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Characterization of regular convolutions

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Abstract. A convolution is a mapping $\mathcal{C}$ of the set Z^+ of positive integers into the set $\mathscr{P}(Z^+)$ of all subsets of Z^+ such that, for any $n \in Z^+$, each member of $\mathcal{C}(n)$ is a divisor of n. If $\mathcal{D}(n)$ is the set of all divisors of n, for any n, then $\mathcal{D}$ is called the Dirichlet's convolution [2]. If $\mathcal{U}(n)$ is the set of all Unitary(square free) divisors of n, for any n, then $\mathcal{U}$ is called unitary(square free) convolution. Corresponding to any general convolution $\mathcal{C}$, we can define a binary relation $\leqslant_{\mathcal{C}}$ on Z^+ by '$m \leqslant_{\mathcal{C}} n$ if and only if $m \in \mathcal{C}(n)$'. In this paper, we present a characterization of regular convolution.

Introduction

A convolution is a mapping $\mathcal{C}$ of the set Z^+ of positive integers into the set $\mathscr{P}(Z^+)$ of subsets of Z^+ such that, for any $n \in Z^+$, $\mathcal{C}(n)$ is a nonempty set of divisors of n. If $\mathcal{C}(n)$ is the set of all divisors of n, for each $n \in Z^+$, then $\mathcal{C}$ is the classical Dirichlet convolution [2]. If

$$\mathcal{C}(n) = \{d \ / \ d|n \text{ and } (d, \frac{n}{d}) = 1\},$$

then $\mathcal{C}$ is the Unitary convolution [1]. As another example if

$$\mathcal{C}(n) = \{d \ / \ d|n \text{ and } m^k \text{ does not divide } d \text{ for any } m \in Z^+\},$$

then $\mathcal{C}$ is the k-free convolution. Corresponding to any convolution $\mathcal{C}$, we can define a binary relation $\leqslant_{\mathcal{C}}$ in a natural way by

$$m \leqslant_{\mathcal{C}} n \quad \text{if and only if} \quad m \in \mathcal{C}(n).$$

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$\leqslant_{\mathcal{C}}$ is a partial order on Z^+ and is called partial order induced by the convolution $\mathcal{C}$ [11], [12]. W. Narkiewicz [2] first proposed the concept of a regular convolution, and in this paper we present a lattice theoretic characterization of regular convolution and prove that the Dirichlet's convolution is the unique regular convolution that induces a lattice structure on $(Z^+, \leqslant_{\mathcal{C}})$.

1. Preliminaries

Let us recall that a partial order on a non-empty set X is defined as a binary relation $\leqslant$ on X which is reflexive ($a \leqslant a$), transitive ($a \leqslant b, b \leqslant c \Longrightarrow a \leqslant c$) and antisymmetric ($a \leqslant b, b \leqslant a \Longrightarrow a = b$) and that a pair $(X, \leqslant)$ is called a partially ordered set (poset) if X is a non-empty set and $\leqslant$ is a partial order on X.

For any $A \subseteq X$ and $x \in X$, x is called a lower(upper) bound of A if $x \leqslant a$ (respectively $a \leqslant x$) for all $a \in A$. We have the usual notations of the greatest lower bound (glb) and least upper bound (lub) of A in X. If A is a finite subset $\{a_1, a_2, \cdots, a_n\}$, the glb of A (lub of A) is denoted by $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ or $\bigwedge_{i=1}^{n} a_i$ (respectively by $a_1 \vee a_2 \vee \cdots \vee a_n$ or $\bigvee_{i=1}^{n} a_i$).

A partially ordered set $(X, \leqslant)$ is called a meet semi lattice if $a \wedge b$ ($=$glb$\{a, b\}$) exists for all a and $b \in X$. $(X, \leqslant)$ is called a join semi lattice if $a \vee b$ ($=$lub$\{a, b\}$) exists for all a and $b \in X$. A poset $(X, \leqslant)$ is called a lattice if it is both a meet and join semi lattice. Equivalently, lattice can also be defined as an algebraic system $(X, \wedge, \vee)$, where $\wedge$ and $\vee$ are binary operations which are associative, commutative and idempotent and satisfying the absorption laws, namely $a \wedge (a \vee b) = a = a \vee (a \wedge b)$ for all $a, b \in X$; in this case the partial order $\leqslant$ on X is such that $a \wedge b$ and $a \vee b$ are respectively the glb and lub of $\{a, b\}$. The algebraic operations $\wedge$ and $\vee$ and the partial order $\leqslant$ are related by

$$a = a \wedge b \iff a \leqslant b \iff a \vee b = b.$$

Throughout the paper Z^+, N, and P denote the set of positive integers, the set of non-negative integers, and set of prime numbers respectively.

Theorem 1 ([12]). *Let $\leqslant_{\mathcal{C}}$ be the binary relation induced by convolution $\mathcal{C}$. Then*

(1) *$\leqslant_{\mathcal{C}}$ is reflexive if and only if $n \in \mathcal{C}(n)$.*
(2) *$\leqslant_{\mathcal{C}}$ is transitive if and only, for any $n \in Z^+$, $\bigcup_{m \in \mathcal{C}(n)} \mathcal{C}(m) \subseteq \mathcal{C}(n)$.*
(3) *$\leqslant_{\mathcal{C}}$ is always antisymmetric.*

Corollary 1 ([12]). *The binary relation $\leqslant_{\mathcal{C}}$ induced by convolution $\mathcal{C}$ on Z^+ is a partial order if and only if $n \in \mathcal{C}(n)$ and $\bigcup_{m\in\mathcal{C}(n)} \mathcal{C}(m) \subseteq \mathcal{C}(n)$ for all $n \in Z^+$.*

Definition 1 ([12]). Let X and Y be non-empty sets and R and S be binary relations on X and Y respectively. A bijection $f\colon X \to Y$ is said to be a relation isomorphism of (X, R) into (Y, S) if, for any elements a and b in X,

$$aRb \text{ in } X \quad \text{if and only if} \quad f(a)Sf(b) \text{ in } Y.$$

Theorem 2 ([12]). *Let $\theta\colon Z^+ \to \sum_P N$ be the bijection defined by*

$$\theta(n)(p) = \textit{the largest } a \textit{ in } N \textit{ such that } p^a \textit{ divides } n,$$

Then a convolution $\mathcal{C}$ is multiplicative if and only if θ is a relation isomorphism of $(Z^+, \leqslant_{\mathcal{C}})$ onto $(\sum_P N, \leqslant_{\mathcal{C}})$.

Theorem 3 ([9], [10]). *For any multiplicative convolution $\mathcal{C}$, $(Z^+, \leqslant_{\mathcal{C}})$ is a lattice if and only if $(N, \leqslant^p_{\mathcal{C}})$ is a lattice for each prime p.*

Now we state the following theorems on co-maximality and prime filters.

Theorem 4 ([5]). *Let $(S, \wedge)$ be any meet semi lattice with smallest element 0 satisfying the descending chain condition. Also, suppose that every proper filter of S is prime. Then the following are equivalent to each other.*

(1) *For any x and $y \in S$, $x \| y \Longrightarrow x \wedge y = 0$.*
(2) *$S - \{0\}$ is a disjoint union of maximal chains.*
(3) *Any two incomparable filters of S are co-maximal.*

Theorem 5 ([5]). *Let $\mathcal{C}$ be any multiplicative convolution such that $(Z^+, \leqslant_{\mathcal{C}})$ is a meet semi lattice. Then any two incomparable prime filters of $(Z^+, \leqslant_{\mathcal{C}})$ are co-maximal if and only if any two incomparable prime filters of $(N, \leqslant^p_{\mathcal{C}})$ are co-maximal, for each $p \in P$.*

Theorem 6 ([3]). *Let p be a prime number. Then every proper filter in $(N, \leqslant^p_{\mathcal{C}})$ is prime if and only if $[p^a)$ is a prime filter in $(Z^+, \leqslant_{\mathcal{C}})$ for all $n > 0$.*

Theorem 7 ([3]). *A filter F of $(Z^+, \leqslant_{\mathcal{C}})$ is prime if and only if there exists unique $p \in P$ such that F^p is a prime filter of $(N, \leqslant^p_{\mathcal{C}})$ and $F^q = N$ for all $q \neq p$ in P and, in this case,*

$$F = \{n \in Z^+ \mid \theta(n)(p) \in F^p\}.$$

Theorem 8 ([3]). *Let F be a filter of $(Z^+, \leqslant_{\mathcal{C}})$. Then $F = [p^a)$ for some prime number p and a positive integer a which is join-irreducible in $(Z^+, \leqslant_{\mathcal{C}})$.*

Definition 2. Any complex valued function defined on the set Z^+ of positive integers is called an arithmetical function. The set of all arithmetical functions is denoted by $\mathscr{A}$.

The following is a routine verification using the properties of addition and multiplication of complex numbers.

Theorem 9. *For any arithmetical functions f and g, define*

$$(f+g)(n) = f(n) + g(n) \quad \textit{and} \quad (f \cdot g)(n) = f(n)g(n)$$

for any $n \in Z^+$.

Then $+$ and $\cdot$ are binary operations on the set $\mathscr{A}$ of arithmetical functions and $(\mathscr{A}, +, \cdot)$ is a commutative ring with unity in which the constant map $\bar{0}$ and $\bar{1}$ are the zero element and unity element respectively.

Definition 3. Let $\mathcal{C}$ be a convolution and f and g arithmetical functions and $\mathscr{C}$ be the field of complex numbers. Define $f\mathcal{C}g \colon Z^+ \to \mathscr{C}$ by

$$(f\mathcal{C}g)(n) = \sum_{d \in \mathcal{C}(n)} f(d)g(\frac{n}{d}).$$

We can consider $\mathcal{C}$ as a binary operation, as defined above, on the set $\mathscr{A}$ of arithmetical functions. W.Narkiewicz proposed the following definition.

Definition 4 ([2]). A convolution $\mathcal{C}$ is called *regular* if the following are satisfied.

(1) $(\mathscr{A}, +, \mathcal{C})$ is a commutative ring with unity, where $+$ is the point-wise addition. This ring will be denoted by $\mathscr{A}_{\mathcal{C}}$.
(2) If f and g are multiplicative arithmetical functions, then so is the product $f\mathcal{C}g$ (f is said to be multiplicative if $f(mn) = f(m)f(n)$.)
(3) The constant function $\bar{1}$, defined by $\bar{1}(n) = 1$ for all $n \in Z^+$, is a unit in the ring $\mathscr{A}_{\mathcal{C}}$.

It can be easily verified that the arithmetical function e, defined by

$$e(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1 \end{cases}$$

is the unity (the identity element with respect to the binary operation $\mathcal{C}$).

W. Narkiewicz proved the following two theorems.

Theorem 10 ([2]). *A convolution $\mathcal{C}$ is regular if and only if the following conditions are satisfied for any m, n and $d \in Z^+$.*

(1) *$\mathcal{C}$ is multiplicative convolution; i.e., $(m,n)=1 \Rightarrow \mathcal{C}(mn)=\mathcal{C}(m)\mathcal{C}(n)$.*
(2) *$d \in \mathcal{C}(m)$ and $m \in \mathcal{C}(n) \Leftrightarrow d \in \mathcal{C}(n)$ and $\frac{m}{d} \in \mathcal{C}(\frac{n}{d})$.*
(3) *$d \in \mathcal{C}(n) \Rightarrow \frac{n}{d} \in \mathcal{C}(n)$.*
(4) *$1 \in \mathcal{C}(n)$ and $n \in \mathcal{C}(n)$.*
(5) *For any prime number p and any $a \in Z^+$, $\mathcal{C}(p^a) = \{1, p^t, p^{2t}, \cdots, p^{rt}\}$, $rt = a$ for some positive integer t and $p^t \in \mathcal{C}(p^{2t})$, $p^{2t} \in \mathcal{C}(p^{3t})$,..., $p^{(r-1)t} \in \mathcal{C}(p^a)$.*

Theorem 11 ([2]). *Let $\mathcal{K}$ be the class of all decompositions of the set of non-negative integers into arithmetic progressions (finite or infinite) each containing 0 and no two progressions belonging to same decomposition have a positive integer in common. Let us associate with each $p \in P$, a member π_p of $\mathcal{K}$. For any $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $p_1, p_2, \cdots, p_r$ are distinct primes and $a_1, a_2, \cdots, a_r \in N$, define*

$$\mathcal{C}(n) = \{p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r} \mid b_i \leqslant a_i, \text{ and } b_i \text{ and } a_i \text{ belong to the same progression in } \pi_{p_i}.\}$$

Then $\mathcal{C}$ is a regular convolution and, conversely every regular convolution can be obtained in this way.

From the above theorems, it is clear that any regular convolution $\mathcal{C}$ is uniquely determined by a sequence $\{\pi_p\}_{p \in P}$ of decompositions of N into arithmetical progressions (finite or infinite) and we denote this by expression $\mathcal{C} \sim \{\pi_p\}_{p \in P}$.

Definition 5. For any two elements a and b in a partially ordered set $(X, \leqslant)$, a is said to be covered by b (b is a cover of a) if $a < b$ and there is no $c \in X$ such that $a < c < b$. This is denoted by $a - < b$.

We note that $\theta \colon Z^+ \to \sum_P N$ defined by

$$\theta(a)(p) = \text{the largest } n \text{ in } N \text{ such that } p^n \text{ divides } a, \quad \text{for any } a \in Z^+ \text{ and } p \in P$$

is a bijection.

2. Main results

In the following two theorems, we prove that any regular convolution $\mathcal{C}$ gives a meet semi lattice structure on $(Z^+, \leqslant_{\mathcal{C}})$ and the convolution $\mathcal{C}$ is completely characterized by certain lattice theoretic properties of $(Z^+, \leqslant_{\mathcal{C}})$. In particular Dirichlet's convolution is the only regular convolution $\mathcal{C}$ which gives a lattice structure on $(Z^+, \leqslant_{\mathcal{C}})$.

Theorem 12. *Let $\mathcal{C}$ be a convolution and $\leqslant_{\mathcal{C}}$ the relation on Z^+ induced by $\mathcal{C}$. Then $\mathcal{C}$ is a regular convolution if and only if the following properties are satisfied.*

(1) *$\theta\colon (Z^+, \leqslant_{\mathcal{C}}) \to \sum_{p\in P}(N, \leqslant^p_{\mathcal{C}})$ is a relation isomorphism.*
(2) *$(Z^+, \leqslant_{\mathcal{C}})$ is a meet semi lattice.*
(3) *Any two incomparable prime filters of $(Z^+, \leqslant_{\mathcal{C}})$ are co-maximal.*
(4) *F is a prime filter of $(Z^+, \leqslant_{\mathcal{C}})$ if and only if $F = [p^a)$ for some $p \in P$ and $a \in Z^+$.*
(5) *For any m and $n \in Z^+$, $m - <_{\mathcal{C}} n \Longrightarrow 1 - <_{\mathcal{C}} \frac{n}{m} \leqslant_{\mathcal{C}} n$.*

Proof. Suppose that $\mathcal{C}$ is a regular convolution. By Theorem 11, $\mathcal{C} \sim \{\pi_p\}_{p\in P}$, where each π_p is a decomposition of N into arithmetic progressions (finite or infinite) in which each progression contains 0 and no positive integer belongs to two distinct progressions. For any $a, b \in N$ and $p \in P$, let us write for convenience,

$$\langle a < b\rangle \in \pi_p \quad \Longleftrightarrow \quad a \text{ and } b \text{ belong to the same progression of } \pi_p.$$

Since $\mathcal{C}$ is regular, $\mathcal{C}$ satisfies properties (1)–(5) of Theorem 10. From (2) and (4) of Theorem 10 and Corollary 1, it follows that $\leqslant_{\mathcal{C}}$ is a partial order on Z^+. Since $\mathcal{C}$ is multiplicative, it follows from Theorem 2 that $\theta\colon Z^+ \to \sum_P N$ is an order isomorphism. Therefore the property (1) is satisfied. For simplicity and convenience, we shall write $\bar{n}$ for $\theta(n)$. For each $n \in Z^+$, $\bar{n}$ is the element in the direct sum $\sum_P N$ defined by

$$\bar{n}(p) = \text{the largest } a \text{ in } N \text{ such that } p^a \text{ divides } n.$$

$n \mapsto \bar{n}$ is an order isomorphism of $(Z^+, \leqslant_{\mathcal{C}})$ onto $\sum_{p\in P}(N, \leqslant^p_{\mathcal{C}})$, where for each $p \in P$, $\leqslant^p_{\mathcal{C}}$ is the partial order on N defined by

$$a \leqslant^p_{\mathcal{C}} b \quad \text{if and only if} \quad p^a \in \mathcal{C}(p^b).$$

For any m and $n \in Z^+$, let $m \wedge n$ be the element in Z^+ defined by

$$\overline{m \wedge n}(p) = \begin{cases} 0 & \text{if } \langle \overline{m}(p), \overline{n}(p)\rangle \notin \pi_p, \\ \min\{\overline{m}(p), \overline{n}(p)\} & \text{otherwise.} \end{cases}$$

for all $p \in P$. If $\langle \overline{m}(p), \overline{n}(p)\rangle \in \pi_p$, then

$$\overline{m}(p) \leqslant^p_{\mathcal{C}} \overline{n}(p) \quad \text{or} \quad \overline{n}(p) \leqslant^p_{\mathcal{C}} \overline{m}(p)$$

and hence $\overline{m \wedge n}(p) \leqslant \overline{m}(p)$ and $\overline{n}(p)$ for all $p \in P$. Therefore $m \wedge n$ is a lower bound of m and n in $(Z^+, \leqslant_{\mathcal{C}})$. Let k be any other lower bound of m and n. For any $p \in P$, if $\langle \overline{m}(p), \overline{n}(p) \rangle \in \pi_p$, then, since

$$\overline{k(p)} \leqslant^p_{\mathcal{C}} \overline{m}(p) \quad \text{and} \quad \overline{k(p)} \leqslant^p_{\mathcal{C}} \overline{n}(p),$$

we have

$$\overline{k(p)} \leqslant^p_{\mathcal{C}} \overline{m \wedge n}(p).$$

If $\langle \overline{m}(p), \overline{n}(p) \rangle \notin \pi_p$, then

$$\overline{k(p)} = 0 = \overline{m \wedge n}(p).$$

Thus $k \leqslant m \wedge n$. Therefore, $m \wedge n$ is the greatest lower bound of m and n in $(Z^+, \leqslant_{\mathcal{C}})$. Thus $(Z^+, \leqslant_{\mathcal{C}})$ is a meet semi lattice and hence the property (2) is satisfied.

To prove (3), by Theorem 5, it is enough if we prove that any two incomparable prime filters if $(N, \leqslant^p_{\mathcal{C}})$ are co-maximal for all $p \in P$. For any positive a and b, if a and b are incomparable in $(N, \leqslant^p_{\mathcal{C}})$, then $\langle a, b \rangle \notin \pi_p$ and hence a and b have no upper bound and therefore $a \vee b$ does not exist in $(N, \leqslant^p_{\mathcal{C}})$. Also, each progression in π_p is a maximal chain in $(N, \leqslant^p_{\mathcal{C}})$ and, for any a and $b \in N$, a and b are comparable if and only if $\langle a, b \rangle \in \pi_p$.

Therefore $(Z^+, \leqslant^p_{\mathcal{C}})$ is a disjoint union of maximal chains. Thus, by Theorem 4, any two incomparable prime filters of $(N, \leqslant^p_{\mathcal{C}})$ are co-maximal. Therefore, by Theorem 5, any two incomparable prime filters of $(Z^+, \leqslant_{\mathcal{C}})$ are co-maximal. This proves (3).

(4) follows from Theorem 6 and Theorem 7 and from the discussion made above.

To prove (5), let m and $n \in Z^+$ such that $m - <_{\mathcal{C}} n$. By Theorem 10 (3), we get that $\frac{m}{n} \leqslant_{\mathcal{C}} n$. Let us write

$$n = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r} \quad \text{and} \quad m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

where $p_1, p_2, \cdots, p_r$ are distinct primes and each $b_i > 0$ such that $0 \leqslant^{p_i}_{\mathcal{C}} a_i \leqslant^{p_i}_{\mathcal{C}} b_i$. Since $m \neq n$, there exists i such that $a_i \leqslant^{p_i}_{\mathcal{C}} b_i$. Now, if $a_j \leqslant^{p_i}_{\mathcal{C}} b_j$ for some $j \neq i$, then the element $k = p_1^{c_1} p_2^{c_2} \cdots p_r^{c_r}$, where

$$c_s = \begin{cases} a_s & \text{if } s \neq i, \\ b_s & \text{if } s = i. \end{cases}$$

will be between m and n (that is, $m <_{\mathcal{C}} k <_{\mathcal{C}} n$) which is a contradiction.

Therefore $a_j = b_j$ for all $j \neq i$ and hence $\frac{n}{m} = p_i^{a_i - b_i}$.

Since $\langle a_i, b_i \rangle \in pi_{p_i}$, there exists $t > 0$ such that

$$b_i = ut \quad \text{and} \quad a_i = vt$$

for some u and v with $v < u$. Also, $vt, (v+1)t, \cdots, ut$ are all in the same progression. Since $m- <_{\mathcal{C}} n$, it follows that $u = v + 1$ and hence $\frac{n}{m} = p_i^t$. Since $0- < t$ in $(N, \leqslant_{\mathcal{C}}^{p_i})$, we get that

$$1- <_{\mathcal{C}} p^t = \frac{n}{m} \leqslant_{\mathcal{C}} n.$$

This proves (5).

Conversely suppose that $\mathcal{C}$ satisfies properties (1)–(5), since $[p^a)$ is prime filter of $(Z^+, \leqslant_{\mathcal{C}})$ for all $p \in P$ and $a \in Z^+$, by Theorem 6, every proper filter of $(N, \leqslant_{\mathcal{C}}^p)$ is prime, for any $p \in P$. Since any two incomparable prime filters of $(Z^+, \leqslant_{\mathcal{C}})$ are co-maximal, by Theorem 8 and Theorem 5, we get that $(Z^+, \leqslant_{\mathcal{C}}^p)$ is a disjoint union of maximal chains.

Fix $p \in P$. Then

$$Z^+ = \coprod_{i \in I} Y_i$$

where each Y_i is a maximal chain in $(Z^+, \leqslant_{\mathcal{C}}^p)$ such that, for any $i \neq j \in I$, $Y_i \cap Y_j = \phi$ and each element of Y_i is incomparable with each element of Y_j. Now, we shall prove that each Y_i is an arithmetical progression (finite or infinite).

Let $i \in I$. Since N is countable, Y_i is at most countable. Also, since $(N, \leqslant_{\mathcal{C}}^p)$ satisfies the descending chain condition, we can express

$$Y_i = \{a_1- <_{\mathcal{C}} a_2- <_{\mathcal{C}} a_3- < \dots\}$$

By using induction on r, we shall prove that $a_r = ra_1$ for all r.

Clearly, this is true for $r = 1$. Assume that $r > 1$ and $a_s = sa_1$ for all $1 \leqslant s < r$. Since $(r-1)a_1 = a_{r-1}- < a_r$ in $(N, \leqslant_{\mathcal{C}}^p)$, we have

$$p^{a_{r-1}}- <_{\mathcal{C}} p^{a_r} \text{ in } (Z^+, \leqslant_{\mathcal{C}})$$

and hence, by condition (5),

$$1- <_{\mathcal{C}} p^{a_r - a_{r-1}} \leqslant_{\mathcal{C}} p^{a_r}.$$

Therefore, $0 \neq a_r - a_{r-1} \leqslant_{\mathcal{C}}^p a_r$ and hence $a_r - a_{r-1} \in Y_i$ (since $a_r \in Y_i$).

Also, since $0- < a_r - a_{r-1}$ in $(N, \leqslant_{\mathcal{C}}^p)$, we have $a_r - a_{r-1} = a_1$ and therefore $a_r = a_{r-1} + a_1 = (r-1)a_1 + a_1 = ra_1$. Hence, for any prime p

and $a \in Z^+$,

$$\mathcal{C}(p^a) = \{1, p^t, p^{2t}, \cdots, p^{st}\} \quad \text{and} \quad st = a$$

for some positive integers t and s and

$$p^t \in \mathcal{C}(p^{2t}), p^{2t} \in \mathcal{C}(p^{3t}), \cdots, p^{(s-1)t} \in \mathcal{C}(p^a).$$

The other conditions given in Theorem 10 are clearly satisfied. Thus, by Theorem 10, $\mathcal{C}$ is a regular convolution. □

Theorem 13. *Let $\mathcal{C}$ be a convolution, then the following conditions are equivalent to each other.*

(1) $(Z^+, \leqslant_{\mathcal{C}})$ *is a lattice.*
(2) $(N, \leqslant_{\mathcal{C}}^p)$ *is a lattice for each* $p \in P$.
(3) $(N, \leqslant_{\mathcal{C}}^p)$ *is a totally ordered set for each* $p \in P$.
(4) *For any* $p \in P$ *and* a *and* $b \in N$, $a \leqslant_{\mathcal{C}}^p b \Longleftrightarrow a \leqslant b$.
(5) *For any* n *and* $m \in Z^+$, $n \leqslant_{\mathcal{C}} m \Longleftrightarrow n$ *divides* m.
(6) $\mathcal{C}(n) =$ *The set of positive divisors of* n.

Proof. Since $\mathcal{C}$ is regular, $\mathcal{C} \sim \{\pi_p\}_{p \in P}$.

(1) $\Longrightarrow$ (2) follows from Theorem 3.

(2) $\Longrightarrow$ (3) Let $p \in P$. Suppose that $(N, \leqslant_{\mathcal{C}}^p)$ is a lattice. If π_p contains two progressions, then choose an element a in one progression S and b in another progression T in π_p. Since $a \leqslant_{\mathcal{C}}^p a \vee b$ and $b \leqslant_{\mathcal{C}}^p a \vee b$, $a \vee b \in S \cap T$. A contradiction.

Therefore π_p contains only one progression, which must be

$$N = \{0 <_{\mathcal{C}}^p 1 <_{\mathcal{C}}^p 2 <_{\mathcal{C}}^p 3 <_{\mathcal{C}}^p \dots\}.$$

Thus $(N, \leqslant_{\mathcal{C}}^p)$ is a totally ordered set.

(3) $\Longrightarrow$ (4) It is trivial.

(4) $\Longrightarrow$ (5) Let m and $n \in Z^+$ and we write $n = \prod_{i=1}^r P_i^{a_i}$ and $m = \prod_{i=1}^r P_i^{b_i}$, where $p_1, p_2, \cdots, p_r$ are distinct primes and $a_i, b_i \in N$. Now,

$$\begin{aligned} n \text{ divides } m &\Longleftrightarrow a_i \leqslant b_i \text{ for all } 1 \leqslant i \leqslant r \\ &\Longleftrightarrow a_i \leqslant_{\mathcal{C}}^p b_i \text{ for all } 1 \leqslant i \leqslant r \\ &\Longleftrightarrow n \leqslant_{\mathcal{C}} m. \end{aligned}$$

(5) $\Longrightarrow$ (6) For any $n \in Z^+$,

$$\mathcal{C}(n) = \{m \in Z^+ \mid m \leqslant_{\mathcal{C}} n\} = \{m \in Z^+ \mid m \text{ divides } n\} = \mathcal{D}(n).$$

(6) $\Longrightarrow$ (1) If $\mathcal{C} = \mathcal{D}$, then $\leqslant_{\mathcal{C}} = \leqslant_{\mathcal{D}}$ and, for any $n, m \in Z^+$,

$$n \wedge m = \gcd\{n, m\}$$

and $n \vee m = \operatorname{lcm}\{n, m\}$ in $(Z^+, \leqslant_{\mathcal{C}})$. $\square$

The above Theorem implies that the Dirichlet's convolution $\mathcal{D}$ is the only regular convolution for which $(Z^+, \leqslant_{\mathcal{C}})$ is a lattice.

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