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SURVEY ARTICLE

Groups with finiteness conditions on some subgroup systems: a contemporary stage

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Dedicated to Leonid Kurdachenko, with heartfelt thanks for his friendship, on the occasion of his 60th birthday

Abstract. This paper gives a brief historical survey of results in which certain systems of subgroups of a group satisfy various finiteness conditions.

1. Introduction

One of the main themes in group theory (finite or infinite) is the study of the influence of systems of subgroups on the structure of a group. The structure of a group depends, to a significant extent, on the presence of a system of subgroups with certain properties, the size of this system, and the interaction of this system with other subgroups. There is a wide variety of cases that can be studied. Sometimes the presence of a single subgroup with given properties can be very influential on the structure of a group whereas, in other cases, a group can have many subgroups with some given property, but the influence of this system of subgroups need not be significant. There are two typical subgroup properties that are often used–properties that are internal to the group such as the properties of being a normal or subnormal subgroup and properties that are external to the group such as the property of belonging to some class $\mathfrak{X}$ of groups. The choices for $\mathfrak{X}$ include the classes of abelian groups and the class of nilpotent groups, to name but two.

A natural approach to studying groups was established. In this approach a group theoretical property $\mathcal{P}$ was chosen and groups in which the system of $\mathcal{P}$-subgroups, $\mathbf{L}_{\mathcal{P}}(G)$, was quite large were considered. For example, $\mathbf{L}_{\mathcal{P}}(G)$ can be taken to coincide with the family of all (proper) subgroups of G. As techniques became established the size of the system $\mathbf{L}_{\mathcal{P}}(G)$ could be gradually reduced so that the system $\mathbf{L}_{non-\mathcal{P}}(G)$ of all non–$\mathcal{P}$–subgroups grew larger.

The first step in this program was taken by R. Dedekind in his classical paper [16] where he described the finite groups, all of whose subgroups are normal. This is precisely the class of finite groups G in which the system $\mathbf{L}_{norm}(G)$, consisting of all normal subgroups, coincides with the system of all subgroups (or in which the system $\mathbf{L}_{non-norm}(G)$ is empty). In their famous paper [63], G. Miller and H. Moreno described the finite groups all of whose proper subgroups are abelian; this class coincides with the class of finite groups G where $\mathbf{L}_{ab}(G)$, the system of all abelian subgroups, coincides with the family of all proper subgroups. Alternatively, this class is the class of finite groups in which the family $\mathbf{L}_{non-ab}(G)$ of all non–abelian subgroups consists of at most one element, the group G. In this setting, we need to mention the remarkable article [73] due to O. Yu. Schmidt, which completely describes the finite groups all of whose proper subgroups are nilpotent. O. Yu. Schmidt continued Dedekind's research in the paper [74] where he described the finite groups G in which the subgroups in the family $\mathbf{L}_{non-norm}(G)$ are conjugate. In [75] he also completely described the finite groups in which $\mathbf{L}_{non-norm}(G)$ is the union of two conjugacy classes.

Many authors have continued these investigations in both finite and infinite groups. The situation is especially promising, but challenging, in infinite group theory. There is a large variety of situations where the concepts "to be quite small" and "to be very large" can be studied with fruitful results. S. N. Chernikov introduced one such effective approach. He began the investigation of groups G where $\mathbf{L}_{non-\mathcal{P}}(G)$ satisfies some natural finiteness condition. Such finiteness conditions include, in particular, such classical finiteness conditions as the minimal and the maximal conditions. In his work [7], he studied groups in which the family $\mathbf{L}_{non-ab}(G)$ satisfies the minimal condition and, in the paper [8], he considered groups G in which the set $\mathbf{L}_{non-norm}(G)$ consists of finite subgroups only.

The main goal of this article is to survey some important developments in the above-mentioned area that have been achieved over the last few decades. This is a huge area of research. Our choice of what to include and that to omit has been guided by our own interests and undoubtedly there are many interesting results that will not be mentioned here. For this reason we have decided to frame our discussion in the context of the most important properties such as normality, subnormality, almost normality, commutativity and their generalizations.

2. Groups with "small" systems of non-normal subgroups

As we mentioned above, the finite groups with all subgroups normal were described by R. Dedekind [16]. Later, R. Baer [2] extended Dedekind's results to infinite groups and the groups obtained have subsequently been termed Dedekind groups. These groups have a very simple structure, namely, a Dedekind group either is abelian or is a group of the type $A \times B \times Q$ where A is an abelian periodic $2'$-group, B is an elementary abelian 2-group, and Q is a quaternion group. In extending these results, S. N. Chernikov [8] considered groups all of whose infinite subgroups are normal, those groups G in which the system $\mathbf{L}_{non-norm}(G)$ of all non-normal subgroups consists of only finite subgroups. Naturally in infinite groups one would expect the structure of the infinite subgroups to play a dominant role. This class of groups is wider then the class of Dedekind groups and S. N. Chernikov obtained the following description of such groups [8].

Theorem 2.1. Let G be an infinite group all of whose infinite subgroups are normal.

(i) If G is non-abelian, then G is periodic;

(ii) If G is locally finite then G is either Dedekind, or G contains a normal Prüfer subgroup K such that G/K is a finite Dedekind group.

The latter result here was obtained by S. N. Chernikov, not just for locally finite groups, but for periodic groups with the additional assumption that the group G has an infinite abelian subgroup. It is a well-known theorem of P. Hall and C. Kulatilaka [28] that every infinite locally finite group contains an infinite abelian subgroup. Sophisticated examples of infinite groups all of whose subgroups are finite have been constructed by A. Yu. Olshanskii [68, § 28], so that some condition such as G being locally finite in Theorem 2.1 is required.

Groups in which $\mathbf{L}_{non-norm}(G)$ satisfies the minimal condition (groups with the condition $\mathbf{Min}-(non-norm)$) have also been studied by S. N. Chernikov [12] where the following results are obtained.

Theorem 2.2. Let G be an infinite group satisfying the condition $\mathbf{Min}-(non-norm)$.

(i) If G is not periodic, then G is abelian.

(ii) If G is locally finite, then G is either Dedekind or Chernikov.

We note that some generalizations of these results have been obtained in the articles [8, 12], but we refrain from discussing these generalizations since they have been described in other surveys (see, [9, 10, 11, 90, 6]). We should also point out the paper [70] where the hypotheses are weakened even further.

The maximal condition is dual to the minimal condition. The groups G in which $\mathbf{L}_{non-norm}(G)$ satisfies the maximal condition (the $\mathbf{Max}-(non-norm)$ condition) were studied in the articles of L. A. Kurdachenko, N. F. Kuzennyi and N. N. Semko [37] and G. Cutolo [13]. We note, that although the class of locally graded groups with the condition $\mathbf{Min}-(non-norm)$ is the union of the classes of Dedekind groups and Chernikov groups, the situation is very different for the class of groups with the condition $\mathbf{Max}-(non-norm)$; here as usual a group is locally graded if every nontrivial finitely generated subgroup has a nontrivial finite image. The main results of the above mentioned articles are captured by the following result.

Theorem 2.3. Let G be a locally graded group with the condition $\mathbf{Max}-(non-norm)$. Then G is a group of one of the following types.

(i) G is an almost polycyclic group;

(ii) G is a Dedekind group;

(iii) $\zeta(G)$ contains a Prüfer p-subgroup P such that G/P is a finitely generated Dedekind group;

(iv) $G = H \times L$ where $H \cong \mathbf{Q}_2$ and L is a finite non-abelian Dedekind group.

Here, $\mathbf{Q}_p$ denotes the additive group of rationals of the form ap^b where p is a prime and a, b are integers. Next we note that if all finitely generated subgroups of a group G are normal, then all subgroups of G are normal in G. By contrast, groups all of whose infinitely generated subgroups are normal are more complicated, where here, by *infinitely generated* we mean not finitely generated. In this case the system $\mathbf{L}_{non-norm}(G)$ consists of only finitely generated subgroups. Such groups have been studied in the articles of L. A. Kurdachenko and V. V. Pylaev [41], G. Cutolo [13], and G. Cutolo and L.A. Kurdachenko [14]. The main results of these papers can be formulated in the following way.

Theorem 2.4. Let G be a group that has an ascending series of subgroups whose factors are locally (soluble-by-finite). Every infinitely generated subgroup of G is normal if and only if G is a group of one of the following types:

(1) G is a Dedekind group;

(2) G contains a normal Prüfer subgroup K such that G/K is a finitely generated Dedekind group;

(3) G satisfies the following conditions:

(3a) the center $\zeta(G)$ contains a Prüfer p-subgroup K such that G/K is a minimax abelian group with finite periodic part;

(3b) $S_p(G/K) = \{p\}$;

(3c) $G/FC(G)$ is torsion-free;

(3d) if A is an abelian subgroup in G, then $A/(A \cap K)$ is finitely generated;

(4) $G = T \times A$, where $A \cong \mathbf{Q}_2$ and T is a finite Dedekind group;

(5) G satisfies the following conditions:

(5a) $G = (A \times T) \rtimes \langle g \rangle$ where $A \cong \mathbf{Q}_p$ for some prime p and T is a finite Dedekind group;

(5b) if T is non-abelian, then $p = 2$;

(5c) the element g induces a power automorphism on the Sylow p-subgroup T_p of the group T;

(5d) there exists $r \in \mathbb{N}$ such that $a^g = a^c$ where $c = p^r$ or $c = -p^r$ for each $a \in AT_{p'}$ (where $T_{p'}$ is a Sylow p'-subgroup of T).

In the articles [3] and [88], new interesting finiteness conditions were introduced, namely the weak minimal and weak maximal conditions for different types of subgroups. These conditions have proved to be very successful, stimulating a lot of research in this area. The results of the published research have been reflected in the survey [34] and so we here consider only the results of that research which are relevant to us here.

Let $\mathfrak{M}$ be some system of subgroups of a group G. We will say that $\mathfrak{M}$ satisfies the weak minimal condition (respectively the weak maximal condition) or that G satisfies the weak minimal condition (respectively the weak maximal condition) on $\mathfrak{M}$-subgroups (which we write as **Min**-∞-$\mathfrak{M}$ or **Max**-∞-$\mathfrak{M}$ respectively) if G has no infinite descending (respectively ascending) chain $\{H_n \mid n \in \mathbb{N}\}$ of $\mathfrak{M}-$subgroups such that the indices $|H_n : H_{n+1}|$ (respectively $|H_{n+1} : H_n|$) are infinite for every $n \in \mathbb{N}$.

If $\mathfrak{M} = \mathbf{L}_{non-norm}(G)$, then we obtain groups with the weak minimal condition (respectively the weak maximal condition) on non-normal subgroups which we denote by **Min**-∞-(non-norm) (respectively **Max**-∞-(non-norm)). The structure of such groups was described by L. A.

Kurdachenko and V. E. Goretskii in [36], where the following result was obtained.

Theorem 2.5. A locally (soluble-by-finite) group G satisfies the condition **Min**-∞-(non-norm) (respectively **Max**-∞-(non-norm)) if and only if G either is a Dedekind group or a minimax group.

A large number of papers have been devoted to the study of groups with restriction on the system $\mathbf{L}_{non-norm}(G)$. Certainly, if all cyclic subgroup of a group are normal, then all subgroups of this group are normal which makes it natural to consider those groups in which $\mathbf{L}_{non-norm}(G)$ consists of only cyclic subgroups. This leads to the study of groups in which all non-cyclic subgroups are normal, a class of groups first studied by S. N. Chernikov [9]. The papers [55]–[59] due to F. N. Liman have also been dedicated to this question. More generally, the works [65, 61], [49]–[54], [71, 72, 76, 78] have been dedicated to the so called metahamiltonian groups–those groups in which $\mathbf{L}_{non-norm}(G)$ consists of only abelian subgroups. Finite metahamiltonian groups were described in [65, 61] and a complete description of metahamiltonian groups was obtained in the work due to N. F Kuzennyi and N. N. Semko [49]–[54],[76].

A natural continuation of these investigations is the exploration of the case when all subgroups of $\mathbf{L}_{non-norm}(G)$ belong to some class which is a natural extension of the class of abelian groups. Thus, in [38, 39], groups have been studied in which the subgroups in the system $\mathbf{L}_{non-norm}(G)$ all have finite derived subgroup (respectively, are FC-groups). The main results of these papers can be summarized as follows.

Theorem 2.6. (i) Let G be a non–nilpotent group with Chernikov derived subgroup K and let D be the divisible part of K. If every non–FC–subgroup of G is normal in G and $C_G(D) \neq G$, then $G = DL$ where L is a subgroup having finite derived subgroup and $D \cap L$ is a finite G–invariant subgroup. Moreover, every non–normal subgroup of G has finite derived subgroup.

(ii) Let G be an almost $FC-$group. If every non–FC–subgroup of G is normal, then either G is an $FC-$subgroup or $[G, G]$ is a Chernikov group.

(iii) Let G be a metabelian group whose non–normal subgroups are FC–groups. If G is not locally nilpotent, then either G is an FC–group or $[G, G]$ is a Chernikov group.

(iv) Let G be a metabelian locally nilpotent group whose non–normal subgroups are FC–groups. Then either G is a Fitting group or $[G, G]$ is a Chernikov group.

(v) Let G be a group whose non–normal subgroups are FC–groups and let F be the FC–center of G. Then either G is an almost FC–group or F is nilpotent with nilpotency class at most 3 and has a normal subgroup H such that $\zeta(F) \leq H, H/\zeta(F)$ is abelian and F/H is a quaternion group.

(vi) Let G be a soluble–by–finite group whose non–normal subgroups are FC–groups. If G is not an FC–group and $G/[G, G]$ is finitely generated, then $[G, G]$ is a Chernikov group.

(vii) Let G be a soluble–by–finite group whose non–normal subgroups are FC–groups. If $G/[G, G]$ is periodic divisible then G is an almost FC–group. In particular, either G is an FC–group or $[G, G]$ is a Chernikov group.

3. Groups with “small” systems of non-(almost normal) subgroups

A subgroup H of a group G is called *almost normal* in G if the set of all conjugates of H, $cl_G(H) = \{H^g \mid g \in G\}$, is finite. If a subgroup H is normal in G, then $cl_G(H) = \{H\}$; thus almost normality is a natural generalization of normality. The subgroup H is almost normal in G if and only if its normalizer $N_G(H)$ has finite index in G, which gives a good justification for the name of these subgroups. It is clear that the intersection of two almost normal subgroups is almost normal and that a subgroup generated by two almost normal subgroups is almost normal. Thus the set $\mathbf{L}_{an}(G)$ of all almost normal subgroups is a lattice but, in contrast to the lattice of normal subgroups, the lattice of almost normal subgroups is not complete. The groups G for which $\mathbf{L}_{an}(G)$ is complete have been considered by L. A. Kurdachenko and S. Rinauro in [42]. There they proved that it is often the case that such groups are central-by-finite, which means that they have a central subgroup of finite index.

The central-by-finite groups here play a role similar to the role that Dedekind groups play in the study of groups G in which $\mathbf{L}_{non-norm}(G)$ is “small”. Two classical results illustrate this very well. If G is a group all of whose subgroups are almost normal then a well-known theorem of B. H. Neumann [66] asserts that G is central-by-finite. Clearly the hypothesis on G here is equivalent to $\mathbf{L}_{non-an}(G) = \emptyset$. Also, if G is a group all of whose abelian subgroups are almost normal then a theorem of I. I. Eremin [23] implies that G is central-by-finite and clearly in this case the system $\mathbf{L}_{non-an}(G)$ consists of non-abelian subgroups.

I. I. Eremin began the study of groups in which the set $\mathbf{L}_{non-an}(G)$ is small by considering the case when it consists of finite subgroups [24] and he obtained some conditions under which these groups are central-by-finite. Later L. A. Kurdachenko, S. S. Levishenko and N. N. Semko [77] described the locally (soluble-by-finite) such groups and we give this description next.

Theorem 3.1. Let G be an infinite locally (soluble-by-finite) group.

(I) If G is non-periodic, then each infinite subgroup of G is almost normal if and only if G is a group of one of the following types:

(Ia) G is central-by-finite;

(Ib) $G = A \rtimes \langle b \rangle$, where $|b| = p$, for some prime p, $A = C_G(A)$ is free abelian of 0-rank $p - 1$ and b induces a rationally irreducible automorphism on A (so every non-trivial $\langle b \rangle$-invariant subgroup of A has finite index in A);

(Ic) G contains a finite normal subgroup F such that G/F is a group of type (Ib).

(II) If G is periodic, then every infinite subgroup of G is almost normal if and only if G is a group of one of the following types:

(IIa) G is central-by-finite;

(IIb) $G = D \rtimes \langle g \rangle$, where $D = C_G(D)$ is a divisible abelian subgroup of special rank $p - 1$, p is a prime, $g^p \in D$ and every proper $\langle g \rangle$-invariant subgroup of D is finite;

(IIc) $G = D \rtimes \langle g \rangle$, where $D = C_G(D)$ is a divisible abelian subgroup of special rank at most $q - 1$, q is the smallest prime in the set $\Pi(\langle g \rangle) = \{|g| : g \text{ is a } p'\text{-element}\}$ and for every element $1 \neq y \in \langle g \rangle$ every proper $\langle y \rangle$-invariant subgroup of D is finite;

(IId) G contains a finite normal subgroup F that G/F is a group of the types (IIb) or (IIc).

These results were generalized by S. Franciosi, F. de Giovanni and L. A. Kurdachenko in [26] to the case when the set $\mathbf{L}_{non-an}(G)$ consists of finitely generated subgroups. For example they showed that if G is a group with an ascending series of subgroups, every factor of which is either locally nilpotent or finite, then if every infinitely generated subgroup of G is almost normal either $G/\zeta(G)$ is finite, or G is a soluble $\mathcal{A}_3$-group.

Almost soluble $\mathcal{A}_3$ -groups in which the set $\mathbf{L}_{non-an}(G)$ consists of finitely generated subgroups occur in several different families that have been described in detail in [26]. The question of S. N. Chernikov concerning the structure of a group G in which the set $\mathbf{L}_{non-an}(G)$ consists of non-cyclic subgroups was also considered in [26].

The groups G in which the set $\mathbf{L}_{non-an}(G)$ satisfies the minimal condition (namely the groups with the condition **Min**-(non-an)) were studied by L. A. Kurdachenko and V. V. Pylaev [40] and the following result was obtained.

Theorem 3.2. (I) A non-periodic group G satisfies the condition **Min**-(non-an) if and only if it is either cental-by-finite or has a finite normal subgroup F such that $H = G/F$ is a group of one of the following types:

(1) $H = A \rtimes \langle b \rangle$, where $|b| = p$, for the prime p, $A = C_H(A)$ is a divisible abelian subgroup of 0-rank $p-1$, and b induces a rationally irreducible automorphism on A;

(2) $H = K \times L$, where K is a divisible Chernikov group and L is a group of type (1).

(II) A locally finite group G satisfies the condition **Min**-(non-an) if and only if it is either a Chernikov group or it is central-by-finite.

Groups in which the set $\mathbf{L}_{non-an}(G)$ satisfies the maximal condition (namely the condition **Max**-(non-an)) were studied by L. A. Kurdachenko, N. F. Kuzennyi and N. N. Semko in the article [37]. These groups are easier to describe. For example, a locally soluble group G satisfies the condition **Max**-(non-an) if and only if it is either polycyclic or central-by-finite.

Groups in which the set $\mathbf{L}_{non-an}(G)$ satisfies the weak minimal condition (respectively the weak maximal condition), namely the groups with the **Min**-∞-(non-an) condition, (respectively **Max**-∞-(non-an)) were studied by G. Cutolo and L. A. Kurdachenko [15]. One of the main results of this paper is the following.

Theorem 3.3. Let the group G have an ascending series of subgroups every factor of which is a locally (soluble-by-finite) group. If G satisfies **Min**-∞-(non-an) (respectively **Max**-∞-(non-an)) then either $G/\zeta(G)$ is finite or G is an almost soluble $\mathcal{A}_3$-group.

To end this section, we note that S. Fransiosi, F. de Giovanni and L. A. Kurdachenko [27] studied groups G in which the set $\mathbf{L}_{non-an}(G)$ consists of subnormal subgroups.

4. Groups with "small" systems of non-subnormal subgroups

It is well known that a finite group G with all subgroups subnormal (so $\mathbf{L}_{non-sn}(G)$ is empty) is nilpotent. In infinite groups the situation is different. There are soluble locally nilpotent groups with trivial center all of whose subgroups are subnormal. Examples of such groups have been constructed by H. Heineken and I. Mohamed [31, 33], B. Hartley [29] and F. Menegazzo [62], among others. Groups with all subgroups subnormal have been studied in many papers and books; the book [60] by J. C. Lennox and S. E. Stonehewer is one excellent reference source. However since that book appeared there has been much activity in this area which we now describe.

First, we must mention the following remarkable result due to W. Möhres [64].

Theorem 4.1. Let G be a group all of whose subgroups are subnormal. Then G is soluble.

As we remarked above, an infinite group with all subgroups subnormal need not be nilpotent. However there are a number of cases in which nilpotence is assured in a group G when all subgroups are subnormal. We summarize some of what is known next.

Theorem 4.2. Let G be a group with all subgroups subnormal. Then G is nilpotent if any one of the following hypotheses also hold.

(1) G is periodic and hypercentral (W. Möhres [64]);

(2) G is periodic and residually finite (H. Smith [82]);

(3) G has a normal nilpotent subgroup A such that G/A is bounded (H. Smith [83]);

(4) G is periodic and residually nilpotent (H. Smith [84], C. Casolo [5]);

(5) G is torsion free (H. Smith [85], C. Casolo [4]).

Some conditions for the nilpotency of a group with all subgroups subnormal are connected to properties of normal closures of elements and have been considered by L. A. Kurdachenko and H. Smith in [48].

Groups in which the set $\mathbf{L}_{non-sn}(G)$ satisfies the minimal condition (the groups with the condition **Min**-(non-sn)) were considered by S. Fransiosi and F. De Giovanni [25]. Under certain other hypotheses, these groups are either Chernikov or are groups with all subgroups subnormal. By contrast, the study of groups in which the set $\mathbf{L}_{non-an}(G)$ satisfies the maximal condition (the groups with the condition **Max**-(non-sn)) turns out to be more interesting and such groups were considered by L. A. Kurdachenko and H. Smith in [45]. The main results of this paper can be summarized in the following theorem. Here we let $B(G)$ denote the Baer radical of the group G, the subgroup of G generated by all cyclic subnormal subgroups of G.

Theorem 4.3. (i) A locally nilpotent group satisfies **Max**-(non-sn) if and only if all of its subgroups are subnormal;

(ii) A locally soluble group G satisfies **Max**-(non-sn) if and only if it has one of the following types:

(1) G is almost polycyclic;

(2) each subgroup of G is subnormal;

(3) $G \neq B(G)$ and $G/B(G)$ is finitely generated, almost abelian and torsion free. In this case $B(G)$ is nilpotent and for every element $g \notin B(G)$ every G-invariant abelian factor of $B(G)$ is finitely generated as a $\mathbb{Z}\langle g\rangle$-module.

A class of groups that is closely related to the class **Max**-(non-sn) is the class of groups G in which the set $\mathbf{L}_{non-sn}(G)$ consists of finitely generated subgroups. These groups have been considered by H. Heineken and L. A. Kurdachenko in [30].

The groups G for which $\mathbf{L}_{non-sn}(G)$ satisfies the weak minimal condition (the groups with **Min**-∞-(non-sn)) were studied by L. A. Kurdachenko and H. Smith in [46]. The main result of this work shows that the situation here is close to that arising for the condition **Min**-(non-sn). For example, it is shown in [46] that if G is a generalized radical group, that is, has an ascending series of subgroups every factor of which is either locally nilpotent or locally finite, then if G satisfies **Min**-∞-(non-sn), either all subgroups of G are subnormal or G is almost soluble and minimax.

L. A. Kurdachenko and H. Smith [47] also discussed those groups G where the set $\mathbf{L}_{non-sn}(G)$ satisfies the weak maximal condition (the groups with **Max**-∞-(non-sn)). In this case the situation is much more complicated. We finish this section by exhibiting a couple of results that illustrate this investigation.

Theorem 4.4. (i) Let G be a locally finite group with **Max**-∞-(non-sn). Then either all subgroups of G are subnormal or G is a Chernikov group;

(ii) Let G be a Baer group with **Max**-∞-(non-sn). Then all subgroups of G are subnormal.

5. Groups with "small" systems of non-abelian subgroups and other restrictions on non-abelian subgroups

As we have already mentioned the description of the finite non-abelian groups all of whose proper subgroups are abelian (the groups G such that

$\mathbf{L}_{non-ab}(G) = \{G\}$), due to G. Miller and H. Moreno [63], is one of the first important results of abstract group theory. Sophisticated examples of such infinite groups have been developed quite recently by A. Yu. Olshanskii (see the book [68, § 28]). These example suggest that it is impossible to expect a complete description of these groups.

Groups in which the set $\mathbf{L}_{non-ab}(G)$ satisfies the minimal condition (the groups with **Min**-(non-ab)) were first discussed by S. N. Chernikov [7]. His results imply that a non-abelian locally soluble group satisfying **Min**-(non-ab) is a Chernikov group, a result that has been extended to locally finite groups by V. P. Shunkov [80].

By contrast, the class of groups with **Max**-(non-ab) (which consists of those groups G in which $\mathbf{L}_{non-ab}(G)$ satisfies the maximal condition) does not coincide with the class of groups satisfying **Max**, even when further stringent hypotheses are added. A simple example here is a group that is a wreath product of a group of prime order with an infinite cyclic group. Groups with **Max**-(non-ab) were considered somewhat later than groups with **Min**-(non-ab) by D. I. Zaitsev and L. A. Kurdachenko [91] and their main result is the following.

Theorem 5.1. *Let G be a locally (soluble-by-finite) non-polycyclic group. Then G satisfies the condition* **Max**-*(non-ab) if and only if it contains a normal abelian subgroup A with the following properties:*

(a) $A = C_G(A)$;

(b) G/A *is an infinitely generated, almost abelian, torsion-free group;*

(c) A *is finitely generated $\mathbb{Z}\langle g\rangle$-module for all $g \in G$.*

The next step in the natural classification process here is the consideration of groups G in which the set $\mathbf{L}_{non-ab}(G)$ satisfies the weak minimal condition (a class denoted by **Min**-∞-(non-ab)) and such groups were studied by D. I. Zaitsev in [89]. The main result of [89] shows that the situation here is similar to the case of the condition **Min**, since it is shown that a non-abelian almost soluble group G satisfies **Min**-∞-(non-ab) if and only if it is almost soluble minimax.

Groups G in which the set $\mathbf{L}_{non-ab}(G)$ satisfies the weak maximal condition (a class denoted by **Max**-∞-(non-ab)) were studied by L. S. Kazarin, L. A. Kurdachenko and I. Ya. Subbotin in [35]. The situation here is more complicated than in the cases of **Max**-(non-ab) and **Min**-∞-(non-ab). However the non-abelian locally finite groups with **Max**-∞-(non-ab) are minimax (and hence Chernikov groups). The description of other classes of groups with $\mathbf{Max} - \infty - (non - ab)$ requires special definitions and terminology and we now discuss these here.

Let G be a group and let A be a normal abelian subgroup of G. Let $a_G(A)$ denote the G–invariant subgroup of A satisfying the conditions:

(i) $a_G(A)$ has an ascending series of G–invariant subgroups whose factors are G–chief;

(ii) $A/a_G(A)$ has no nontrivial minimal G–invariant subgroups.

Let R be a ring and let A be an R–module. We say that A is R–minimax if A has a finite series of submodules, every factor of which is either Artinian or Noetherian. Generalized radical groups were studied in [35] and the following theorems represent the main results of that work.

Theorem 5.2. Let G be a non–abelian generalized radical group, let A be a maximal normal abelian subgroup of G and let T be the periodic part of A. Suppose that either $a_G(A)$ does not contain T or that $r_0(A)$ is infinite. Then G satisfies **Max-**∞-(non-ab) if and only if the following conditions hold:

(i) G/A is torsion–free, finitely generated and abelian–by–finite;

(ii) For each element $g \in G \backslash A$ the $\mathbb{Z}\langle g\rangle$–module A is minimax.

Theorem 5.3. Let G be a non–abelian generalized radical group, let A be a maximal normal abelian subgroup of G and let T be the periodic part of A. Suppose that A is non– minimax, $T \leq a_G(A)$ and $r_0(A)$ is finite. Then G satisfies **Max-**∞-(non-ab) if and only if the following conditions hold:

(i) A/T is minimax;

(ii) G/A is torsion–free;

(iii) $G/A = L$ contains a normal subgroup $K = H/A$ of finite index such that either K is abelian and minimax or $K = C \leftthreetimes D$ where $C = C_K(C)$ is abelian and minimax, $D = C_K(D)$ is abelian and finitely generated;

(iv) For each element $g \in G \backslash A$ the $\mathbb{Z}\langle g\rangle$–module T is Artinian.

When the maximal normal abelian subgroup A is minimax these results take a particularly pleasing form, as follows.

Theorem 5.4. Let G be a non–abelian generalized radical group and let A be a maximal normal abelian subgroup of G. Suppose that A is minimax. Then G satisfies **Max-**∞-(non-ab) if and only if G is soluble–by–finite and minimax.

Those groups G having finite derived subgroup (the so-called BFC–groups) are a natural generalization of abelian groups. We let $\mathbf{L}_{non-BFC}(G)$ denote the family of groups all of whose subgroups have infinite derived subgroups. In the papers [19, 20] of M. R. Dixon and L. A. Kurdachenko, the groups in which the family $\mathbf{L}_{non-BFC}(G)$ satisfies the maximal condition (the groups satisfying **Max**-(non-BFC)) were studied. The following theorems describe the main results of [19]. If G is a group and $\mathfrak{F}$ is the class of finite groups then we let $G^{\mathfrak{F}}$ denote the finite residual of G, the intersection of all subgroups of G of finite index in G.

Theorem 5.5. (1) Let G be a locally finite group satisfying **Max**-(non-BFC). If $G/G^{\mathfrak{F}}$ is finite and $[G,G]$ is infinite, then G is a Chernikov group.

(2) Let G be a locally FC–group satisfying **Max**-(non-BFC). If $G/G^{\mathfrak{F}}$ is not finitely generated and $[G,G]$ is infinite, then G has a series of normal subgroups $F \leq T \leq L \leq G$ such that

(i) F is finite;

(ii) $T = FD$ where $D = G^{\mathfrak{F}} \leq \zeta(G)$ is a divisible Chernikov p–subgroup for some prime p;

(iii) G/T is torsion–free abelian and L/T is finitely generated;

(iv) L/F is abelian, G/L is a Prüfer p–group and $T/F = [G/F, G/F]$;

(v) The p–rank of D is at most the 0–rank of L/T;

(vi) If H is a subgroup of G having infinite derived subgroup, then $D \leq H$.

Theorem 5.6. Let G be a non–periodic nilpotent locally FC–group satisfying **Max**-(non-BFC). If $G/G^{\mathfrak{F}}$ is finitely generated and $[G,G]$ is infinite, then

(i) $G/G^{\mathfrak{F}}$ is a BFC–group;

(ii) $G^{\mathfrak{F}} = P \times Q$ where P, Q are Prüfer p–groups for some prime p;

(iii) $G/C_G(G^{\mathfrak{F}})$ is a torsion–free abelian group and $[G^{\mathfrak{F}}, g] = P \leq \zeta(G)$ for each $g \in C \backslash C_G(G^{\mathfrak{F}})$.

Theorem 5.7. Let G be a non–nilpotent locally FC–group satisfying **Max**-(non-BFC). If $G/G^{\mathfrak{F}}$ is finitely generated and $[G,G]$ is infinite then either G contains a finite normal subgroup F such that G/F is nilpotent, or G has a series of normal subgroups $D \leq C \leq G$ such that

(i) $D = G^{\mathfrak{F}}$ is a divisible Chernikov p–subgroup for some prime p;

(ii) G/D is a $BFC-$group;

(iii) $C = C_G(D)$ and if $g \in G\backslash C$ then every proper $\langle g \rangle$–invariant subgroup of D is finite;

(iv) If P/C is the periodic part of G/C then either P/C has order p or P/C is a cyclic p'–group. In the former case, D has rank $p-1$, and in the latter case D has rank at most $q-1$ for all primes q dividing the order of P/C. Furthermore L/F is abelian, G/L is a Prüfer p–group and $T/F = [G/F, G/F]$.

(v) The p–rank of D is at most the 0–rank of L/T.

(vi) If H is a subgroup of G having infinite derived subgroup, then $D \le H$.

The following two theorems constitute the main results of [20].

Theorem 5.8. Let G be a finitely generated soluble–by–finite group satisfying **Max**-(non-BFC). Suppose that S is the soluble radical of G and that K is that term of the derived series of S such that S/K is finitely generated but $K/[K,K]$ is not finitely generated. Suppose also that K is a BFC–group. Then G has a series of normal subgroups $F \le A \le G$ such that

(i) F is finite;

(ii) A/F is abelian;

(iii) G/A is finitely generated, abelian–by–finite and torsion–free;

(iv) A/F is a finitely generated $\mathbb{Z}\langle g \rangle$–module for each element $g \in G\backslash A$.

Theorem 5.9. Let G be a finitely generated soluble–by–finite group satisfying **Max**-(non-BFC). Suppose that S is the soluble radical of G and that K is that term of the derived series of S such that S/K is finitely generated but $K/[K,K]$ is not finitely generated. Suppose also that $[K,K]$ is infinite. Then G has a series of normal subgroups $F \le T \le A \le G$ such that

(i) F is finite;

(ii) $T = FD$ where $D \le \zeta(A)$ is a divisible Chernikov p–subgroup for some prime p;

(iii) A/T is abelian and torsion–free;

(iv) G/A is finitely generated, abelian–by–finite and torsion–free;

(v) A/T is a finitely generated $\mathbb{Z}\langle g\rangle$–module for each element $g \in G\backslash A$;

(vi) if the subgroup H has infinite derived subgroup, then H contains D.

In the paper [21] a more general situation has been considered, that in which the family $\mathbf{L}_{non-FC}(G)$, of all non–FC–subgroups, satisfies the maximal condition. This is the class of groups with **Max**-(non-FC). If G is a locally FC–group then the set of elements of finite order forms a subgroup, the torsion subgroup of G, which we denote by $\mathbf{Tor}(G)$. The derived subgroup of an FC–group is well–known to be periodic, so if G is a locally FC–group then $G/\mathbf{Tor}(G)$ is torsion-free abelian. The following result holds.

Theorem 5.10. (i) Let G be a locally FC–group satisfying **Max**-(non-BFC). If $G/\mathbf{Tor}(G)$ is not finitely generated then either G is an FC–group or G satisfies **Max**-(non-BFC);

(ii) Let G be a locally FC–group satisfying **Max**-(non-BFC). If G is soluble, then either G is an FC–group or G satisfies **Max**-(non-BFC).

6. Groups with "small" systems of non-nilpotent subgroups and related topics

As we have already mentioned, the description of the finite non-nilpotent groups in which all proper subgroups are nilpotent (the groups with $\mathbf{L}_{non-nil}(G) = \{G\}$) was obtained by O.Yu. Schmidt [73]. We next consider some generalizations of this work concerned with infinite groups. First, we discuss the work of M. F. Newman and J. Wiegold [67]), who studied the class of groups G in which, for some fixed natural number k, $\mathbf{L}_{non-nil(k)}(G) = \{G\}$. They showed that then G can be generated by at most $k+1$ elements. Furthermore, they proved that if $\mathbf{L}_{non-nil(k)}(G) = \{G\}$ or $\mathbf{L}_{non-nil}G = \{G\}$ then $G/\mathbf{Fratt}(G)$ is a non-abelian simple group. Also, for a simple group G in which $\mathbf{L}_{non-nil(k)}(G) = \{G\}$ or $\mathbf{L}_{non-nil}(G) = \{G\}$, it is the case that

(a) every pair of maximal subgroups of G has trivial intersection;

(b) if $1 \neq x \in G$, then there is an element g, such $\langle g^{-1}xg, x\rangle = G$;

(c) G has no involutions.

Examples of such simple groups have been obtained by A.Yu. Olshanskii [68, Section 28].

Following D. I. Zaitcev [86], we call a nilpotent group G, of nilpotency class k, *stably nilpotent* if every infinite subgroup of G (in particular G itself) of nilpotency class k contains a proper subgroup of nilpotency class k. It turns out that every infinite nilpotent group of nilpotency class k contains a proper infinite subgroup of nilpotency class k and that if G is a nilpotent torsion–free group, then every nontrivial subgroup is stably nilpotent. This means that if G is a nilpotent group in which $\mathbf{L}_{nil(k)}(G) = \{G\}$ then G is finite. These results are due to D. I. Zaitsev [86].

D. I. Zaitsev continued this investigation in [87] where he showed that a locally nilpotent group G which has a nilpotent subgroup of nilpotency class k contains a stably nilpotent subgroup of class k if and only if G is not a Chernikov group (D.I. Zaitcev [87]). A. N. Ostylovskii took up a similar theme in [69] where locally finite groups (and more generally binary finite groups) were considered. He proved in this case that if every infinite subgroup of G that has infinite index either satisfies **Min** or has nilpotency class at most k then either G satisfies **Min** or is nilpotent of class at most k. Furthermore, if G is a binary finite group that is not of nilpotency class at most k and if the set $\mathbf{L}_{nil(k)}(G)$ satisfies the weak minimal condition, then G is a Chernikov group. The examples constructed by A. Yu. Olshanskii leave little hope of obtaining a complete description of finitely generated such groups. However, a recent amazing theorem of Asar [1] shows that every locally graded group, all of whose proper subgroups are nilpotent, is soluble.

The following result supplements this theorem of Asar very nicely and was obtained by H. Smith in [81].

Theorem 6.1. Let G be a soluble non-nilpotent group all of whose proper subgroups are nilpotent. If G has no maximal subgroups, then the following conditions hold:

(a) G is a countable p-group for some prime p;

(b) $G/[G,G]$ is a Prufer p-group;

(c) every subgroup of G is subnormal;

(d) $[G,G]^p \neq [G,G]$ and every hypercentral subgroup G is abelian; in particular, $[G,G] = \gamma_n(G)$ for all $n \geq 2$;

(e) $\zeta(G)$ contains all divisible subgroups of G;

(f) $C_G([G,G])$ is abelian and $[G,G]$ is a non-essential subgroup (thus if $H[G,G] = G$ then $H = G$ for every $H \leq G$); in particular G contains no proper subgroups of finite index;

(g) if H is finite subgroup of $[G,G]$, then $H^G \neq [G,G]$;

(h) the hypercenter of G coincides with its center.

H. Smith [81] also initiated the investigation of those groups G in which the set $\mathbf{L}_{non-nil}(G)$ satisfies the maximal, minimal, weak minimal and weak maximal conditions. For example a locally nilpotent torsion-free group satisfying one of these conditions is nilpotent.

The groups G for which $\mathbf{L}_{non-nil}(G)$ satisfies the maximal condition have also been studied by M. R. Dixon and L. A. Kurdachenko in [17, 18]. The first article deals with locally nilpotent such groups, while the second article is dedicated to soluble groups. Here are the main results of the first article.

Theorem 6.2. Let G be a locally nilpotent group satisfying the condition **Max**-(non-nil) and let T be its periodic part. If G is non-nilpotent and $G/G^{\mathfrak{F}}$ is infinitely generated, then the following conditions hold:

(a) $G^{\mathfrak{F}} \leq T$ and $T/G^{\mathfrak{F}}$ is finite;

(b) G/T is a nilpotent maximal subgroup and $S_p(G/T) = \{p\}$ for some prime p;

(c) $G^{\mathfrak{F}}$ is a p-subgroup;

(d) G contains a nilpotent normal subgroup U such that G/U is a Prüfer p-group;

(e) if S is a non-nilpotent subgroup of G, then $G = US$.

Theorem 6.3. Let G be a locally nilpotent group satisfying the condition **Max**-(non-nil). If G is non-nilpotent, $G/G^{\mathfrak{F}}$ is finitely generated and $G^{\mathfrak{F}}$ is nilpotent, then the following conditions hold:

(a) $G^{\mathfrak{F}}$ is a divisible Chernikov group;

(b) every proper G-invariant subgroup of G is finite;

(c) $[G, G^{\mathfrak{F}}] = G^{\mathfrak{F}}$.

Theorem 6.4. Let G be a locally nilpotent group satisfying the condition **Max**-(non-nil) and let T be its periodic part. If G is non-nilpotent and non-minimax and $G/G^{\mathfrak{F}}$ is infinitely generated, then the following conditions hold:

(a) $G^{\mathfrak{F}}$ is periodic;

(b) $G^{\mathfrak{F}}$ has no proper subgroup of finite index;

(c) $G^{\mathfrak{F}}$ is non-nilpotent while all its proper subgroups are nilpotent;

(d) G is soluble;

(e) $G^{\mathfrak{F}}$ is nilpotent-by-Chernikov.

In particular, $G^{\mathfrak{F}}$ is a p-subgroup for some prime p, and it has an ascending series of nilpotent G-invariant subgroups

$$\langle 1\rangle = A_0 \leq A_1 \leq \cdots \leq A_n \leq \cdots \leq \bigcup_{n\in\mathbb{N}} A_n = G^{\mathfrak{F}}.$$

The main results of the article [18] are as follows.

Theorem 6.5. Let G be a locally (soluble–by–finite) group satisfying **Max**-(non-nil) and let L be the locally nilpotent radical of G. If $L \neq G$, then L is nilpotent and G/L is a finitely generated abelian–by–finite group.

Theorem 6.6. Let G be a locally (soluble–by–finite) group satisfying **Max**-(non-nil) and let L be the locally nilpotent radical of G. Suppose that L is not finitely generated. Let g be an element of $G\backslash L$ such that gL has infinite order. Let

$$\langle 1\rangle = L_0 \leq L_1 \leq \cdots \leq L_n = L$$

be the upper central series of L and for $k > n$ write $\mathrm{L}_k = \mathrm{L}_n$. Then

(i) $\langle L, g\rangle$ is non–nilpotent and finitely generated;

(ii) there exists $m < n$ such that $\langle L_m, g\rangle$ is locally nilpotent but $\langle L_{m+1}, g\rangle$ is not locally nilpotent;

(iii) every factor L_{j+1}/L_j is finitely generated as a $\mathbb{Z}\langle g\rangle$–module for $j \geq m$;

(iv) either $\langle L_m, g\rangle$ is nilpotent or L_m contains a central divisible Chernikov subgroup D such that L_m/D is finitely generated and $D = [D, g]$.

The study of those groups G in which the set $\mathbf{L}_{non-nil}(G)$ satisfies the weak maximal condition was initiated by L. A. Kurdachenko, P. Shumyatskii and I. Ya. Subbotin in [44]. The main results of this paper are as follows.

Theorem 6.7. Let G be a locally finite group satisfying **Max**-∞-(non-nil). If G is not locally nilpotent, then G is Chernikov.

Theorem 6.8. Let G be a locally nilpotent group satisfying **Max**-∞-(non-nil). Suppose that G is non–nilpotent. Then G satisfies the following conditions:

(i) $G/G^{\mathfrak{F}}$ is nilpotent and minimax;

(ii) the set $\Pi(G)$ is finite;

(iii) the subgroup $G^{\mathfrak{F}}$ is periodic;

(iv) if the subgroup $M = [G^{\mathfrak{F}}, G^{\mathfrak{F}}]$ is non–nilpotent, then $G^{\mathfrak{F}}/M$ is a divisible Chernikov group and every proper G–invariant subgroup of M is nilpotent. In particular, M is a p–subgroup for some prime p, and M has an ascending series of G–invariant subgroups

$$\langle 1 \rangle = A_0 \leq A_1 \leq \cdots \leq A_n \leq \cdots \leq \bigcup_{n \in \mathbb{N}} A_n = M$$

such that every subgroup A_n is nilpotent.

Theorem 6.9. Let G be a generalized radical group satisfying **Max**-∞-(non-nil) and let L be the locally nilpotent radical of G. Then L is nontrivial. If L is nilpotent, then G/L is minimax, soluble–by–finite and almost torsion–free.

Theorem 6.10. Let G be a generalized radical group satisfying **Max**-∞-(non-nil) and let L be the locally nilpotent radical of G. If L is non–nilpotent, then G satisfies the following conditions:

(i) $G/G^{\mathfrak{F}}$ is minimax, soluble–by–finite and almost torsion–free;

(ii) the set $\Pi(G)$ is finite;

(iii) $G^{\mathfrak{F}} = L^{\mathfrak{F}}$ is periodic;

(iv) if the subgroup $M = [G^{\mathfrak{F}}, G^{\mathfrak{F}}]$ is non–nilpotent, then $G^{\mathfrak{F}}/M$ is a divisible Chernikov group and every proper G–invariant subgroup of M is nilpotent.

The study of groups satisfying **Max**-∞-(non-nil) was continued in the article [43] of L. A. Kurdachenko and N. N. Semko, where they described the hypercentral groups of this kind. Here is the main result of that work:

Theorem 6.11. Let G be a hypercentral group satisfying **Max**-∞-(non-nil). Suppose that G is non–nilpotent and non–minimax. Then G contains a finite normal subgroup F such that $G/F \leq M \times L$ where M is a hypercentral minimax group and L satisfies the following conditions:

(i) L is a hypercentral non–nilpotent, non–minimax group satisfying $\mathbf{Max} - \infty - (non - nil)$;

(ii) the periodic part P of the group L has a central series of L– invariant subgroups

$$\langle 1 \rangle = L_0 \leq L_1 \leq \cdots \leq L_n = P$$

such that the factors L_{j+1}/L_j , for $0 \leq j \leq n-2$, are elementary abelian and P–quasifinite, L_n/L_{n-1} is finite, and, in particular, P is a bounded nilpotent subgroup;

(iii) L/P is abelian–by–finite and minimax.

The groups satisfying the dual condition **Min**-∞-(non-nil), the weak minimal condition on non–nilpotent subgroups, were studied in the paper [22] by M. R. Dixon, M. J. Evans, and H. Smith.

The main result of this paper is as follows.

If G is a group then let $G^{\mathfrak{M}}$ denote the minimax residual of G, the intersection of all normal subgroups whose quotient is minimax.

Theorem 6.12. Let G be a locally (soluble–by–finite) group. Then G satisfies **Min**-∞-(non-nil) if and only if one of the following conditions hold:

(i) G is nilpotent;

(ii) G is minimax;

(iii) G is locally nilpotent, $G^{\mathfrak{M}}$ is nilpotent, $G/G^{\mathfrak{M}}$ is minimax and every non–nilpotent non-minimax subgroup of G contains $G^{\mathfrak{M}}$.

These authors also described certain other features of locally nilpotent groups with **Min**-∞-(non-nil).

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