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RESEARCH ARTICLE

On the genus of the annihilator graph of a commutative ring*

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Abstract. Let R be a commutative ring and $Z(R)^*$ be its set of non-zero zero-divisors. The *annihilator graph* of a commutative ring R is the simple undirected graph $\mathrm{AG}(R)$ with vertices $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}(xy) \neq \mathrm{ann}(x) \cup \mathrm{ann}(y)$. The notion of annihilator graph has been introduced and studied by A. Badawi [7]. In this paper, we determine isomorphism classes of finite commutative rings with identity whose $\mathrm{AG}(R)$ has genus less or equal to one.

1. Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. I. Beck[8] began the study of associating a graph called the zero-divisor graph $\Gamma_0(R)$ to a commutative ring R and was mainly interested in the coloring of the zero-divisor graph. For a commutative ring R, the *zero-divisor graph* is the simple graph with R as the vertex set and two distinct elements x and y are adjacent if and only if $xy = 0$ [8]. D. F. Anderson and P. S. Livingston[3] modified and

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studied the zero-divisor graph $\Gamma(R)$ as the graph with vertex set as the nonzero zero-divisors $Z(R)^*$ of R and two distinct elements $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. Thereafter, several graphs have been associated with commutative rings. These graphs exhibit the interplay between the algebraic properties of R and graph theoretical properties of the associated graph. For $a \in R$, let $\mathrm{ann}(a) = \{d \in R : da = 0\}$ be the annihilator of $a \in R$. In 2014, A. Badawi [7] introduced the *annihilator graph* $\mathrm{AG}(R)$ as the simple graph with vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}(xy) \neq \mathrm{ann}(x) \cup \mathrm{ann}(y)$. One can see that the zero-divisor graph $\Gamma(R)$ is a subgraph of the annihilator graph $\mathrm{AG}(R)$.

The main objective of topological graph theory is to embed a graph into a surface. Let S_k denote the sphere with k handles, where k is a nonnegative integer, that is, S_k is an oriented surface of genus k. The genus of a graph G, denoted $g(G)$, is the minimal integer n such that the graph can be embedded in S_n. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. A graph G with genus 0 is called a planar graph, whereas a graph G with genus 1 is called a toroidal graph. Further note that if H is a subgraph of a graph G, then $g(H) \leqslant g(G)$. For details on the notion of embedding a graph in a surface, see [26].

Many research articles have appeared on the genus of zero divisor graphs of commutative rings. In particular, there are many papers [1, 2, 9, 15, 24], where the planarity of zero-divisor graphs has been discussed. The question addressed in these papers is this:For which finite commutative rings R is $\Gamma(R)$ planar? A partial answer was given in [1], but the question remained open for local rings of order 32. In [15], and independently in both [9] and [24], it was shown that no local ring of order 32 has the planar zero divisor graph. Furthermore, Smith [15] gave a complete list of finite planar rings; this list included the 2 infinite families $\mathbb{Z}_2 \times F$ and $\mathbb{Z}_3 \times F$, where F is any finite field, and the 42 other isomorphism classes of finite planar rings. H.J. Wang determined rings of the forms $\mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_r^{\alpha_r}}$ and $\frac{\mathbb{Z}_n[x]}{\langle x^m \rangle}$ that have genus at most one [24, Theorems 3.5 and 3.11]. Further H.J. Wang and N.O. Smith obtained all commutative rings whose zero divisor graph has genus at most one [23, Theorem 3.6.2].

Note that the zero divisor graph $\Gamma(R)$ is a subgraph of $\mathrm{AG}(R)$. In [7], it has been shown that for any reduced ring R that is not an integral domain, $\mathrm{AG}(R) = \Gamma(R)$ if and only if R has exactly two distinct minimal prime ideals [7, Theorem 3.6]. Note that using the proof of this result,

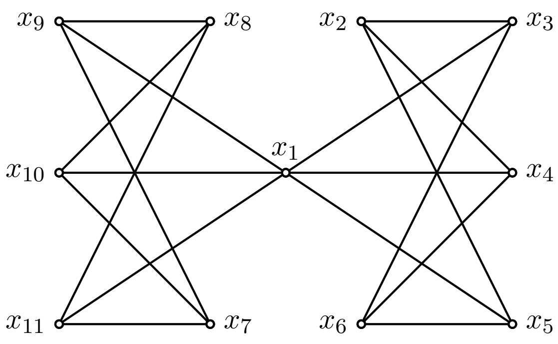

Figure 1. Graph G.

one can establish that for any reduced ring, $\mathrm{AG}(R)$ is complete if and only if, $\Gamma(R)$ is complete if and only if, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. If $G = K_{1,n}$ where $n \geqslant 1$, then G is a star graph. P_n denotes the path of length n for $n \geqslant 1$. A graph G is said to be *unicyclic* if it contains a unique cycle. Given a connected graph G, we say that a vertex v of G is a cut vertex if $G - v$ is disconnected. For a subset S of vertices of G, the *induced subgraph* of G is the subgraph with vertex set S together with edges whose both ends are in S and is denoted by $< S >$. A block is a maximal connected subgraph of G having no cut vertices. A result of Battle, Harary, Kodama, and Youngs states that the genus of a graph is the sum of the genera of its blocks[6]. For example, the graph G in Figure 1 has two blocks, both isomorphic to $K_{3,3}$, and so has genus 2 [25, C. Wickham].

Throughout this paper, we assume that R is a finite commutative ring with identity, $Z(R)$ its set of zero-divisors and $\mathrm{Nil}(R)$ its set of nilpotent elements, $R^{\times}$ its group of units, $\mathbb{F}_q$ denotes the field with q elements, and $R^* = R - \{0\}$. The following results are useful for further reference in this paper.

Theorem 1. [23, Theorem 3.5.1] *Let $(R, \mathfrak{m})$ be a finite local ring which is not a field. Then $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following 29 rings:*

$$\mathbb{Z}_4,\quad \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle},\quad \mathbb{Z}_9,\quad \frac{\mathbb{Z}_3[x]}{\langle x^2\rangle},\quad \mathbb{Z}_8,\quad \frac{\mathbb{Z}_2[x]}{\langle x^3\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2\rangle},\quad \frac{\mathbb{Z}_2[x,y]}{\langle x^2, xy, y^2\rangle},$$
$$\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2\rangle},\quad \frac{\mathbb{F}_4[x]}{\langle x^2\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle},\quad \mathbb{Z}_{25},\quad \frac{\mathbb{Z}_5[x]}{\langle x^2\rangle},\quad \mathbb{Z}_{16},\quad \frac{\mathbb{Z}_2[x]}{\langle x^4\rangle},$$
$$\frac{\mathbb{Z}_4[x]}{\langle x^2-2, x^4\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^3-2, x^4\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^3+x^2-2, x^4\rangle},\quad \frac{\mathbb{Z}_2[x,y]}{\langle x^3, xy, y^2-x^2\rangle},$$
$$\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2x\rangle},\quad \frac{\mathbb{Z}_8[x]}{\langle x^2-4, 2x\rangle},\quad \frac{\mathbb{Z}_4[x,y]}{\langle x^3, x^2-2, xy, y^2-2, y^3\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^2\rangle},$$
$$\frac{\mathbb{Z}_4[x,y]}{\langle x^2, y^2, xy-2\rangle},\quad \frac{\mathbb{Z}_2[x,y]}{\langle x^2, y^2\rangle},\quad \mathbb{Z}_{27},\quad \frac{\mathbb{Z}_3[x]}{\langle x^3\rangle},\quad \frac{\mathbb{Z}_9[x]}{\langle x^2-3, x^3\rangle},\quad \frac{\mathbb{Z}_9[x]}{\langle x^2+3, x^3\rangle}.$$

One can have the following theorem from Theorem 3.7 [15].

Theorem 2. [15, Theorem 3.7] *Let $R = F_1 \times \cdots \times F_n$ be a finite ring, where each F_i is a field and $n \geqslant 2$. Then $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_2 \times F,\quad \mathbb{Z}_3 \times \mathbb{F},\quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,$$

where F is a finite field.

Theorem 3. [23, Theorem 3.5.2] *Let $(R, \mathfrak{m})$ be a finite local ring which is not a field. Then $g(\Gamma(R)) = 1$ if and only if R is isomorphic to one of the following 17 rings:*

$$\mathbb{Z}_{49},\quad \frac{\mathbb{Z}_7[x]}{\langle x^2\rangle},\quad \frac{\mathbb{Z}_2[x,y]}{\langle x^3, xy, y^2\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^3, \quad 2x\rangle},\quad \frac{\mathbb{Z}_4[x,y]}{\langle x^3, x^2-2, xy, y^2\rangle},$$
$$\frac{\mathbb{Z}_8[x]}{\langle x^2, 2x\rangle},\quad \frac{\mathbb{F}_8[x]}{\langle x^2\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^3+x+1\rangle},\quad \frac{\mathbb{Z}_4[x,y]}{\langle 2x, 2y, x^2, xy, y^2\rangle},\quad \frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z\rangle^2},$$
$$\mathbb{Z}_{32},\quad \frac{\mathbb{Z}_2[x]}{\langle x^5\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^3-2, x^5\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^4-2, x^5\rangle},\quad \frac{\mathbb{Z}_8[x]}{\langle x^2-2, x^5\rangle},$$
$$\frac{\mathbb{Z}_8[x]}{\langle x^2-2x+2, x^5\rangle},\quad \frac{\mathbb{Z}_8[x]}{\langle x^2+2x-2, x^5\rangle}.$$

Theorem 4. [11, 12, Theorem 6.3] *Let G be a connected graph. Then G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2$, C_4, C_5.*

Theorem 5. [3, Lemma 2.12] *Let R be a finite commutative ring. If $\Gamma(R)$ has exactly one vertex adjacent to every other vertex and no other adjacent vertices, then either $R \cong \mathbb{Z}_2 \times F$, where F is a finite field with $|F| \geqslant 3$, or R is local with maximal ideal $\mathfrak{m}$ satisfying $\frac{R}{\mathfrak{m}} \cong \mathbb{Z}_2$, $\mathfrak{m}^3 = 0$ and $|\mathfrak{m}^2| \leqslant 2$. Thus $|\Gamma(R)|$ is either p^n or $2^n - 1$ for some prime p and integer $n \geqslant 1$.*

Theorem 6. [3, Theorem 2.10] *Let R be a finite commutative ring. If $\Gamma(R)$ is complete, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is local with char $R = p$ or p^2, and $|\Gamma(R)| = p^n - 1$, where p is prime and $n \geqslant 1$.*

Theorem 7. [3, Theorem 2.13] *Let R be a finite commutative ring with $|\Gamma(R)| \geqslant 4$. Then $\Gamma(R)$ is a star graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a finite field. In particular, if $\Gamma(R)$ is a star graph, then $\Gamma(R) = p^n$ for some prime p and integer $n \geqslant 0$. Conversely, each star graph of order p^n can be realized as $\Gamma(R)$.*

Theorem 8. [7, Theorem 3.6] *Let R be a reduced commutative ring that is not an integral domain. Then $\mathrm{AG}(R) = \Gamma(R)$ if and only if R has exactly two distinct minimal prime ideals.*

Theorem 9. [7, Theorem 3.10] *Let R be a nonreduced commutative ring with $|\,\mathrm{Nil}(R)^*| \geqslant 2$ and let $\mathrm{AG}_N(R)$ be the (induced) subgraph of $\mathrm{AG}(R)$ with vertices $\mathrm{Nil}(R)^*$. Then $\mathrm{AG}_N(R)$ is complete.*

2. Basic properties of annihilator graph

In this section, we state some basic observations of the annihilator graph. Especially, we identify the annihilator ideal graph of small order and in particular we list out all local rings R with $|R| \leqslant 7$, for which the annihilator graph $\mathrm{AG}(R)$ is complete.

Remark 1. Let $R = F_1 \times F_2$ where F_1 and F_2 are finite fields. Then R is reduced with exactly two distinct minimal prime ideals. Hence, by Theorem 8, $\mathrm{AG}(R) \cong \Gamma(R) = K_{|F_1^*|,|F_2^*|}$.

Remark 2. Let $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$ be a local ring that is not a field and $|Z(R)^*| \geqslant 3$. By Theorem 9, $\mathrm{AG}(R)$ is complete and hence $\mathrm{gr}(\mathrm{AG}(R)) = 3$. On the other hand, let R be a finite commutative ring with identity but not a field. Since R is finite, $R \cong R_1 \times \cdots \times R_n$, where each R_i is a local ring. If $n \geqslant 3$, then $(1, 0, \ldots, 0) - (0, 1, 0, \ldots, 0) - (0, 0, 1, 0, \ldots, 0) - (1, 0, \ldots, 0)$ is a cycle in $\mathrm{AG}(R)$ and hence $\mathrm{gr}(\mathrm{AG}(R) = 3$.

$\lvert Z(R)^*\rvert$	Local ring R	AG(R)
1	$\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$	K_1
2	$\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$	K_2
3	$\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2-2\rangle}$	K_3
3	$\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{Z_4[x]}{\langle x^2+x+1\rangle}$	K_3
3	$\frac{\mathbb{Z}_4[x]}{<x,2>^2}$, $\frac{\mathbb{Z}_2[x,y]}{<x,y>^2}$	K_3
4	$\mathbb{Z}_{25}$, $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$	K_4

TABLE 1.

Remark 3. Note that $\Gamma(R)$ is a subgraph of AG(R). D.F. Anderson et al., [2] gave all zero-divisor graphs of order $\leqslant 4$. Using this, we give in Table 1, all commutative local rings R for which $|Z(R)^*| \leqslant 4$ and AG(R) is complete. One can note that there are only two rings $\mathbb{Z}_2 \times \mathbb{F}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ with $|Z(R)^*| \leqslant 4$ which are not mentioned in Table 1 since $\Gamma(\mathbb{Z}_2 \times \mathbb{F}_4) = K_{1,3}$ and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3) = K_{2,2}$ (refer Remark 1).

S. P. Redmond [13, 14] has given all local commutative rings R with $|Z(R)^*| \leqslant 7$. Using the list given in [13, 14], Table 2 provides all finite commutative local rings, for which $6 \leqslant |Z(R)^*| \leqslant 7$ and AG(R) is complete.

$\lvert Z(R)^*\rvert$	Local Ring R	AG(R)
6	$\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[x]}{\langle x^2\rangle}$	K_6
7	$\mathbb{Z}_{16}$, $\frac{\mathbb{Z}_2[x]}{\langle x^4\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+2x+2\rangle}$	K_7
7	$\frac{\mathbb{Z}_4[x]}{\langle x^3-2,2x^2,2x\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^3,xy,y^2\rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle 2x,x^2\rangle}$	K_7
7	$\frac{\mathbb{Z}_4[x]}{\langle x^3,2x^2,2x\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^2-2,xy,y^2,2x,2y\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+2x\rangle}$	K_7
7	$\frac{\mathbb{Z}_8[x]}{\langle 2x,x^2+4\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2-xy\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^2,y^2-xy,xy-2,2x,2y\rangle}$	K_7
7	$\frac{\mathbb{Z}_4[x,y]}{\langle x^2,y^2,xy-2,2x,2y\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2\rangle}$,	K_7
7	$\frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z\rangle^2}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^2,y^2,xy,2x,2y\rangle}$, $\frac{\mathbb{F}_8[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3+x+1\rangle}$	K_7

TABLE 2.

Theorem 10. *Let R be a finite commutative ring with identity that is not a field. Then* AG(R) *is a tree if and only if R is isomorphic to one of the following 5 rings:*

$$\mathbb{Z}_4, \quad \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}, \quad \mathbb{Z}_9, \quad \frac{\mathbb{Z}_3[x]}{\langle x^2\rangle} \quad or \quad \mathbb{Z}_2 \times F,$$

where F is a finite field.

Proof. Since R is finite, $R \cong R_1 \times \cdots \times R_n$, where each R_i is a local ring. Suppose $\mathrm{AG}(R)$ is a tree. In view Remark 2, $n \leqslant 2$.

Suppose $R \cong R_1 \times R_2$. If $Z(R_1)^* \neq \{0\}$, then there exist $x_1, y_1 \in Z(R_1)^*$ such that $x_1y_1 = 0$ and $|R_1^\times| \geqslant 2$. Let $x = (0,1)$, $y = (x_1, 0)$ $z = (y_1, 1)$ and $w = (1,0)$. Then $x, y, z, w \in Z(R)^*$ and $(x_1, 1) \in \mathrm{ann}(zw)$ where as $(x_1, 1)$ is neither in $\mathrm{ann}(z)$ nor in $\mathrm{ann}(w)$. Hence $x-y-z-w-x$ is a cycle in $\mathrm{AG}(R)$, a contradiction. Hence R_1 and R_2 are fields and so $\mathrm{AG}(R) \cong K_{|R_1|-1,|R_2|-1}$. Since $\mathrm{AG}(R)$ is tree, $|R_1| = 2$ or $|R_2| = 2$ and so $R \cong \mathbb{Z}_2 \times F$, where F is a finite field.

Suppose $R \cong R_1$. Since R is not a field, $Z(R)^* \neq \{0\}$. Here $R = R_1$ is a local ring and so $\mathrm{AG}(R)$ is complete. Hence by the assumption viz., $\mathrm{AG}(R)$ is a tree, we get that $|Z(R)^*| \leqslant 2$. Hence $R \cong \mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$, or $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$.

The converse can be ascertained from Table 1 and Remark 1. □

Theorem 11. *Let R be a finite commutative ring with identity that is not a field. Then* AG(R) *is unicyclic if and only if R is isomorphic to one of the following 8 rings:*

$$\mathbb{Z}_8, \quad \frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}, \quad \frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2\rangle}, \quad \frac{\mathbb{F}_4[x]}{\langle x^2\rangle}, \quad \frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle},$$
$$\frac{\mathbb{Z}_4[x]}{\langle x, 2\rangle^2}, \quad \frac{\mathbb{Z}_2[x,y]}{\langle x, y\rangle^2} \quad or \quad \mathbb{Z}_3 \times \mathbb{Z}_3.$$

Proof. Sufficient part follows from Table 1 and Remark 1.

Conversely, assume that $\mathrm{AG}(R)$ contains a unique cycle of length $\ell \geqslant 3$. Since R is finite, $R \cong R_1 \times \cdots \times R_n$, where each R_i is a local ring. Suppose $n \geqslant 3$. Let $x_1 = (1,0,0,\ldots,0)$, $x_2 = (0,1,0,\ldots,0)$, $x_3 = (0,0,1,0,\ldots,0)$, $y_1 = (0,1,1,0\ldots,0)$, $y_2 = (1,0,1,0\ldots,0)$. Then $x_1, x_2, x_3, y_1, y_2 \in Z(R)^*$ and $x_1-x_2-x_3-x_1$ as well as $x_1-y_1-y_2-x_2-x_1$ are two distinct cycles in $\mathrm{AG}(R)$, a contradiction. Hence $n \leqslant 2$.

Case 1. Suppose $n = 2$. If $Z(R_1) \neq \{0\}$, then there exist $x, y \in Z(R_1)^*$ such that $xy = 0$. Since R_1 is local, R_1 contains at least one unit $u_1 \in R_1^\times$

apart from the identity. Then $(x, 0) - (y, 1) - (1, 0) - (0, 1) - (x, 0)$ and $(x, 0) - (y, 1) - (u_1, 0) - (0, 1) - (x, 0)$ are cycles in $\mathrm{AG}(R)$, a contradiction. Hence R_1 and R_2 are fields and so $\mathrm{AG}(R) \cong K_{|R_1|-1,|R_2|-1}$. Since $\mathrm{AG}(R)$ is unicyclic, $R_1 \cong \mathbb{Z}_3$ and $R_2 \cong \mathbb{Z}_3$.

Case 2. Suppose $n = 1$. Here R is a local ring but not a field. By Theorem 9, $\mathrm{AG}(R)$ is complete. Since $\mathrm{AG}(R)$ is unicyclic, $|Z(R)^*| = 3$ and by Table 1, R is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x, 2 \rangle^2}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x, y \rangle^2}$. □

Note that any complete graph is a split graph with any single vertex as an independent set and all the other vertices induce a clique. Hence, if R is a local ring, then $\mathrm{AG}(R)$ is a split graph. Now, we characterize all nonlocal rings R for which $\mathrm{AG}(R)$ is a split graph.

Theorem 12. *Let R be a finite commutative nonlocal ring with identity and $|Z(R)^*| \geqslant 2$. Then $\mathrm{AG}(R)$ is a split graph if and only if $R \cong \mathbb{Z}_2 \times F$, where F is a finite field.*

Proof. Suppose $R = Z_2 \times F$, where F is a finite field. By Remark 1, $\mathrm{AG}(R) = K_{1,|F^*|}$ and hence $\mathrm{AG}(R)$ is a split graph.

Conversely, suppose $\mathrm{AG}(R)$ is a split graph. Since R is finite, $R \cong R_1 \times \cdots \times R_n$ where each R_i is local for $1 \leqslant i \leqslant n$ and $n \geqslant 2$. If $n \geqslant 3$, then $(1, 0, 0, \ldots, 0) - (0, 1, 0, \ldots, 0) - (1, 0, 1, 0 \ldots, 0) - (0, 1, 1, 0, \ldots, 0) - (1, 0, 0, \ldots, 0)$ is a cycle of length 4 in $\mathrm{AG}(R)$ and by Theorem 4, $\mathrm{AG}(R)$ is not split, a contradiction. Hence $n = 2$.

If $Z(R_1) \neq \{0\}$, then there exists an element $x \in Z(R_1)^*$ such that $xy = 0$ for some $y \in Z(R_1)^*$ and so $(x, 0) - (0, 1) - (1, 0) - (y, 1) - (x, 0)$ is a cycle of length 4 in $\mathrm{AG}(R)$, a contradiction. Hence R_1, R_2 are fields and so $\mathrm{AG}(R) \cong K_{|R_1|-1,|R_2|-1}$. Since $\mathrm{AG}(R)$ is split, either $|R_1| - 1 = 1$ or $|R_2| - 1 = 1$ and so $R \cong \mathbb{Z}_2 \times F$ where F is a finite field. □

Theorem 13. *Let R be a finite commutative ring with identity that is not a field. Then*

(i) $\mathrm{gr}(\mathrm{AG}(R)) = \infty$ *if and only if* $R \cong \mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, *or* $\mathbb{Z}_2 \times F$, *where F is a finite field;*

(ii) $\mathrm{gr}(\mathrm{AG}(R)) = 4$ *if and only if* $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$, *or* $F_1 \times F_2$, *where F_1, F_2 are finite fields with $|F_1| \geqslant 3$ and $|F_2| \geqslant 3$;*

(iii) $\mathrm{gr}(\mathrm{AG}(R)) = 3$ *if and only if R is not isomorphic to the rings in (i) and (ii).*

Proof. (i) Suppose $\mathrm{gr}(\mathrm{AG}(R)) = \infty$. Then $\mathrm{AG}(R)$ contains no cycles and so $\mathrm{AG}(R)$ is a tree. Remaining part of the proof follows from Theorem 10.

(ii) Suppose $\mathrm{gr}(\mathrm{AG}(R)) = 4$. By Remark 2, R cannot be local. Also $|Z(R)^*| \geqslant 4$. Let $R \cong R_1 \times \cdots \times R_n$ where each R_i is a local ring. If $n \geqslant 3$, by Remark 2, $\mathrm{gr}(\mathrm{AG}(R)) = 3$, a contradiction and hence $n = 2$.

Suppose R is not reduced. Then $|Z(R_i)^*| \neq \{0\}$ for some i. If $|Z(R_1)^*| \geqslant 2$, then there exist $x, y \in Z(R_1)^*$ such that $x \neq y$ and $xy = 0$. From this, we get that $(x,0) - (y,0) - (0,1) - (x,0)$ is a cycle of length 3 in $\mathrm{AG}(R)$, a contradiction. Thus $|Z(R_i)^*| \leqslant 1$ for $i = 1, 2$. If $|Z(R_i)^*| = 1$ for $i = 1, 2$, then $R_i \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. Suppose $R \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, then $(2,0) - (0,1) - (2,2)$ is a cycle in $\mathrm{AG}(R)$. Note that all the remaining cases produce the same $\mathrm{AG}(R)$. Hence in all the cases $\mathrm{gr}(\mathrm{AG}(R)) = 3$, a contradiction. This shows that $|Z(R_1)^*| = 1$ or $|Z(R_2)^*| = 1$.

If $|Z(R_1)^*| = 1$ and $|Z(R_2)^*| = 0$, then $R_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ and R_2 is a field. If $|R_2| \geqslant 3$, then $(2,0) - (2,1) - (2,x) - (2,0)$ for some $1 \neq x \in R_2^*$ is a cycle of length three in $\mathrm{AG}(R)$, a contradiction. Hence $|R_2| = 2$ and so $R_2 \cong \mathbb{Z}_2$ and so $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$.

Suppose R is reduced. Then R_1 and R_2 are fields and so $\mathrm{AG}(R) \cong K_{|R_1|-1, |R_2|-1}$. Since $\mathrm{gr}(\mathrm{AG}(R)) = 4$, $|R_1| \geqslant 3$ and $|R_2| \geqslant 3$.

Converse part of (ii) is trivial.

Part (iii) now follows directly from the above two cases. □

Corollary 1. *Let R be a finite commutative ring with identity that is not a field. Then $\mathrm{AG}(R)$ is a complete bipartite graph if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_9, \quad \frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2 \quad \text{or} \quad F_1 \times F_2,$$

where F_1 and F_2 are finite fields.

Proof. We only need to prove the necessary part. Suppose $\mathrm{AG}(R)$ is a complete bipartite graph. Then $\mathrm{AG}(R)$ does not contains a cycle of odd length and so girth of $\mathrm{AG}(R)$ cannot be odd. By Theorem 13, $\mathrm{gr}(\mathrm{AG}(R) = 4$ or $\mathrm{gr}(\mathrm{AG}(R)) = \infty$. By the assumption that $\mathrm{AG}(R)$ is complete bipartite, $\mathrm{AG}(R) \ncong K_1$. Hence R is isomorphic to one of the following rings: $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$, or $F_1 \times F_2$, where F_1 and F_2 are fields. □

3. Planar annihilator graphs

In this section, we characterize finite commutative rings R for which $\mathrm{AG}(R)$ is planar. The following are known regarding the genus.

Lemma 1 ([26]). *A connected graph G is planar if and only if G contains no subdivision of either K_5 or $K_{3,3}$.*

Lemma 2 ([26]). *Let n be a positive integer and for real number x, $\lceil x \rceil$ is the least integer that is the greater than or equal to x. Then $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ if $n \geqslant 3$. In particular, $g(K_n) = 1$ if $n = 5, 6, 7$.*

Lemma 3 ([26]). *Let m, n be positive integers and for real number x, $\lceil x \rceil$ is the least integer that is the greater than or equal to x. Then $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ if $m, n \geqslant 2$. In particular, $g(K_{4,4}) = g(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$ and $g(K_{5,4}) = g(K_{6,4}) = g(K_{m,4}) = 2$ if $m = 7, 8, 9, 10$.*

Lemma 4 ([26, Euler formula]). *If G is a finite connected graph with n vertices, m edges, and genus g, then $n - m + f = 2 - 2g$, where f is the number of faces created when G is minimally embedded on a surface of genus g.*

Theorem 14. *Let $(R, \mathfrak{m})$ be a finite commutative local ring with identity. Then $\mathrm{AG}(R)$ is planar if and only if R is isomorphic to one of the following 13 rings:*

$$\mathbb{Z}_4, \quad \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}, \quad \mathbb{Z}_9, \quad \frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}, \quad \mathbb{Z}_8, \quad \frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}, \quad \frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2 \rangle}, \quad \frac{\mathbb{F}_4[x]}{\langle x^2 \rangle},$$
$$\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}, \quad \frac{\mathbb{Z}_4[x]}{\langle x, 2 \rangle^2}, \quad \frac{\mathbb{Z}_2[x,y]}{\langle x, y \rangle^2}, \quad \mathbb{Z}_{25} \quad \textit{or} \quad \frac{\mathbb{Z}_5[x]}{\langle x^2 \rangle}.$$

Proof. By Lemma 1, $\mathrm{AG}(R)$ is planar if and only if $\mathrm{AG}(R)$ contains no subdivision of either K_5 or $K_{3,3}$. Hence $|Z(R)^*| \leqslant 4$. Now proof follows from Table 1. □

Theorem 15. *Let $R = R_1 \times \cdots \times R_n$ be a finite commutative nonlocal ring, where each R_i is a local ring and $n \geqslant 2$. Then $\mathrm{AG}(R)$ is planar if and only if R is isomorphic either of the following rings:*

$$\mathbb{Z}_2 \times F, \quad \mathbb{Z}_3 \times F, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2 \quad \textit{or} \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

where F is a finite field.

Proof. Note that $\mathrm{AG}(\mathbb{Z}_2 \times F) = K_{1,|F^*|}$ and $\mathrm{AG}(\mathbb{Z}_3 \times F) = K_{2,|F^*|}$ and so by Lemma 3, they are planar. Since $\mathrm{AG}(\mathbb{Z}_4 \times \mathbb{Z}_2) = \mathrm{AG}(\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2) \cong K_{2,3}$, they are also planar. As per the embedding given in Figure 2, $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is planar.

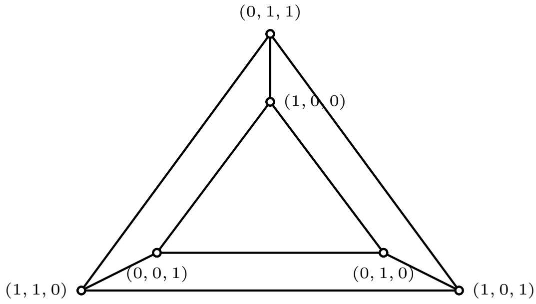

FIGURE 2. $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.

Conversely assume that $\mathrm{AG}(R)$ is planar. Suppose $n \geqslant 4$. Let

$$x_1 = (1,0,0,0,\ldots,0), \quad x_2 = (0,1,0,0,\ldots,0), \quad x_3 = (1,1,0,0,\ldots,0),$$
$$y_1 = (0,0,1,0,\ldots,0), \quad y_2 = (0,0,0,1,0,\ldots,0), \quad y_3 = (0,0,1,1,0,\ldots,0).$$

Then $\Omega = \{x_1, x_2, x_3, y_1, y_2, y_3\} \subseteq Z(R)^*$ and the subgraph of $\mathrm{AG}(R)$ induced by Ω contains $K_{3,3}$ as a subgraph, a contradiction. Hence $n \leqslant 3$.

Case 1. $n = 2$. Suppose $\mathfrak{m}_i \neq \{0\}$ for all $i = 1, 2$. Then $|R_i| \geqslant 4$ and $|R_i^\times| \geqslant 2$. Let a_i, b_i be two distinct elements in R_i^* other than identity for $i = 1, 2$. Let $d_1 = (1, 0)$, $d_2 = (a_1, 0)$, $d_3 = (b_1, 0)$, $g_1 = (0, 1)$, $g_2 = (0, a_2)$, $g_3 = (0, b_2) \in Z(R)^*$. Then $\Omega_1 = \{d_1, d_2, d_3, g_1, g_2, g_3\} \subseteq Z(R)^*$ and the subgraph of $\mathrm{AG}(R)$ induced by Ω_1 contains $K_{3,3}$ as a subgraph, a contradiction. Hence $\mathfrak{m}_i = \{0\}$ for some i.

Without loss of generality, we assume that $\mathfrak{m}_2 = \{0\}$. Then R_2 is a field.

Suppose $\mathfrak{m}_1 \neq \{0\}$. We claim that $|\mathfrak{m}_1^*| = 1$. If not, $|\mathfrak{m}_1^*| \geqslant 2$ and $|R_1^\times| \geqslant 3$. Note that $|R_2| \geqslant 2$. For $a, b \in \mathfrak{m}_1^*$ with $ab = 0$, and two distinct units $u_1, u_2 \in R_1^\times$, let $z_1 = (a, 1)$, $z_2 = (b, 1)$, $z_3 = (0, 1)$, $w_1 = (1, 0)$, $w_2 = (u_1, 0)$, $w_3 = (u_2, 0) \in Z(R)^*$. Then $\Omega_2 = \{z_1, z_2, z_3, w_1, w_2, w_3\} \subseteq Z(R)^*$ with $z_3 w_i = 0$ in R for $i = 1, 2, 3$. Clearly $z_2 \in \mathrm{ann}(z_1 w_1)$, $z_2 \notin \mathrm{ann}(z_1) \cup \mathrm{ann}(w_1)$, $z_2 \in \mathrm{ann}(z_1 w_2)$, $z_2 \notin \mathrm{ann}(z_1) \cup \mathrm{ann}(w_2)$ and so z_1 is adjacent to both w_1 and w_2 in $\mathrm{AG}(R)$. Further $z_1 \in \mathrm{ann}(z_2 w_1)$, $z_1 \notin \mathrm{ann}(z_2) \cup \mathrm{ann}(w_1)$, $z_1 \in \mathrm{ann}(z_2 w_2)$, $z_1 \notin \mathrm{ann}(z_2) \cup \mathrm{ann}(w_2)$ and so z_2 is adjacent to both w_1 and w_2 in $\mathrm{AG}(R)$. From this, we observe that $K_{3,3}$ is a subgraph of $\mathrm{AG}(R)$, a contradiction. Hence $|\mathfrak{m}_1^*| = 1$ and so $R_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.

Suppose $|R_2| \geqslant 3$. For $d \in \mathfrak{m}_1^*$ with $d^2 = 0$ and, $1 \neq v_1 \in R_1^\times$ and $1 \neq v_2 \in R_2^*$, let $s_1 = (d, 1), s_2 = (d, v_2), s_3 = (0, 1), p_1 = (d, 0), p_2 =$

$(v_1, 0)$, $p_3 = (1, 0)$. Then $\Omega_3 = \{s_1, s_2, s_3, p_1, p_2, p_3\} \subseteq Z(R)^*$ and, $p_1 s_1 = p_1 s_2 = 0$ in R and $s_3 p_i = 0$ in R for $i = 1, 2, 3$. Clearly $s_2 \in \text{ann}(s_1 p_2)$, $s_2 \notin \text{ann}(s_1) \cup \text{ann}(p_2)$, $s_2 \in \text{ann}(s_1 p_3)$, $s_2 \notin \text{ann}(s_1) \cup \text{ann}(p_3)$ and so s_1 is adjacent to both p_2 and p_3 in $\text{AG}(R)$. Also $s_1 \in \text{ann}(s_2 p_2)$, $s_1 \notin \text{ann}(s_2) \cup \text{ann}(p_2)$, $s_1 \in \text{ann}(s_2 p_3)$, $s_1 \notin \text{ann}(s_2) \cup \text{ann}(p_3)$ and so s_2 is adjacent to both p_2 and p_3 in $\text{AG}(R)$. From this, we get that $K_{3,3}$ is a subgraph of $\text{AG}(R)$, a contradiction. Hence $|R_2| = 2$ and so $R_2 \cong \mathbb{Z}_2$.

Suppose $\mathfrak{m}_1 = \{0\}$. Then R_1 is a field and by Remark 1, $\text{AG}(R) \cong K_{|R_1|-1,|R_2|-1}$. By Lemma 3, $R \cong \mathbb{Z}_2 \times F$ or $\mathbb{Z}_3 \times F$, where F is a finite field.

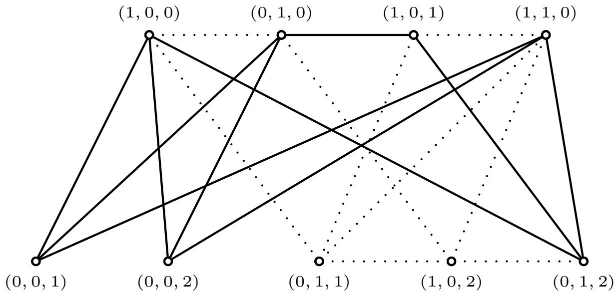

FIGURE 3. $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$.

Case 2. $n = 3$. Suppose $|R_i| \geqslant 4$ for some i. Without loss of generality, we assume that $|R_1| \geqslant 4$. Let u_1, u_2, u_3 be three distinct nonzero elements in R_1. Let $h_1 = (u_1, 0, 0)$, $h_2 = (u_2, 0, 0)$, $h_3 = (u_3, 0, 0)$, $t_1 = (0, 1, 0)$, $t_2 = (0, 0, 1)$, $t_3 = (0, 1, 1) \in Z(R)^*$. Then $h_i t_j = 0$ for all $i, j = 1, 2, 3$ and so $K_{3,3}$ is a subgraph of $\text{AG}(R)$, a contradiction. Hence $|R_i| \leqslant 3$ is field for $i = 1, 2, 3$ and so $R_i \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ for $i = 1, 2, 3$. Since $\Gamma(R)$ is a subgraph of $\text{AG}(R)$ and $\text{AG}(R)$ is planar, by Theorem 2, the possibilities for R are $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Note that the edges in dark lines of Figure 3 constitute a subdivision of $K_{3,3}$ and so by Lemma 1, $\text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ is not planar. Thus $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. □

4. Genus of AG(R)

In this section, we characterize isomorphism classes of finite commutative rings R whose $\text{AG}(R)$ has genus one. First, let us characterize finite commutative local rings R for which genus of $\text{AG}(R)$ is one.

Theorem 16. *Let $(R, \mathfrak{m})$ be a finite commutative local ring with identity that is not a field. Then $g(\text{AG}(R)) = 1$ if and only if R is isomorphic to* *one of the following 22 rings:*

$$\mathbb{Z}_{49},\quad \frac{\mathbb{Z}_7[x]}{\langle x^2\rangle},\quad \mathbb{Z}_{16},\quad \frac{\mathbb{Z}_2[x]}{\langle x^4\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^4, x^2-2\rangle},\quad \frac{\mathbb{Z}_2[x]}{\langle x^3-2, x^4\rangle},$$
$$\frac{\mathbb{Z}_4[x]}{\langle x^4, x^3+x^2-2\rangle},\quad \frac{\mathbb{Z}_2[x]}{\langle x^3, x^2-2x\rangle},\quad \frac{\mathbb{Z}_2[x,y]}{\langle x^3, xy, y^2-x^2\rangle},\quad \frac{\mathbb{Z}_8[x]}{\langle x^2-4, 2x\rangle},$$
$$\frac{\mathbb{Z}_4[x,y]}{\langle x^3, xy, x^2-2, y^2-2, y^3\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^2\rangle},\quad \frac{\mathbb{Z}_4[x,y]}{\langle x^2, y^2, xy-2\rangle},\quad \frac{\mathbb{Z}_2[x,y]}{\langle x^2, y^2\rangle},$$
$$\frac{\mathbb{Z}_2[x,y]}{\langle x^2, y^2, xy\rangle},\quad \frac{\mathbb{Z}_4[x]}{\langle x^3, 2x\rangle},\quad \frac{\mathbb{Z}_4[x,y]}{\langle x^3, x^2-2, xy, y^2\rangle},\quad \frac{\mathbb{Z}_8[x]}{\langle x^2, 2x\rangle},\quad \frac{F_8[x]}{\langle x^2\rangle},$$
$$\frac{\mathbb{Z}_4[x]}{\langle x^3+x+1\rangle},\quad \frac{\mathbb{Z}_4[x,y]}{\langle 2x, 2y, x^2, y^2, xy\rangle},\quad \frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z\rangle^2}.$$

Proof. By Theorem 9, $\mathrm{AG}(R)$ is complete. By Lemma 2, $5 \leqslant |Z(R)^*| \leqslant 7$. Note that there is no local ring R with $|Z(R)^*| = 5$. Now the proof follows from Table 2. □

Remark 4. Note that if $R \cong R_1 \times \cdots \times R_n$ is a commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and $n \geqslant 2$, then $K_{5,6}$ is a subgraph of $\mathrm{AG}(R)$ and hence $g(\mathrm{AG}(R)) \geqslant 2$. Thus if $g(\mathrm{AG}(R)) = 1$, then one of the components R_i must be a field.

Theorem 17. *Let $R = R_1 \times \cdots \times R_n$ be a finite commutative nonlocal ring, where each R_i is a local ring and $n \geqslant 2$. Then $g(\mathrm{AG}(R)) = 1$ if and only if R is isomorphic to one of the following 7 rings:*

$$\mathbb{F}_4\times\mathbb{F}_4,\quad \mathbb{F}_4\times\mathbb{Z}_5,\quad \mathbb{Z}_5\times\mathbb{Z}_5,\quad \mathbb{F}_4\times\mathbb{Z}_7,\quad \mathbb{Z}_4\times\mathbb{Z}_3,$$
$$\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}\times\mathbb{Z}_3 \quad \text{or} \quad \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3.$$

Proof. Assume that $g(\mathrm{AG}(R)) = 1$.

Suppose $n \geqslant 4$. Let

$$x_1 = (1,1,0,0,\ldots,0),\quad x_2 = (0,1,0,1,\ldots,1),\quad x_3 = (0,0,1,0,\ldots,0),$$
$$x_4 = (0,1,1,0,\ldots,0),\quad x_5 = (1,0,1,\ldots,1),\quad x_6 = (1,1,0,1,\ldots,1),$$
$$x_7 = (1,1,1,0,\ldots,0),\quad x_8 = (1,0,1,0,\ldots,0),\quad x_9 = (0,0,0,1,\ldots,1),$$
$$x_{10} = (1,0,0,1,\ldots,1),\quad x_{11} = (0,1,\ldots,1).$$

Then $\Omega = \{x_1,\ldots,x_{11}\} \subseteq Z(R)^*$ with $x_1x_3 = x_2x_3 = x_3x_6 = x_1x_9 = x_7x_9 = x_8x_9 = 0$. Clearly $x_5 \in \mathrm{ann}(x_1x_4)$, $x_5 \notin \mathrm{ann}(x_1)\cup\mathrm{ann}(x_4)$, $x_{11} \in \mathrm{ann}(x_1x_5)$, $x_{11} \notin \mathrm{ann}(x_1)\cup\mathrm{ann}(x_5)$, $x_5 \in \mathrm{ann}(x_2x_4)$, $x_5 \notin \mathrm{ann}(x_2)\cup$

$\operatorname{ann}(x_4)$, $x_7 \in \operatorname{ann}(x_2x_5)$, $x_7 \notin \operatorname{ann}(x_2) \cup \operatorname{ann}(x_5)$, $x_4 \in \operatorname{ann}(x_5x_6)$, $x_4 \notin \operatorname{ann}(x_5) \cup \operatorname{ann}(x_6)$, $x_5 \in \operatorname{ann}(x_6x_4)$, $x_5 \notin \operatorname{ann}(x_6) \cup \operatorname{ann}(x_4)$, $x_{11} \in \operatorname{ann}(x_1x_{10})$, $x_{11} \notin \operatorname{ann}(x_1) \cup \operatorname{ann}(x_{10})$, $x_5 \in \operatorname{ann}(x_1x_{11})$, $x_5 \notin \operatorname{ann}(x_1) \cup \operatorname{ann}(x_{11})$, $x_{11} \in \operatorname{ann}(x_8x_{10})$, $x_{11} \notin \operatorname{ann}(x_8) \cup \operatorname{ann}(x_{10})$, $x_6 \in \operatorname{ann}(x_{11}x_8)$, $x_6 \notin \operatorname{ann}(x_{11}) \cup \operatorname{ann}(x_8)$, $x_{10} \in \operatorname{ann}(x_{11}x_7)$, $x_{10} \notin \operatorname{ann}(x_{11}) \cup \operatorname{ann}(x_7)$, $x_{11} \in \operatorname{ann}(x_{10}x_7)$ and $x_{11} \notin \operatorname{ann}(x_{10}) \cup \operatorname{ann}(x_7)$. From all the above observations, Note that the subgraph induced by Ω in $\mathrm{AG}(R)$ contains G given in Figure 1 as a subgraph and so $g(\mathrm{AG}(R)) \geqslant 2$. Hence $n \leqslant 3$.

Case 1. $n = 3$. Suppose R_1 and R_2 are not fields. Then as mentioned in Remark 4, $K_{5,6}$ is a subgraph of $\mathrm{AG}(R)$ and so $g(\mathrm{AG}(R)) \geqslant 2$, a contradiction. Hence at least two of the components $R_i, (1 \leqslant i \leqslant 3)$ must be fields. Without loss of generality, let us assume that R_2 and R_3 are fields. Then $|R_2| \geqslant 2$ and $|R_3| \geqslant 2$. Suppose R_1 is not a field. Then $|\mathfrak{m}_1| \geqslant 2$ and $|R_1^\times| \geqslant 2$. For $z \in \mathfrak{m}_1^*$ with $\operatorname{ann}(z) = \mathfrak{m}_1$ and $u \in R_1^\times$ and $1 \neq u$, let $x_1 = (z,1,0), x_2 = (0,1,0), x_3 = (z,0,0), x_4 = (1,0,0)$, $x_5 = (u,0,0)$, $x_6 = (0,0,1,)$, $x_7 = (0,1,1), x_8 = (z,1,1), x_9 = (1,0,1)$, $x_{10} = (z,0,1)$, $x_{11} = (u,0,1)$ and $\Omega_1 = \{x_1,\ldots,x_{11}\}$. Without much difficulty, one can check that the subgraph induced by Ω_1 in $\mathrm{AG}(R)$ contains G given in Figure 1 and so $g(\mathrm{AG}(R)) \geqslant 2$, a contradiction. Hence R_1 must be a field. Since $\mathrm{AG}(R)$ is non-planar, by Theorem 15, $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose at least two of the components $R_i(i \leqslant i \leqslant 3)$ contain at least three elements. Without loss of generality, assume that $|R_2| \geqslant 3$ and $|R_3| \geqslant 3$. For $1 \neq a \in R_3^*$ and $1 \neq b \in R_2^*$, let $x_1 = (1,1,0), x_2 = (0,1,0), x_3 = (0,0,1)$, $x_4 = (0,0,a), x_5 = (1,0,1)$, $x_6 = (0,1,0)$, $x_7 = (1,b,0), x_8 = (1,0,0), x_9 = (0,1,a)$, $x_{10} = (0,1,1)$, $x_{11} = (0,b,1)$ and $\Omega_2 = \{x_1,\ldots,x_{11}\}$. Then the subgraph of $\mathrm{AG}(R)$ induced by Ω_2 contains G in Figure 1 and so $g(\mathrm{AG}(R)) \geqslant 2$, a contradiction. Hence two of the components R_i must be $\mathbb{Z}_2$. Without loss of generality, we assume that $R_1 = R_2 = \mathbb{Z}_2$ and $|R_3| \geqslant 3$.

Suppose $|R_3| \geqslant 4$. Let u and v be two distinct non-zero elements in R_3^* other than identity. Let $x_1 = (1,0,0)$, $x_2 = (1,0,u)$, $x_3 = (0,1,1)$, $x_4 = (0,1,v)$, $x_5 = (0,1,u)$, $x_6 = (1,0,v)$, $x_7 = (1,1,0)$, $x_8 = (0,1,0)$, $x_9 = (0,0,1)$, $x_{10} = (0,0,u)$, $x_{11} = (1,0,v)$ and $\Omega_3 = \{x_1,\ldots,x_{11}\}$. Then the subgraph of $\mathrm{AG}(R)$ induced by Ω_3 contains G given in Figure 1 and so $g(\mathrm{AG}(R)) \geqslant 2$, a contradiction. Hence $R_3 = \mathbb{Z}_3$ and so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

Case 2. $n = 2$. If R_1 and R_2 are not fields, then as mentioned in Remark 4, $K_{5,6}$ is a subgraph of $\mathrm{AG}(R)$ and so $g(\mathrm{AG}(R)) \geqslant 2$, a contradiction. Hence one of the components R_i must be a field. Without loss generality, we assume that R_2 is a field and so $|R_2| \geqslant 2$. By Theorem 15, $R \not\cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$.

We claim that, if $|\mathfrak{m}_1^*| \geqslant 2$, then $g(\mathrm{AG}(R)) \geqslant 2$.

Suppose $|\mathfrak{m}_1^*| = 2$. By the facts given in Table 1, $R_1 \cong \mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{\langle x^3\rangle}$ and hence $|R_1^\times| = 6$. Let $\mathfrak{m}_1^* = \{a, b\}$, $R_1^\times = \{u_1, \ldots, u_6\}$ and $y_1 = (a, 1)$, $y_2 = (b, 1)$, $y_3 = (0, 1)$, $x_1 = (a, 0)$, $x_2 = (b, 0)$, $x_3 = (u_1, 0)$, $x_4 = (u_2, 0)$, $x_5 = (u_3, 0)$, $x_6 = (u_4, 0)$, $x_7 = (u_5, 0)$, $x_8 = (u_6, 0)$. Then $\Omega_4 = \{y_1, y_2, y_3, x_1, \ldots, x_8\} \subseteq Z(R)^*$ and the subgraph of $\mathrm{AG}(R)$ induced by Ω_4 contains $K_{3,8}$. By Lemma 3, $g(\mathrm{AG}(R)) \geqslant 2$, a contradiction.

Suppose $|\mathfrak{m}_1^*| \geqslant 3$. Then $|R_1^\times| \geqslant 4$. Let $d, e, f \in \mathfrak{m}_1^*$ such that $de = df = 0$ and $\{v_1, \ldots, v_4\} \subseteq R_1^\times$. Consider $\Omega_5 = \{z_1, \ldots, z_7, w_1, \ldots, w_4\} \subseteq Z(R)^*$, where $z_1 = (d, 0)$, $z_2 = (e, 0)$, $z_3 = (f, 0)$, $z_4 = (v_1, 0)$, $z_5 = (v_2, 0)$, $z_6 = (v_3, 0)$, $z_7 = (v_4, 0)$, $w_1 = (d, 1)$, $w_2 = (e, 1)$, $w_3 = (f, 1)$, $w_4 = (0, 1)$. Then the subgraph induced by Ω_5 of $\mathrm{AG}(R)$ contains $K_{4,7}$ and by Lemma 3, $g(\mathrm{AG}(R)) \geqslant 2$, a contradiction.

Hence we conclude that $|\mathfrak{m}_1| \leqslant 2$. From this $R_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$ when $|\mathfrak{m}_1| = 2$ and R_1 must be a field when $|\mathfrak{m}_1| = 1$. By Theorem 15, $R_2 \not\cong \mathbb{Z}_2$ and so $|R_2| \geqslant 3$.

Suppose $|\mathfrak{m}_1| = 2$ and $|R_2| \geqslant 5$. For $a \in \mathfrak{m}_1^*$ with $a^2 = 0$ and $u_1, u_2 \in R_1^\times$, $e_j \in R_2^*$, let $x_1 = (a, 0)$, $x_2 = (u_1, 0)$, $x_3 = (u_2, 0)$, $x_4 = (a, e_1), x_5 = (a, e_2)$, $x_6 = (a, e_3)$, $x_7 = (a, e_4)$, $x_8 = (0, e_1)$, $x_9 = (0, e_2)$, $x_{10} = (0, e_3)$, $x_{11} = (0, e_4)$ and $\Omega_6 = \{x_1, \ldots, x_{11}\} \subset Z(R)^*$. Then the subgraph induced by Ω_6 of $\mathrm{AG}(R)$ contains $K_{3,8}$ and by Lemma 3, $g(\mathrm{AG}(R)) \geqslant 2$, a contradiction. Hence R_2 is isomorphic to either $\mathbb{Z}_3$ or $\mathbb{F}_4$.

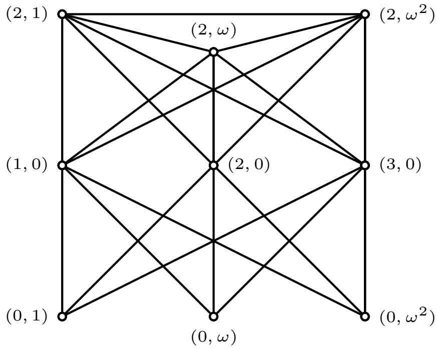

Figure 4. $\mathrm{AG}(\mathbb{Z}_4 \times \mathbb{F}_4) \cong \mathrm{AG}\left(\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{F}_4\right)$.

Consider the case that $R_2 \cong \mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$. One can observe from Figure 4 that $K_{3,6}$ is a subgraph of $\mathrm{AG}(R)$. By Lemma 3, $g(K_{3,6}) = 1$ and hence one can fix an embedding of $K_{3,6}$ on the surface of torus. By Euler's formula, there are 9 faces in the embedding of $K_{3,6}$, say $\{S_1, \ldots, S_9\}$. Let n_i be the length of the face S_i. Note that $\sum\limits_{i=1}^{9} n_i = 36$ and $n_i \geqslant 4$ for every i. Thus $n_i = 4$ for every i. Let $U = \{(2,1),(2,\omega),(2,\omega^2)\} \subset V(K_{3,6})$. Further, the subgraph G of $\mathrm{AG}(R)$ induced by the vertices in U is K_3, $E(G) \cap E(K_{3,6}) = \varnothing$. Since K_3 cannot be embedded in the torus along with an embedding with only rectangles as faces, one cannot have an embedding of G and $K_{3,6}$ together in the torus. This implies that $g(\mathrm{AG}(R)) \geqslant 2$. Hence $R_2 \not\cong \mathbb{F}_4$ and so R is isomorphic to either $\mathbb{Z}_4 \times \mathbb{Z}_3$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_3$.

Suppose $|\mathfrak{m}_1| = 1$ and in this case both R_1 and R_2 are fields. Note that $\mathrm{AG}(R) \cong K_{|R_1|-1,|R_2|-1}$. Since $g(\mathrm{AG}(R)) = 1$ and by Lemma 3, R is isomorphic to one of the following rings $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$ or $\mathbb{F}_4 \times \mathbb{Z}_7$.

Converse follows from Remark 1, Lemma 3, Figure 5 and Figure 6. □

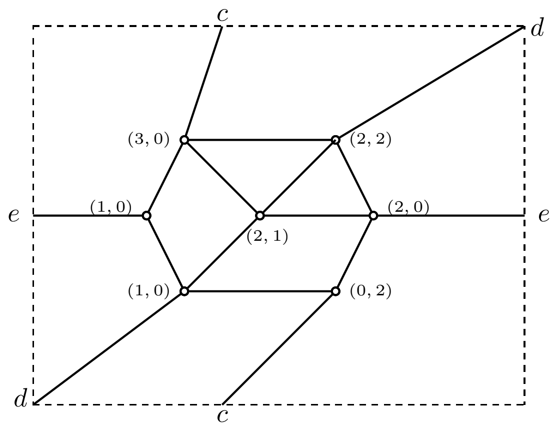

FIGURE 5. Embedding of $\mathrm{AG}(\mathbb{Z}_4 \times \mathbb{Z}_3) \cong \mathrm{AG}\left(\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_3\right)$ in S_1.

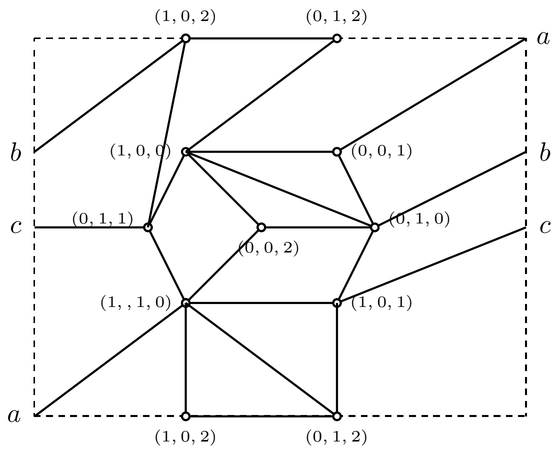

FIGURE 6. Embedding of $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ in S_1.

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