite groups is a Schur class. Ya. D. Polovitsky [53] proved that the class of Chernikov groups is also a Schur class. The following theorem generalizes these two results.

Theorem 56 ([37]). *Let G be a group. Suppose that $G/\zeta(G)$ is a minimax group. Then the derived subgroup $[G, G]$ is also minimax.*

Other examples of Schur classes were developed in the paper [11]. Denote by $\mathfrak{R}_0$ be the class of all groups having a finite series of subnormal subgroups whose factors either are locally finite or infinite cyclic.

Theorem 57 ([11]). *The following classes are Schur classes: $\mathfrak{S}_0\mathfrak{F}$, $\mathfrak{S}^{\triangle}\mathfrak{F}$, $\mathfrak{R}_0$, layer-finite groups and layer-Chernikov groups.*

Let G be a group. If H is a subgroup of G, then we denote the set of all conjugates of H in G by $Cl_G(H) = \{H^g \mid g \in G\}$. This set is called *the conjugacy class of H in G*. B. H. Neumann in the paper [48] characterized the groups G such that $Cl_G(H)$ is finite for every subgroup H of G: these are the central–by–finite groups. Since $|Cl_G(H)| = |G : N_G(H)|$, the finiteness of this index is equivalent to that its core, and this remark allow to consider Neumann's result in larger classes of groups. Given a subgroup H of a group G, *the normalizerof the conjugacy class of H in G* is by definition

$$N_G(Cl_G(H)) = \bigcap_{g\in G} N_G(H^g) = \bigcup_{g\in G} N_G(H)^g.$$

Let $\mathfrak{X}$ be a class of groups. The group G is said to have *$\mathfrak{X}$–classes of conjugate subgroups* if $G/N_G(Cl_G(H)) \in \mathfrak{X}$ for every subgroup H of G ([42]). If $\mathfrak{I}$ is the class of all identity groups, then it is clear that G has $\mathfrak{I}$–classes of conjugate subgroups if and only if every subgroup of G is normal, i.e. G is a Dedekind group. If $\mathfrak{X} = \mathfrak{F}$, then we clearly obtain the groups with finite conjugacy classes of subgroups, that is the groups considered formerly by B. H. Neumann. Therefore, the next natural step is to consider classes of infinite groups close to that of finite groups and that were well-studied from different points of view. The first candidates are the formation $\mathfrak{C}$ of all Chernikov groups and the formation $\mathfrak{P}$ of all polycyclic–by–finite groups. In the paper [54], Ya. D. Polovizky had considered groups with Chernikov classes of conjugate subgroups and had proved that a periodic group with Chernikov classes of conjugate subgroups is central–by–Chernikov. In contrast to this, in the paper [41], it was constructed an example showing that in general that fact is not true. The description of groups with Chernikov classes of conjugate subgroups gives the following result.

Theorem 58 ([41])**.** *Let G be a group with Chernikov classes of conjugate subgroups. Then the following assertions hold.*

(1) G is abelian–by–Chernikov.

(2) $G/C_G(Tor(A))$ is finite.

(3) $[G, G]$ is Chernikov.

However for groups with polycyclic–by–finite classes of conjugate subgroups we recover B. H. Neumann's characterization.

Theorem 59 ([42])**.** *A group G has polycyclic–by–finite classes of conjugate subgroups if and only if the factor-group $G/\zeta(G)$ is polycyclic–by–finite.*

References

[1] J. Alcázar, J. Otal, *Sylow subgroups of groups with Chernikov conjugacy classes*, J. Algebra **110** (1987), 507-513.

[2] R. Baer, *Sylow theorems of infinite groups*, Duke Math. J. **6** (1940), 598-614.

[3] R. Baer, *Finiteness properties of groups*, Duke Math. J. **15** (1948), 1021-1032.

[4] S. N. Chernikov, *Infinite layer-finite groups*, Math. Sb. **22** (1948), 101-133.

[5] S. N. Chernikov, *On complementability of Sylow π–subgroups in some classes of infinite groups*, Math. Sb. **37** (1955), 557-566.

[6] S. N. Chernikov, *On groups with finite conjugate classes*, Doklady AN SSSR **114** (1957), 1177-1179.

[7] S. N. Chernikov, *On a structure of groups with finite conjugate classes*, Doklady AN SSSR **115** (1957), 60-63.

[8] S. N. Chernikov, *On layer-finite groups*, Math. Sb. **45** (1958), 415-416.

[9] A. P. Ditsman, *On p–groups*, Doklady Akad. Nauk SSSR **15** (1937), 71-76.

[10] S. Franciosi, F. de Giovanni, M. J. Tomkinson, *Groups with polycyclic–by–finite conjugacy classes*, Bolletino UMI **4B** (1990), 35-55.

[11] S. Franciosi, F. de Giovanni, L. A. Kurdachenko, *The Schur property and groups with uniform conjugacy classes*, J. Algebra **174** (1995), 823-847

[12] P. A. Golberg, *Sylow π–subgroups of locally normal groups*, Math. Sb. **19** (1946), 451-460.

[13] M. González, J. Otal, *P. Hall's covering group and embedding of countable CC–groups*, Comm. Algebra **18** (1990), 3405-3412.

[14] M. González, J. Otal, *Embedding theorems for residually Chernikov CC–groups*, Proc. Amer. Math. Soc. **123** (1995), 2383-2332.

[15] M. González, J. Otal, *The extension of results due to Gorchakov and Tomkinson from FC–groups to CC–groups*, J. Algebra **185** (1996), 314-328.

[16] Yu. M. Gorchakov, *On embedding of locally normal groups in direct products of finite groups*, Doklady AN SSSR **138** (1961), 26-28.

[17] Yu. M. Gorchakov, *On locally normal groups*, Doklady AN SSSR **147**, (1962), 537-539.

[18] Yu. M. Gorchakov, *On locally normal groups*, Math. Sb. **67** (1965), 244-254.

[19] Yu. M. Gorchakov, *Locally normal groups*, Sib. Math. J. **12** (1971), 1259-1272.

[20] Yu. M. Gorchakov, *Theorems of Prüfer-Kulikov type*, Algebra and Logic **13** (1974), 655-661.

[21] Yu. M. Gorchakov, *Subgroups of direct products*, Algebra and Logic **15** (1976), 622-627.

[22] Yu. M. Gorchakov, *The groups with finite classes of conjugacy elements*, Nauka, Moskow 1978.

[23] P. Hall, *Periodic FC–groups*, J. London Math. Soc. **34** (1959), 289-304.

[24] M. I. Kargapolov, *To theory of semisimple locally normal groups*, in *Scientific reports of higher school*, Fiz.- math. Sciences **6** (1958), 3-7.

[25] L. A. Kurdachenko, *FC–groups whose periodic part can be embedded in a direct product of finite groups*, Math. Notes **21** (1977), 6-12.

[26] L. A. Kurdachenko, *Structure of FC–groups whose periodic part can be embedded in a direct product of finite groups*, Math. Notes **25** (1979), 10-15.

[27] L. A. Kurdachenko, *Embeddability of an FC–group in the direct product of certain finite groups and a torsion-free abelian group*, Math. Notes **29** (1981), 186-193.

[28] L. A. Kurdachenko, *Some conditions for embeddability of an FC–group in a direct product of finite groups and a torsion-free abelian group*, Math. Sb. **42** (1982), 499-514.

[29] L. A. Kurdachenko, *On some non-periodic FC–groups*, in *Groups and systems of their subgroups*, Math. Institute Kiev- 1983, 92-100.

[30] L. A. Kurdachenko, *FC–groups with bounded periodic part*, Ukrain. Math. J. **35** (1983), 324-327.

[31] L. A. Kurdachenko, *Central factorization of FC–groups*, in *The structure of groups and their subgroup characterization*, Math. Institute Kiev 1984, 73-85.

[32] L. A. Kurdachenko, *Non-periodic FC–groups and related classes of locally normal groups and abelian groups without torsion*, Sib. Math. J. **27** (1986), 227-236.

[33] L. A. Kurdachenko, *FC–groups of countable torsion-free rank*, Math. Notes **40** (1986), 16-30.

[34] L. A. Kurdachenko, *The nilpotent FC–groups of class two*, Ukrain. Math. J. **39** (1987), 255-259.

[35] L. A. Kurdachenko, *On some classes of FC–groups*, in *The researches of groups with the restrictions on subgroups*, Math. Institute Kiev 1988, 34-41.

[36] L. A. Kurdachenko, *On some classes of periodic FC–groups*, Contemporary Math. **131** (1992), 499-509.

[37] L. A. Kurdachenko, *On groups with minimax conjugacy classes*, in *Infinite groups and adjoining algebraic structures*, Naukova Dumka, Kiev 1993, 160-177.

[38] L. A. Kurdachenko, *On normal closures of elements in generalized FC–groups*, in *Infinite groups 94 (Ravello 1994)*,Walter de Gruyter, Berlin 1996, 141-151.

[39] L. A. Kurdachenko, J. Otal, *Frattini properties of groups with minimax conjugacy classes*, Quaderni di Matematica **8** (2001), 221-237.

[40] L. A. Kurdachenko, J. Otal, *FC–groups all of whose factor groups are residually finite*, Comm. Algebra **31** (2003), 1235-1251.

[41] L. A. Kurdachenko, J. Otal, *Groups with Chernikov classes of conjugate subgroups*, J. Group Th. **8** (2005), 93-108.

[42] L. A. Kurdachenko, J. Otal, P. Soules, *Groups with polycyclic–by–finite conjugate classes of subgroups*, Comm. Algebra **32** (2004), 4769-4784.

[43] L. A. Kurdachenko, I. Ya. Subbotin, *Groups with restrictions for cocentralizers of elements*, Comm. Algebra **24** (1996), 1173-1187.

[44] L. A. Kurdachenko, J. Otal, M. J. Tomkinson, *FC–groups whose central factor group can be embedded in a direct product of finite groups*, Comm Algebra **28** (2000), 1343-1350.

[45] A. I. Maltsev, *On groups of finite rank*, Mat. Sb. **22** (1948), 351-352.

[46] B. H. Neumann, *Groups with finite classes of conjugate elements*, Proc. London Math. Soc. **1** (1951), 178-187.

[47] B. H. Neumann, *Groups covered by permutable subsets*, J. London Math. Soc. **29** (1954), 236-248.

[48] B. H. Neumann, *Groups with finite classes of conjugate subgroups*, Math. Z. **63** (1955), 76-96.

[49] J. Otal, J. M. Peña, *Characterizations of the conjugacy of Sylow p–subgroups of CC–groups*, Proc. Amer. Math. Soc. **106** (1989), 605-610.

[50] J. Otal, J. M. Peña, *Sylow theory of CC–groups*, Rendiconti Semin. Mat. Univ. Padova **86** (1991), 105-118.

[51] J. Otal, J. M. Peña, *Sylow theory of CC–groups: a survey*, London Math. Soc. Lecture Notes Ser. **160** (1991), 400-407.

[52] J. Otal, J. M. Peña, M. J. Tomkinson, *Locally inner automorphisms of CC–groups*, J. Algebra **141** (1991), 382-398.

[53] Ya. D. Polovicky, *Groups with extremal classes of conjugate elements*, Sib. Math. J. **5** (1964), 891-895.

[54] Ya. D. Polovicky, *The periodic groups with extremal classes of conjugate abelian subgroups*, Izvestiua Vysshyh Uchebnyh Zavedenij. Math. **4** (1977), 95-101.

[55] I. Schur, *Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **127** (1904), 20-50.

[56] M. J. Tomkinson, *Formation of locally soluble FC–groups*, Proc. London Math. Soc. **19** (1969), 675-708.

[57] M. J. Tomkinson, *Local conjugate classes*, Math. Z. **108** (1969), 202-212.

[58] M. J. Tomkinson, *Local conjugate classes*, Arch. Math. **20** (1969), 567-571.

[59] M. J. Tomkinson, *$\mathfrak{F}$–injectors of locally soluble FC–groups*, Glasgow Math. J. **10** (1969), 130-136.

[60] M. J. Tomkinson, *Extraspecial sections of periodic FC–groups*, Compositio Math. **31** (1975), 285-303.

[61] M. J. Tomkinson, *Residually finite periodic FC–groups*, J. London Math. Soc. **16** (1977), 221-228.

[62] M. J. Tomkinson, *On the commutator subgroup of a periodic FC–group*, Arch. Math. **31** (1978), 123-125.

[63] M. J. Tomkinson, *A characterization of residually finite periodic FC–groups*, Bull. London Math. Soc. **13** (1981), 133-137.

[64] M. J. Tomkinson, *FC–groups*, Pitman, Boston 1984.

[65] M. J. Tomkinson, *FC–groups: Recent Progress*, in *Infinite groups 1994*, Proceedings of the international conference, held in Ravello, Italy, May 23-27, 1994, Walter de Gruyter, Berlin 1996, 271-285.

Contact information

J. Otal Departamento de Matemáticas – IUMA, Universidad de Zaragoza, Pedro Cerbuna 12. 50009 Zaragoza, SPAIN
E-Mail: otal@unizar.es

N. N. Semko Department of Mathematics, National State Tax Service Academy of Ukraine, 09200 Irpen, UKRAINE.
E-Mail: n_semko@mail.ru

Received by the editors: 07.07.2009
and in final form 07.07.2009.