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A characterization via graphs of the soluble groups in which permutability is transitive

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Dedicated to Professor Leonid Andreevich Kurdachenko on the occasion of his sixtieth birthday

Abstract. There are different ways to associate to a group a certain graph. In this context, it is interesting to ask for the relations between the structure of the group, given in group-theoretical terms, and the structure of the graphs, given in the language of graph theory.

In this paper we recall some properties of the groups in which permutability is a transitive relation and present a new characterisation of the class of soluble groups in which permutability is a transitive relation in graph-theoretical terms.

1. Permutability

In this paper, we will consider only finite groups. The notation and definitions is the usual in the scope of group theory, see for instance [17].

Perhaps the story of groups in which normality or permutability is transitive can be traced back to the results of Dedekind [15]. He studied the groups in which all subgroups are normal. These groups receive now the name of *Dedekind groups*. He proved:

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Theorem 1. *A group G has all subgroups normal if and only if either G is abelian, or G is a direct product of a quaternion group of order* 8*, an elementary abelian* 2*-group, and an abelian group of odd order.*

Recall that two subgroups H and K of a group G are said to *permute* when $HK = KH$. This is equivalent to affirming that HK is a subgroup of G. A subgroup H of a group G is said to be *permutable* in G when H permutes with all subgroups of G. Sometimes, we require a subgroup H of a group G to permute not necessarily with all subgroups of G, but only with a selected family of subgroups of G. This is the case of the *S-permutable* subgroups, which are the subgroups which permute with all Sylow subgroups of the group. It is well-known that normal subgroups are permutable, and obviously permutable subgroups are S-permutable. However, the converses are not true, as we can see with an extraspecial group of order p^3 and exponent p^2 for a prime $p > 2$, which has permutable subgroups which are not normal, and the dihedral group of order 8, which has S-permutable subgroups which are not normal.

As Dedekind did with groups in which all subgroups are normal, Iwasawa studied the groups in which all subgroups are permutable in [22].

Theorem 2. *A group whose subgroups are permutable is a nilpotent group in which for every Sylow p-subgroup P, either P is a direct product of a quaternion group and an elementary abelian* 2*-group, or P contains an abelian normal subgroup A and an element $b \in P$ such that $P = A\langle b\rangle$ and there exists a natural number s, with $s \geq 2$ if $p = 2$, such that $a^b = a^{1+p^s}$ for every $a \in A$.*

It is a well-known fact that normality is not in general a transitive relation. This motivates the introduction of subnormality, which is the transitive closure of normality. Groups in which both relations coincide are the groups in which normality is transitive and receive the name of *T-groups*. The relation of permutability is not transitive in general. However, a result of Ore [25] shows that permutable subgroups are subnormal. Hence the groups in which permutability and subnormality coincide are the groups in which permutability is transitive and are called *PT-groups*. Finally, according to a result of Kegel [23], S-permutable subgroups are subnormal. Hence the groups in which S-permutable subgroups are subnormal coincide with the groups in which S-permutability is a transitive relation and are called *PST-groups*. All T-groups are PT-groups, and all PT-groups are PST-groups. These containments are proper, because the extraspecial groups of order p^3 and exponent p^2 for a prime p are PT-groups, but not T-groups, and the dihedral group of order 8 is a PST-group which is not a PT-group. On the other hand, Dedekind groups are T-groups, Iwasawa p-groups for a prime p are PT-groups, and all nilpotent groups are PST-groups.

These classes of groups have been widely studied during the last years, especially in the soluble universe, with many characterisations available. We will begin with the classical characterisations. Agrawal [3] characterised soluble PST-groups as follows:

Theorem 3. *A group G is a soluble PST-group if and only if the nilpotent residual L of G is a Hall subgroup of odd order of G such that G acts on L as a group of power automorphisms.*

Here a power automorphism of a group X is an automorphism which fixes every subgroup of X. This is equivalent to affirming that every element x is sent by a power automorphism to a power of x.

If we impose in Agrawal's characterisation the fact that the Sylow subgroups are Iwasawa, we obtain the characterisation of Zacher [30] of soluble PT-groups. If we force the Sylow subgroups to be Dedekind, we obtain the characterisation of Gaschütz [18] of soluble T-groups.

These classical characterisations have the virtue of showing that these classes consist of supersoluble groups in the soluble universe and that the unique difference between all these three classes is the Sylow structure. This will be made clear with the help of local characterisations. In this context, we say that a class of groups $\mathfrak{X}$ is *local* when for every prime p there is a generalisation $\mathfrak{X}_p$ of $\mathfrak{X}$ such that $\mathfrak{X} = \bigcap_{p\in\mathbb{P}} \mathfrak{X}_p$. The notion of p-soluble group, p-supersoluble group, and p-nilpotent group show, respectively, that the classes of all soluble groups, all supersoluble groups, and all nilpotent groups are local. From now on, p will denote a fixed, but arbitrarily chosen, prime number.

Robinson [26] introduced the property $\mathcal{C}_p$ as follows: A group G satisfies $\mathcal{C}_p$ when if a subgroup H is contained in a Sylow p-subgroup P of G, then H is normal in the normaliser $\mathrm{N}_G(P)$. He characterised the soluble T-groups as the groups satisfying the property $\mathcal{C}_p$ for every prime p. Bryce and Cossey [14] defined a property $\mathcal{D}_p$ as follows: A group satisfies $\mathcal{D}_p$ when all Sylow p-subgroups of G are Dedekind and the chief factors of G of order divisible by p are cyclic and, as modules for G by conjugation, form a unique isomorphism class. They showed that $\mathcal{D}_p$-groups are $\mathcal{C}_p$-groups and that a group G is a soluble T-group if and only if it satisfies $\mathcal{D}_p$ for all primes p. It is worth noting that the classes $\mathcal{D}_p$ and $\mathcal{C}_p$ coincide in the p-soluble universe.

For the class of soluble PT-groups, Beidleman, Brewster, and Robinson [10] introduced a property $\mathcal{X}_p$: A group G satisfies $\mathcal{X}_p$ when if a subgroup H is contained in a Sylow p-subgroup P of G, then H is permutable in the normaliser $N_G(P)$. They proved that a group G is a soluble PT-group if, and only if, it satisfies $\mathcal{X}_p$ for all primes p.

Alejandre, the first author, and Pedraza-Aguilera [4] introduced the property $\mathcal{U}_p^*$ as follows: A group G satisfies $\mathcal{U}_p^*$ when G is all chief factors of G of order divisible by p are cyclic and isomorphic when regarded as G-modules by conjugation. They proved that a group is a soluble PST-group when it satisfies $\mathcal{U}_p^*$ for all primes p. Note that this property $\mathcal{U}_p^*$ generalises the property $\mathcal{D}_p$ given by Bryce and Cossey. The unique difference is that here we do not impose conditions to the Sylow p-subgroups. The first and the third authors introduced in [7] and [6] a new property $\mathcal{Y}_p$: A group G satisfies $\mathcal{Y}_p$ when for every pair of p-subgroups H and K of G such that $H \leq K$, H is S-permutable in $N_G(K)$. They showed that a group is a soluble PST-group if and only G satisfies $\mathcal{Y}_p$ for all primes p. We note that $\mathcal{U}_p^*$-groups satisfy $\mathcal{Y}_p$ and that both properties coincide in the p-soluble universe.

We must note that the easy way to generalise the properties $\mathcal{C}_p$ and $\mathcal{X}_p$ (a group satisfies $\mathcal{Y}_p^*$ when if H is contained in a Sylow p-subgroup P of G, then H is S-permutable in $N_G(P)$) does not yield a sufficiently strong property to characterise soluble PST-groups. The symmetric group of degree 4 is an example of a group satisfying $\mathcal{Y}_p^*$ for all primes p which is not a PST-group. The main difference is that the conditions $\mathcal{C}_p$ and $\mathcal{D}_p$ imply that the group has Dedekind and Iwasawa Sylow p-subgroups, respectively. These are very strong conditions and impose major restrictions on the structure of the group, in particular, the properties $\mathcal{C}_p$ and $\mathcal{X}_p$ are inherited by subgroups. For the property $\mathcal{Y}_p$, we do not have such restrictions and we must impose in the definition that the property is closed under taking subgroups. Furthermore, as shown in [7], $\mathcal{C}_p$-groups coincide with the $\mathcal{Y}_p$-groups with Dedekind Sylow p-subgroups and $\mathcal{X}_p$-groups are exactly the $\mathcal{Y}_p$-groups with Iwasawa Sylow p-subgroups.

There are also characterisations of T-, PT-, and PST-groups for finite, not necessarily soluble, groups. A long list of references is available in the paper of Beidleman and Ragland [11]. There also exist some descriptions of PT-groups in some universes of infinite groups, like the ones given by the first author, Kurdachenko, and Pedraza in [9] or the first author, Kurdachenko, Pedraza, and Otal in [8].

2. Graphs and groups

We will consider only graphs which are undirected, simple (that is, with no parallel edges), and without loops. These graphs will be characterised by the set of vertices and the adjacency relation between the vertices. Only basic concepts about graphs will be needed for this paper. They can be found in any book about graph theory or discrete mathematics, for example, [16].

Given a group G, there are many ways to associate a graph to G in such a way the vertices are associated with families of elements or subgroups and in which two vertices are adjacent if and only if they satisfy a certain relation. Since the graph can be studied in terms of graph theory, we can ask about the structure of the group (in terms of group theory) and the structure of the graph (in terms of graph theory). In other terms, we are interested in characterising certain properties of the group in terms of some properties of the graph, or, more in general, to study the influence of a property the group on the structure of the graph or the influence of a property of the graph on the structure of the group. This has been a fruitful topic in the last years. We will present some results which illustrate these ideas.

We can begin with the *commutativity graph* or *commuting graph*. This graph has as vertices the elements of the group, and two vertices are adjacent when they commute (as elements of the group). It is obvious that the groups in which the commutativity graph is complete (in the sense that every two vertices are adjacent) are exactly the abelian groups. This graph has been used to study simple groups since the paper of Stellmacher [27].

The following variation of the previous example gives a characterisation for the groups in which all subgroups are permutable. We will call it the *graph of permutability of cyclic subgroups*. Given a group G, consider the graph in which the vertices are the cyclic subgroups of G and in which every two vertices are adjacent when they permute. A group has all subgroups permutable if and only if the graph of permutability of cyclic subgroups is complete. A similar graph with vertices the non-normal subgroups was studied by Bianchi, Gillio, and Verardi (see [13, 12, 19]).

Another graph which has deserved a lot of attention is the *prime graph*. In this graph, the vertices are the prime numbers dividing the order of the group G and two different vertices p and q are connected when G possesses an element of order pq. As a matter of example, in the cyclic group of order 6, this graph is complete, but in the symmetric group of degree 3, this graph has two isolated vertices. The first references of the prime graph known to the authors correspond to Gruenberg and Kegel in an unpublished manuscript, and Williams, who studied the number of connected components of the prime graph of finite groups (see [20, 28, 29]).

For our purposes, a generalisation of the prime graph introduced by Abe and Iiyori in [2] becomes interesting. Given a group G, we construct the graph Γ_G in the following way: The vertices of Γ_G are, like in the prime graph, the prime numbers dividing the order of G, but two different vertices p and q are adjacent when G has a soluble subgroup of order divisible by pq. Abe and Iiyori [2] proved:

Theorem 4. *If G is a non-abelian simple group, then Γ_G is connected, but not complete.*

This theorem is used in the proof of our characterisation of soluble PT-groups to prove that the groups in which our graph is complete must be soluble.

Herzog, Longobardi, and Maj [21] have considered the graph whose vertices are the non-trivial conjugacy classes of a group G and in which two non-trivial conjugacy classes C and D of G are connected if there exist $c \in C$ and $d \in D$ such that $cd = dc$. They show that if G is a soluble group, then this graph has at most two connected components, each of diameter at most 15. They also study the structure of the groups in which there are no edges between non-central conjugacy classes and the relation between this graph and the prime graph.

Of course, not all graph properties can be reduced to graph completeness. For instance, the *non-commuting graph* of a non-abelian group G is defined as follows: its vertices are the non-central elements of G, and two vertices are adjacent when they do not commute. This graph has been studied by Neumann [24]. Abdollahi, Akbari, and Maimani [1] proved the following result about this graph:

Theorem 5. *If G is a non-abelian group, then its non-commuting graph G is connected, Hamiltonian, its diameter is 2 and its girth is 3. Moreover, this graph is planar if and only if G is isomorphic to the symmetric group of degree 3 or to a non-abelian group of order 8.*

3. A characterisation of soluble PT-groups with graphs

Now we are in a position to present a characterisation of soluble PT-groups in terms of graphs.

Motivated by the results of Herzog, Longobardi, and Maj [21], given a group G, we consider a graph Γ whose vertices are the conjugacy classes of cyclic subgroups of G and in which two vertices are adjacent when there are representatives of both conjugacy classes which permute. In other words, $\mathrm{Cl}_G(\langle x\rangle)$ is adjacent to $\mathrm{Cl}_G(\langle y\rangle)$ if we can find an element $g \in G$ such that $\langle x\rangle$ permutes with $\langle y^g\rangle$.

The main result of [5] is:

Theorem 6. *G is a soluble PT-group if and only if the graph Γ is complete.*

References

[1] A. Abdollahi, S. Akbari, H. R. Maimani, *Non-commuting graph of a group*. J. Algebra, M.**298**, 2006, pp.468–492.

[2] S. Abe, N. Iiyori, *A generalization of prime graphs of finite groups*, Hokkaido Math. J., N.**29**, 2000, pp.391–407.

[3] R. K. Agrawal, *Finite groups whose subnormal subgroups permute with all Sylow subgroups*, Proc. Amer. Math. Soc., N.**47** (1975), pp.77–83.

[4] M. J. Alejandre, A. Ballester-Bolinches, M. C. Pedraza-Aguilera, *Finite soluble groups with permutable subnormal subgroups*, J. Algebra, N.**240**, 2001, pp.705–722.

[5] A. Ballester-Bolinches, J. Cossey, R. Esteban-Romero, *Finite groups in which a graph related to permutability is complete*, preprint.

[6] A. Ballester-Bolinches, R. Esteban-Romero, *Sylow permutable subnormal subgroups of finite groups II*, Bull. Austral. Math. Soc., N.**64**, 2001,, pp.479–486.

[7] A. Ballester-Bolinches, R. Esteban-Romero, *Sylow permutable subnormal subgroups of finite groups*, J. Algebra, N.**251**, 2002, pp.727–738.

[8] A. Ballester-Bolinches, L. A. Kurdachenko, J. Otal, T. Pedraza, *Infinite groups with many permutable subgroups*, Rev. Matem. Iberoam., N.**24**, 2008, pp.745–764.

[9] A. Ballester-Bolinches, L. A. Kurdachenko, T. Pedraza, *On periodic radical groups in which permutability is a transitive relation*, J. Pure Appl. Algebra, N.**210**, 2007, pp.665–671.

[10] J. C. Beidleman, B. Brewster, D. J. S. Robinson, *Criteria for permutability to be transitive in finite groups*, J. Algebra, **222**, 1999, pp.400–412.

[11] J. C. Beidleman, M. F. Ragland, *The intersection map of subgroups and certain classes of finite groups*, Ricerche Mat., **56**, 2007, pp.217–227.

[12] M Bianchi, A. Gillio, L. Verardi, *Subgroup-permutability and affine planes*, Geometriae Dedicata, N.**85**, 2001, pp.147–155.

[13] M. Bianchi, A. Gillio Berta Mauri, L. Verardi, *Finite groups and subgroup-permutability*, Ann. Mat. Pura Appl., IV. Ser., N.**169**, 1995, pp.251–268.

[14] R. A. Bryce, J. Cossey, *The Wielandt subgroup of a finite soluble group*, J. London Math. Soc., N.**40**, 1989, pp.244–256.

[15] R. Dedekind, *Über Gruppen, deren sámtliche Teiler Normalteiler sind*, Math. Ann., N.**48**, 1897, pp.548–561.

[16] R. Diestel, *Graph Theory*, Springer, 2007.

[17] K. Doerk, T. Hawkes. *Finite Soluble Groups*, volume 4 of *De Gruyter Expositions in Mathematics*, Walter de Gruyter, 1992.

[18] W. Gaschütz, *Gruppen, in denen das Normalteilersein transitiv ist*, J. reine angew. Math., N.**198**, 1957, pp.87–92.

[19] A. Gillio Berta Mauri, L. Verardi, *On finite groups with a reducible permutability-graph*, Ann. Mat. Pura Appl., IV. Ser., N.**171**, 1996, pp.275–291.

[20] K. W. Gruenberg, *Free abelianised extensions of finite groups*, In C. T. C. Wall, editor, *Homological group theory. Proc. Symp., Durham 1977*, volume 36 of *Lond. Math. Soc. Lect. Note Ser.*, pp.71–104, Cambridge, 1979. London Mathematical Society, Cambridge Univ. Press.

[21] M. Herzog, P. Longobardi, M. Maj, *On a commuting graph on conjugacy classes of subgroups*, preprint.

[22] K. Iwasawa, *Über die endlichen Gruppen und die Verbände ihrer Untergruppen*, J. Fac. Sci. Imp. Univ. Tokyo Sect. I, N.**4**, 1941, pp.171–199

[23] O. H. Kegel, *Sylow-Gruppen und Subnormalteiler endlicher Gruppen*, Math. Z., N.**78**, 1962, pp.205–221.

[24] B. Neumann, *A problem of Paul Erdős on groups*, J. Austral. Math. Soc. Ser. A, N.**21**, 1976, pp.467–472.

[25] O. Ore, *Contributions to the theory of groups of finite order*, Duke Math. J., N.**5**, 1939, pp.431–460.

[26] D. J. S. Robinson, *A note on finite groups in which normality is transitive*, Proc. Amer. Math. Soc., N.**19**, 1968, pp.933–937.

[27] B. Stellmacher. *Einfache Gruppen, die von einer Konjugiertenklasse vno Elementen der Ordnung drei erzeugt werden*, J. Algebra, N.**30**, 1974, pp.320–354

[28] J. S. Williams. *The prime graph components of finite groups*, In B. Cooperstein and G. Mason, editors, *The Santa Cruz Conference on Finite Groups (Proc. Symp. Pure Math., June 25–July 20, 1979*, volume 37 of *Proc. Symp. Pure Math.*, pp.195–196, Providence, RI, USA, 1980. American Mathematical Society (AMS).

[29] J. S. Williams, *Prime graph components of finite groups*, J. Algebra, N.**69**, 1981, pp.487–513.

[30] G. Zacher, *I gruppi risolubli finiti in cui i sottogruppi di composizione coincidono con i sottogrupi quasi-normali*, Atti Accad. Naz. Lincei Rend. cl. Sci. Fis. Mat. Natur. (8), N.**37**, 1964, pp.150–154.

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