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RESEARCH ARTICLE

A note on a problem due to Zelmanowitz

Virgínia Silva Rodrigues and Alveri Alves Sant'Ana

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Abstract. In this paper we consider a problem due to Zelmanowitz. Specifically, we study under what conditions a uniform compressible module whose nonzero endomorphisms are monomorphisms is critically compressible. We give a positive answer to this problem for the class of nonsingular modules, quasi-projective modules and for modules over rings which are in a certain class of rings which contains at least the commutative rings and the left duo rings.

Dedicated to Professor Miguel Ferrero
on occasion of his 70-th anniversary

Introduction

The notions of compressible modules (a module is called compressible if it can be embedded in any of its nonzero submodules) and critically compressible modules (a compressible module is called critically compressible if it can not be embedded in any proper factor module) appeared in the theory of primitive rings in an attempt to extend the Jacobson density theorem, see [6] and [7]. In these papers Zelmanowitz succeed to extend the entire theory of primitive rings to the larger class of weakly primitive rings (rings that possess a faithful critically compressible module) and introduced the associated class of rings. Also, in [4], the author focused his attention on the extended density theorem for superrings and in the same way it was necessary the above concepts.

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For example in [6], Zelmanowitz claimed that a *"compressible uniform module whose nonzero endomorphisms are monomorphisms would be critically compressible"*. Later in [7] he said that he was unable either to prove or to disprove the statement. In [2] the author called the above statement *"Zelmanowitz's Conjecture"*. In this paper we prefer to enunciate it as a question. So we have the following:

Zelmanowitz's question: *Under what conditions a compressible uniform module whose nonzero endomorphisms are monomorphisms is critically compressible?*

In [2] it was proved that for modules over commutative rings the above question has a positive answer and the concepts of the compressible and critically compressible modules are equivalent in the case of modules over duo rings. Besides that, self-similar modules (in [3], these modules are also called isomorphically compressible) with some additional hypothesis become critically compressible. In [3], the author presents some conditions for compressible modules to be simple, for example, a compressible module is simple if it has a simple submodule. In this special case we can see that in the class of modules over semi-Artinian rings, this question is easily answered.

This paper is organized in three sections. In section 1 we give preliminary definitions and we show some results that allow us to rewrite the hypotheses of the Zelmanowitz's question. In section 2 we give an affirmative answer to the problem in the class of quasi-projective modules. Using the equivalent hypotheses that were obtained in section 1, we extend some results given in [2]. Section 3 is strongly related with the primeness condition of modules. Therein it is defined a class $\mathcal{C}$ of rings such that the Zelmanowitz's question is answered affirmatively. Also, using a suitable property given in ([5], 3.13) we give a positive answer to the problem.

1. Reformulating Zelmanowitz's question

Throughout this paper, it is assumed that R is an associative ring with an identity element. Unless otherwise indicated modules are unitary left modules and homomorphisms are written as right operators. If N is a submodule of M, we write $N \leq M$ and if N is an essential submodule of M then we write $N \trianglelefteq M$. A *partial endomorphism* of a module M is a homomorphism from a submodule of M into M.

Firstly we recall some definitions. A nonzero R-submodule N of a module M is called *rational* or *dense* in M if $Hom_R(X/N, M) = 0$, for any $N \leq X \leq M$. An R-module M is called *monoform* if every submodule is dense. This definition is equivalent to the second condition in the next proposition.

Proposition 1.1. [7, Prop. 1.1] *The following conditions are equivalent for a compressible module M:*

(i) *M is critically compressible;*

(ii) *Every nonzero partial endomorphism of M is a monomorphism.*

An R-module M is called *polyform* if every essential submodule of M is dense in M and M is *uniform* if every submodule of M is essential. It follows from [5], that for an R-module M the following statements are equivalent:

(i) *M is polyform;*

(ii) *For any submodule $K \leq M$ and for every nonzero homomorphism $f : K \to M$, $Ker(f)$ is not essential in K.*

Now we are able to give results that enable us to reformulate the Zelmanowitz's question. An R-module M is *retractable* if $Hom_R(M, X) \neq 0$ for every $X \leq M$. Often we change the hypothesis compressible by retractable and this last class of modules is larger than the first.

It is clear that compressible modules are retractable, but the converse is not true. Indeed, if D is a division ring and V a finite dimensional D-vector space, then V is a retractable D-module but it is clearly not compressible. We start with the following result:

Proposition 1.2. *Suppose that M is a retractable R-module. If every nonzero $f \in End(M)$ is a monomorphism, then every nonzero element of $Hom_R(M, N)$ is a monomorphism, for any nonzero submodule N of M. In particular, M is compressible.*

Proof. Let N be a nonzero submodule of M and $g : M \to N$ a nonzero homomorphism which there exists because M is retractable. Considering the canonical inclusion $i : N \hookrightarrow M$, gi is a monomorphism and obviously g is a monomorphism. The last part is clear. □

Since every endomorphism of M is also a partial endomorphism of M, it follows from the last result that the Proposition 1.1 can be extended to the setting of retractable modules.

Proposition 1.3. *Let M be a retractable R-module. The following statements are equivalent:*

(i) *M is critically compressible;*

(ii) *Every nonzero partial endomorphism of M is a monomorphism.*

It will be necessary the following results to give another formulation to the Zelmanowitz's question which is more useful for our purposes.

Theorem 1.4. *Let M be an R-module. The following conditions are equivalent:*

(i) *M is compressible and every nonzero endomorphism of M is a monomorphism;*

(ii) *M is compressible and $End(M)$ is a domain;*

(iii) *M is retractable and every nonzero endomorphism of M is a monomorphism;*

(iv) *M is retractable and $End(M)$ is a domain.*

Proof. It is clear that (i)⇒(ii) and (ii)⇒(iv). The implication (iv)⇒(iii) is an easy observation and (iii)⇒(i) follows from Proposition 1.2. □

Proposition 1.5. *Let M be a retractable uniform module such that $End(M)$ is a domain. Then M is critically compressible if and only if M is polyform.*

Proof. Suppose that M is polyform. Since a module is polyform and uniform if and only if it is a monoform module (see [5, 11.3 and 11.1]), by using that M is retractable, we have that M is critically compressible from Proposition 1.3. Conversely, if M is critically compressible, then by Proposition 1.3 it is monoform, and hence polyform. □

Now we are ready to reformulate our problem.

Zelmanowitz's question 1.6. *Under what conditions a retractable uniform R-module M such that $End(M)$ is a domain would be a polyform module?*

Here, we answer a question of Christian Lomp who asked us about a possible extended Zelmanowitz's question: "Is every compressible uniform module whose endomorphism ring is a domain a monoform module?" Note that it follows from Theorem 1.4 that the Lomp's question is equivalent to the Zelmanowitz's question.

Although the next proposition is given in [4], we would like to present a more direct proof of this result.

Proposition 1.7. *Let M be a retractable R-module. Suppose that M is a nonsingular uniform module. Then M is critically compressible.*

Proof. By Proposition 1.3 it is enough to prove that M is monoform. Let N be a nonzero submodule of M and P a submodule of M such that $N \leq P \leq M$. We can see that P/N is a singular R-module because M is uniform and it follows immediately that $Hom_R(P/N, M) = 0$ (see [1, Proposition 1.20 (a)]). □

As a final comment in this section we would like to observe that in [7], the author claimed that if M is a compressible module then M is singular or M is nonsingular. By Proposition 1.7 the Zelmanowitz's question has an affirmative answer in the class of nonsingular modules. So an another question arises: under the Zelmanowitz's hypotheses is a singular module critically compressible? We are not able to answer this question but we believe that it is false.

2. Zelmanowitz's question and fully retractability

In this section, we generalize results given in [2] and we show that in the class of quasi-projective modules the Zelmanowitz's question has an affirmative answer. We recall the concept of fully retractable that was given in [9].

Definition 2.1. *A module M is said to be fully retractable if for every nonzero submodule N of M and every nonzero element $g \in Hom_R(N, M)$ we have $Hom_R(M, N)g \neq 0$.*

Clearly, if M is fully retractable then M is retractable. Under some additional conditions we can get the reverse of the last implication as we will see later on.

According to [2], a nonzero R-module M is called *self-similar* if every nonzero submodule of M is isomorphic to M. It is clear that self-similar modules are fully retractable, but the converse is not true. Indeed, for instance $\mathbb{Z}_4$ is a fully retractable $\mathbb{Z}$-module which is not self-similar.

The next two propositions generalize Theorems 4.1 and 4.2 of [2], respectively.

Proposition 2.2. *If M is fully retractable such that $End(M)$ is a domain, then M is polyform.*

Proof. Suppose that M is not polyform. Then there exist a nonzero submodule K of M and a nonzero homomorphism $f : K \to M$ such that $Ker(f) \trianglelefteq K$. But we have that $Hom_R(M, K)f \neq 0$. So, there exists $0 \neq g : M \to K$ such that $gf \neq 0$ and it follows that gf is a monomorphism (see Theorem 1.4). Thus g is a monomorphism.

Now, $0 = Ker(gf) = g^{-1}(Ker(f)) \cong Ker(f) \cap Im(g)$, because g is a monomorphism. Since $Ker(f) \trianglelefteq K$, we have necessarily $Im(g) = 0$ which is a contradiction. □

In the next proposition we denote by $\widehat{M}$ the injective hull of M in $\sigma[M]$ (see [5, p. 37]).

Proposition 2.3. *Let M be a retractable module such that every nonzero $f \in Hom_R(M, \widehat{M})$ is a monomorphism. Then M is critically compressible.*

Proof. It follows from Proposition 1.3 and from the fact that $\widehat{M}$ is M-injective. □

We can see easily from the above proposition that if M is quasi-injective then the Zelmanowitz's question has a positive answer.

Proposition 2.4. *Let M be a retractable uniform module such that $End(M)$ is a domain. Then the following conditions are equivalent:*

(i) *M is critically compressible;*

(ii) *M is polyform;*

(iii) *M is fully retractable.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows directly from Proposition 1.5. (iii) $\Rightarrow$ (ii) follows from Proposition 2.2. Now we prove (ii) $\Rightarrow$ (iii). Since M is polyform and uniform, it is monoform as it was seen before. Therefore if X is a nonzero submodule of M and g is a nonzero homomorphism from X to M, then it is a monomorphism. Since M is retractable, $Hom_R(M, X)$ is nonzero and we have that $Hom_R(M, X)g \neq 0$. □

Now, we give one more class of modules where the Zelmanowitz's question has an affirmative answer. We refer to [5] for a definition of quasi-projective module.

Theorem 2.5. *Suppose that M is a quasi-projective module satisfying the $Zelmanowitz's$ hypotheses. Then M is polyform.*

Proof. We have that M is retractable. According to ([8], Proposition 2.2), it follows that M is a retractable module if and only if M is fully retractable. From Proposition 2.4 we have that M is polyform. □

3. Primeness condition

In this section we consider the class $\mathcal{C}$ of rings such that for every prime left R-module M and for any nonzero elements $x, y \in M$ one has $Ann_R(x) = Ann_R(y)$.

We recall that a ring R is called *left (right) duo ring* if every left (right) ideal of R is an ideal of R. Obviously commutative rings are left (right) duo rings. The next result shows that left (right) duo rings are also in $\mathcal{C}$.

Proposition 3.1. *Let R be a left duo ring. Then $R \in \mathcal{C}$.*

Proof. Let ${}_RM$ be a prime module. Since R is a left duo ring, for any $0 \neq x \in M$, $Ann_R(x) = \{r \in R : rx = 0\}$ is an ideal of R. Then for every $r \in Ann_R(x)$ and $a \in R$, $ra \in Ann_R(x)$ and this implies that $r \in Ann_R(Rx)$. Thus $Ann_R(x) \subseteq Ann_R(Rx)$ and obviously $Ann_R(x) = Ann_R(Rx)$. Since M is a prime module, we have $Ann_R(Rx) = Ann_R(Ry)$ for every nonzero $x, y \in M$, and it follows that $Ann_R(x) = Ann_R(Rx) = Ann_R(Ry) = Ann_R(y)$. □

Now we are able to give our result.

Theorem 3.2. *Let R be in $\mathcal{C}$, M a retractable R-module such that $End(M)$ is a domain. Then M is a polyform module.*

Proof. Firstly we note that by Proposition 1.2, M is a compressible module. Then it is easy to see that M is a prime module.

Let K, L be submodules of M such that $K \leq L \leq M$ and $K \trianglelefteq M$, and suppose that $\alpha \in Hom(L/K, M)$. We need to prove that $\alpha = 0$. By contradiction, we suppose that there exists $l \in L$ such that $(l + K)\alpha \neq 0$. Since $Rl \neq 0$ we can consider the canonical projection $\pi : L \to L/K$. Thus we have $(l)\pi \notin K$ so that $\pi\alpha \neq 0$. Since M is a retractable module such that every nonzero endomorphism of M is a monomorphism, we have by Proposition 1.2 that there exists a monomorphism $g : M \to Rl$. So we can consider the following composition:

$$\varphi : M \xrightarrow{g} L \xrightarrow{\pi} L/K \xrightarrow{\alpha} M.$$

Since $g \neq 0$, there exists $m \in M$ such that $(m)g \neq 0$ and so, there exists $r \in R$ such that $(m)g = rl \neq 0$. Thus, $r \notin Ann_R(l) = Ann_R(x)$ for every $x \in M \setminus \{0\}$, by hypothesis. Therefore $r((l + K)\alpha) \neq 0$ and so $rl \notin K$. In this way we had proved that $(m)\varphi \neq 0$ and it follows that φ needs to be a monomorphism by our hypothesis. On the other hand, since $K \trianglelefteq M$ we have $(M)g \cap K \neq 0$ and so $0 \neq (K)g^{-1} \subseteq Ker\,(\varphi)$, a contradiction. Therefore we need to have $\alpha = 0$ and the result follows. □

Corollary 3.3. *The Zelmanowitz's question has an affirmative answer for modules over rings which are in* $\mathcal{C}$.

In ([5], 3.13), it was given a property of an R-module M which is important for primeness conditions. In our case this property has an important role because it allows us to give an answer to this question in an another case. It is the following:

($*$) For any nonzero submodule K of M, $Ann_R(M/K) \not\subset Ann_R(M)$, i.e., there is $r \in R \backslash Ann_R(M)$ such that $rM \subset K$.

Theorem 3.4. *Let M be a retractable module satisfying ($*$) and such that every nonzero endomorphism of M is a monomorphism. Then M is monoform.*

Proof. By Proposition 1.2, M is compressible and so it is a prime module. Let L be a nonzero submodule of M and $f : L \to M$ such that $0 \neq IM \subseteq Ker(f)$ for some ideal I of R. Then $I((L)f) = 0$ and hence $(L)f$ must be zero. This is clear because M is a prime module.

Now we prove that M is monoform. Let K, L be nonzero submodules of M such that $K \leq L \leq M$, we need to show that $Hom_R(L/K, M) = 0$. By contradiction suppose that there exists a nonzero $g : L/K \to M$. Considering the canonical projection $\pi : L \to L/K$, $f = \pi g : L \to M$ is nonzero. Since that M satisfy ($*$), there exists $r \in R$ such that $0 \neq rM \subset K \subseteq Ker(f)$. Taking $I = (r)$ the ideal generated by r, we have $0 \neq IM \subseteq K \subseteq Ker(f)$ and according to the previous paragraph f needs to be zero, an absurd. Therefore $Hom_R(L/K, M) = 0$ and M is monoform. □

Moreover if M is a uniform module in the above theorem, then M is critically compressible and so the Zelmanowitz's question is affirmatively answered in this context.

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References

[1] Goodearl, K.R., *"Ring Theory - Nonsingular Rings and Modules"*, Monographs and Textbooks in Pure and Applied Mathematics, 33 (Marcel Dekker, Inc., New York - Basel, 1976).

[2] Jeong, J-W., *"On critically compressible modules"*, Master thesis, Kyungpook National University, 1998.

[3] Limarenko, S.V., *"Compressible Modules"*, Moscow Univ. Math. Bull., vol. 60, no. 3, pp. 26-29, 2005.

[4] Limarenko, S.V., *"Weakly primitive superrings"*, J. Math. Sci., vol. 139, no.4, pp. 6723-6752, 2006.

[5] Wisbauer, R., *"Modules and Algebras - Bimodule Structure and Group Actions on Algebras"*, Pitman Monographs and Surveys in Pure and Appl. Math., 81 (Addison-Wesley, Longman, Harlow, 1996).

[6] Zelmanowitz, J., *"An extension of the Jacobson density theorem"*, Bull. Amer. Math. Soc., vol. 82, no. 4, pp. 551-553, 1976.

[7] Zelmanowitz, J., *"Weakly primitive rings"*, Comm. Algebra, 9(1), pp. 23-45, 1981.

[8] Zelmanowitz, J., *"Correspondences of closed submodules"*, Proc. Amer. Math. Soc., vol. 124, no. 10, pp. 2955-2960, 1996.

[9] Zelmanowitz, J., *"Density for polyform modules"*, Contemporary Math. 259 (2000), 563-569.

Contact information

V. Rodrigues
Departamento de Matemática
Universidade Federal de Santa Catarina
88040-900 - Florianópolis - Brazil
E-Mail: virginia@mtm.ufsc.br

A. Sant'Ana
Instituto de Matemática
Universidade Federal do Rio Grande do Sul
91509-900 - Porto Alegre - Brazil
E-Mail: alveri@mat.ufrgs.br

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