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The central polynomials for the finite dimensional Grassmann algebras

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Abstract. In this note we describe the central polynomials for the finite dimensional unitary Grassmann algebras G_k over an infinite field F of characteristic $\neq 2$. We exhibit a set of generators of $C(G_k)$, the T-space of the central polynomials of G_k in a free associative F-algebra.

Dedicated to Professor Miguel Ferrero on occasion of his 70-th anniversary

Introduction

Central polynomials of algebras with polynomial identities are of fundamental importance in PI-theory. The existence of proper central polynomials for the matrix algebras $M_n(F)$ over a field F was conjectured by Kaplansky, and confirmed by means of direct constructions by Formanek [5] and by Razmyslov [14]. One can find further references about central polynomials of PI algebras in [1], [4] and [8].

However, an explicit description of the vector space of *all* central polynomials was obtained for very few algebras so far (in the results mentioned above *some* central polynomials for the corresponding algebras

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were constructed). The module structure of the centre of the generic matrix algebra of order 2 was given by Formanek [6], and generators for the central polynomials for $M_2(F)$ were exhibited by Okhitin in [13]; both results were obtained assuming the base field F of characteristic 0. For an infinite field F, $\operatorname{char} F = p \neq 2$, generating sets for the central polynomials for $M_2(F)$ were described in [2]. Very recently in [1] the central polynomials of the infinite dimensional Grassmann algebra G over an infinite field F of characteristic $\neq 2$ were described. In fact, this is an almost complete list of known results concerning an explicit description of the central polynomials in a given algebra.

In this note we describe the central polynomials of the finite dimensional Grassmann algebras G_k over an infinite field F, $\operatorname{char} F \neq 2$. We exhibit a set of generators of the T-space $C(G_k)$ of the central polynomials of G_k.

Let us give the precise definitions. Let F be a field and let $F_1\langle X\rangle$ be the free unitary associative algebra over F on the free generating set $X = \{x_0, x_1, x_2, \ldots\}$. A polynomial $f(x_1, \ldots, x_n) \in F_1\langle X\rangle$ is a *polynomial identity* in an F-algebra A if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. An ideal I of $F_1\langle X\rangle$ is called a *T-ideal* if I is closed under all endomorphisms of $F_1\langle X\rangle$. If A is an algebra then its polynomial identities form a T-ideal $T(A)$ in $F_1\langle X\rangle$; conversely, for every T-ideal I in $F_1\langle X\rangle$ there is an algebra A such that $I = T(A)$, that is, I is the ideal of all polynomial identities satisfied in A. We refer to [3], [4], [10] and [15] for the terminology and basic results concerning PI algebras.

A vector subspace V of $F_1\langle X\rangle$ is called a *T-space* if V is closed under all (algebra) endomorphisms of $F_1\langle X\rangle$. A set $S \subset V$ *generates* V *as a T-space* if V is the minimal T-space in $F_1\langle X\rangle$ containing S. Therefore V is the span of all polynomials $f(g_1, \ldots, g_n)$ where $f \in S$ and $g_i \in F_1\langle X\rangle$. Note that if I is a T-ideal in $F_1\langle X\rangle$ then T-spaces and T-ideals can be defined in the quotient algebra $F_1\langle X\rangle/I$ in a natural way. In recent years T-spaces turned out to be objects of intensive study, see [9] for an account.

The polynomial $f(x_1, \ldots, x_n)$ is called *a central polynomial* for A if $f(a_1, \ldots, a_n) \in Z(A)$, the centre of A, for every $a_i \in A$. The central polynomials for a given algebra A form a T-space $C(A)$ in $F_1\langle X\rangle$. However, not every T-space can be obtained as the T-space of the central polynomials for some algebra. In fact the central polynomials for a given algebra A are closed under multiplication, and so they form a T-subalgebra in $F_1\langle X\rangle$.

Let V be the vector space over a field F of characteristic $\neq 2$, with a countable infinite basis e_1, e_2, … and let V_k denote the subspace of V generated by $e_1, \ldots, e_k$ $(k = 2, 3, \ldots)$. Let G and G_k denote the unitary Grassmann algebras of V and of V_k respectively. Then as a vector space G has a basis that consists of 1 and of all monomials $e_{i_1}e_{i_2}\dots e_{i_k}$, $i_1 < i_2 < \cdots < i_k$, $k \geq 1$. The multiplication in G is induced by $e_ie_j = -e_je_i$ for all i and j. The algebra G_k is the subalgebra of G generated by $e_1, \dots, e_k$, and $\dim G_k = 2^k$.

Let $a, b, c \in A$, we denote by $[a,b] = ab - ba$ the commutator of a and b, and we set $[a,b,c] = [[a,b],c]$.

Krakowski and Regev [11] described the polynomial identities of G when $\operatorname{char} F = 0$, and several authors described the generators of $T(G)$ in the general case. Let T be the T-ideal in $F_1\langle X\rangle$ generated by the triple commutator $[x_1, x_2, x_3]$.

Proposition 1 ([7, 11, 12], see also [3, 4, 8, 10]). *Let F be an infinite field of characteristic $\neq 2$. Then $T(G) = T$.*

The description of the polynomial identities of G_k can be obtained easily from the proof of Proposition 1, see for instance [3, 4] if $\operatorname{char} F = 0$, and [7] if $\operatorname{char} F \neq 2$. Let $T(G_k)$ be the T-ideal of the polynomial identities of G_k and let T_n be the T-ideal generated by the polynomials $[x_1, x_2]\dots[x_{2n-1}, x_{2n}]$ and $[x_1, x_2, x_3]$.

Proposition 2 ([7]). *Let F be an infinite field of characteristic $\neq 2$. Then $T(G_k) = T_n$ where $n = [k/2] + 1$, $[a]$ being the integer part of the rational number a.*

Very recently the central polynomials for the infinite dimensional Grassmann algebra G were described in [1]. Let

$$q(x_1, x_2) = x_1^{p-1}[x_1, x_2]x_2^{p-1}$$

and let, for each $s \geq 1$,

$$q_s = q_s(x_1, \dots, x_{2s}) = q(x_1, x_2)q(x_3, x_4)\dots q(x_{2s-1}, x_{2s}).$$

Theorem 3 ([1]). *Over an infinite field F of characteristic $p > 2$, the vector space $C(G)$ of the central polynomials of G is generated (as a T-space in $F_1\langle X\rangle$) by the polynomial $x_0[x_1, x_2, x_3]$ and by the polynomials*

$$x_0^p\ ,\ x_0^p q_1\ ,\ x_0^p q_2\ , \dots ,\ x_0^p q_n\ , \dots .$$

Proposition 4 ([1]). *If $\operatorname{char} F = 0$ then the T-space $C(G)$ is generated by 1, $x_0[x_1, x_2, x_3]$ and $[x_1, x_2]$.*

In this note we deal with the central polynomials for the finite dimensional Grassmann algebras G_k. Our main results are as follows.

Theorem 5. *Over an infinite field F of a characteristic $p > 2$ the vector space $C(G_k)$ of the central polynomials of G_k is generated (as a T-space in $F_1\langle X\rangle$) by the polynomials*

$$x_0[x_1, x_2, x_3], \qquad x_0[x_1, x_2] \ldots [x_{2n-3}, x_{2n-2}]$$

and by the polynomials

$$x_0^p\ ,\ x_0^p q_1\ ,\ x_0^p q_2\ , \ldots,\ x_0^p q_{n-2}, \qquad n = [k/2] + 1.$$

Proposition 6. *If $\mathrm{char}\, F = 0$ then the T-space $C(G_k)$ is generated by 1, $x_0[x_1, x_2, x_3]$, $[x_1, x_2]$ and $x_0[x_1, x_2] \ldots [x_{2n-3}, x_{2n-2}]$ where $n = [\frac{k}{2}] + 1$.*

We deduce Theorem 5 and Proposition 6 from the following proposition of independent interest.

Proposition 7. *Let F be an infinite field of characteristic $\neq 2$. Then, for each $k \geq 2$, $C(G_k) = C(G) + T_{n-1}$, where $n = [\frac{k}{2}] + 1$.*

1. Proof of the main results

To prove our results we need the following well-known properties of the T-ideal T (see, for instance, [3, 10, 7]).

Lemma 8. *Let F be a field. For all g, g_1, g_2, g_3, $g_4 \in F_1\langle X\rangle$ we have the following:*

(i) $[g_1, g_2] + T$ *is central in* $F_1\langle X\rangle/T$;

(ii) $[g_1, g_2][g_3, g_4] + T = -[g_1, g_3][g_2, g_4] + T$;

(iii) $[g_1, g_2][g_3, g_4] + T = T$ *if* $g_i = g_j$ *for some* i *and* j, $i \neq j$.

Let B be the set of all polynomials in $F_1\langle X\rangle$ of the form

$$x_{i_1}^{n_1} x_{i_2}^{n_2} \ldots x_{i_s}^{n_s} [x_{j_1}, x_{j_2}] \ldots [x_{j_{2r-1}}, x_{j_{2r}}]$$

where $s, r \geq 0$, $i_1 < i_2 < \ldots < i_s$, $j_1 < j_2 < \ldots < j_{2r}$, $n_k > 0$ for all k. Note that $1 \in B$ because 1 is of the form above for $s = r = 0$. Let, for each $n \geq 1$, B_n be the subset of B consisting of all elements with $0 \leq r < n$, that is, of elements of B whose "commutator part" $[x_{j_1}, x_{j_2}] \ldots [x_{j_{2r-1}}, x_{j_{2r}}]$ contains less than n commutators. The next proposition is well-known. It follows immediately, for instance, from [3, Theorem 4.3.11 (i) and the proof of Theorem 5.1.2 (i)].

Proposition 9. *Let F be an infinite field of characteristic $\neq 2$. Then the F-vector space $F_1\langle X\rangle/T$ has a basis $\{b + T \mid b \in B\}$ and the vector space $F_1\langle X\rangle/T_n$ has a basis $\{b + T_n \mid b \in B_n\}$.*

First we prove Proposition 7. Note that $C(G) + T_{n-1} \subseteq C(G_k)$. Indeed, $C(G) \subset C(G_k)$ because $T \subset T_n$ and $C(G)/T_n$ and $C(G_k)/T$ are the centres of $F_1\langle X\rangle/T_n$ and of $F_1\langle X\rangle/T$, respectively. On the other hand, $T_{n-1} \subset C(G_k)$ because the elements of T_{n-1}/T_n are central in $F_1\langle X\rangle/T_n$. Indeed, T_{n-1}/T_n is spanned by elements of the form $h + T_n$, where $h = g_0[g_1, g_2] \dots [g_{2n-3}, g_{2n-2}]$ ($g_i \in F_1\langle X\rangle$). Since $[g, g'] + T$ is central in $F_1\langle X\rangle/T$ for all g, g', for each t we have

$$[h, x_t] + T = [g_0, x_t][g_1, g_2] \dots [g_{2n-3}, g_{2n-2}] + T \in T_n/T,$$

that is, $[h, x_t] \in T_n$. Hence, $h + T_n$ is central in $F_1\langle X\rangle/T_n$ and so is each element of T_{n-1}/T_n.

Thus, to prove Proposition 7 it suffices to check that

$$C(G_k) \subseteq C(G) + T_{n-1}.$$

Let f be an arbitrary element of $C(G_k)$. By Proposition 9, the set $\{b+T \mid b \in B\}$ is an F-basis of the algebra $F_1\langle X\rangle/T$ so

$$f + T = \sum \alpha_i b_i^{(1)} + \sum \beta_i b_i^{(2)} + T$$

where, for all i, α_i, $\beta_i \in F$, $b_i^{(1)} \in B_{n-1}$ and $b_i^{(2)} \in B \setminus B_{n-1}$. Equivalently,

$$f = \sum \alpha_i b_i^{(1)} + \sum \beta_i b_i^{(2)} + f_1$$

where α_i, β_i, $b_i^{(1)}$ and $b_i^{(2)}$ are as above and $f_1 \in T$. Note that $\sum \beta_i b_i^{(2)} \in T_{n-1}$ and $f_1 \in T \subset T_{n-1}$ so $(\sum \beta_i b_i^{(2)} + f_1) \in T_{n-1}$. Hence, to prove that $f \in C(G) + T_{n-1}$ it suffices to check that $g = \sum \alpha_i b_i^{(1)} \in C(G)$ or, equivalently, that $[g, x_t] \in T$ for all t.

Let

$$b_i^{(1)} = x_{i_1}^{m_1} \dots x_{i_s}^{m_s} [x_{j_1}, x_{j_2}] \dots [x_{j_{2r-1}}, x_{j_{2r}}].$$

Then

$$[b_i^{(1)}, x_t] + T = [x_{i_1}^{m_1} \dots x_{i_s}^{m_s}, x_t][x_{j_1}, x_{j_2}] \dots [x_{j_{2r-1}}, x_{j_{2r}}] + T.$$

Note that if A is an associative ring then

$$[v_1 v_2 \dots v_l, u] = \sum_{i=1}^{l} v_1 \dots v_{i-1} [v_i, u] v_{i+1} \dots v_l.$$

Also recall that $[g, g'] + T$ is central in $F_1\langle X\rangle/T$ for all g, g'. Hence we obtain that $[b_i^{(1)}, x_t] + T$ equals

$$\sum_{l=1}^{s} m_l \; x_{i_1}^{m_1} \dots x_{i_l}^{m_l - 1} \dots x_{i_s}^{m_s} [x_{i_l}, x_t][x_{j_1}, x_{j_2}] \dots [x_{j_{2r-1}}, x_{j_{2r}}] + T.$$

Further, it follows from the items ii) and iii) of Lemma 8 that, for all $g_i \in F_1\langle X\rangle$ and for each permutation σ on the set $\{1, 2, \dots, 2u\}$,

$$[g_1, g_2] \dots [g_{2u-1}, g_{2u}] + T = \pm [g_{\sigma(1)}, g_{\sigma(2)}] \dots [g_{\sigma(2u-1)}, g_{\sigma(2u)}] + T$$

and

$$[g_1, g_2] \dots [g_{2u-1}, g_{2u}] + T = T$$

if $g_i = g_j$ for some i and j, $i \neq j$. Therefore we can rewrite $[b_i^{(1)}, x_t] + T$ as a linear combination of elements of the form

$$x_{i_1}^{m'_1} \dots x_{i_s}^{m'_s} [x_{j'_1}, x_{j'_2}] \dots [x_{j'_{2r+1}}, x_{j'_{2r+2}}] + T,$$

where $j'_1 < j'_2 < \dots < j'_{2r+2}$. Since $b_i^{(1)} \in B_{n-1}$, we have $r < n-1$ so each element above belongs to B_n.

Thus, for each i,

$$[b_i^{(1)}, x_t] + T = \sum \gamma_{ij} b_{ij}^{(3)} + T,$$

where $\gamma_{ij} \in F$, $b_{ij}^{(3)} \in B_n$. It follows that

$$[g, x_t] + T = \sum \mu_{i'} b_{i'} + T \tag{1}$$

where $\mu_{i'} \in F$, $b_{i'} \in B_n$ for all i'.

Note that $g \in C(G_k)$. Indeed, as we observed above, $T_{n-1} \subset C(G_k)$ so $(\sum \beta_i b_i^{(2)} + f_1) \in C(G_k)$. Also $f \in C(G_k)$ so $g = f - (\sum \beta_i b_i^{(2)} + f_1) \in C(G_k)$.

Since $g \in C(G_k)$, we have $[g, x_t] + T_n = T_n$. On the other hand, (1) implies $[g, x_t] + T_n = \sum \mu_{i'} b_{i'} + T_n$ because $T \subset T_n$. It follows that $\sum \mu_{i'} b_{i'} + T_n = T_n$. Since $\{b + T_n \mid b \in B_n\}$ is a basis of $F_1\langle X\rangle/T_n$ over F, we have $\mu_{i'} = 0$ for all i'. Then, by (1), $[g, x_t] + T = T$ for all t, that is, $g \in C(G)$.

Thus,

$$f = g + (\sum \beta_i b_i^{(2)} + f_1) \in C(G) + T_{n-1},$$

as required. This completes the proof of Proposition 7.

Now we prove Theorem 5. Recall that $\operatorname{char} F = p > 2$. By Proposition 7, $C(G_k) = C(G) + T_{n-1}$, where $n = [\frac{k}{2}] + 1$. It can be easily seen that as a T-space T_{n-1} is generated by

$$x_0[x_1, x_2, x_3] \tag{2}$$

and

$$x_0[x_1, x_2][x_3, x_4] \ldots [x_{2n-3}, x_{2n-2}]. \tag{3}$$

Since, by Theorem 3, the T-space $C(G)$ is generated by (2) and by the set

$$x_0^p \ , \ x_0^p \, q_1 \ , \ \ldots \ , \ x_0^p \, q_s \ , \ \ldots , \tag{4}$$

the T-space $C(G_k) = C(G) + T_{n-1}$ is generated by (2), (3) and the set (4). Notice that $x_0^p \, q_s \in T_{n-1}$ for all $s \geq n-1$ because, by Lemma 8,

$$x_0^p \, q_s + T = x_0^p \ \ x_1^{p-1}[x_1, x_2]x_2^{p-1} \ldots x_{2s-1}^{p-1}[x_{2s-1}, x_{2s}]x_{2s}^{p-1} + T$$

$$= x_0^p x_1^{p-1} x_2^{p-1} \ldots x_{2s}^{p-1}[x_1, x_2] \ldots [x_{2s-1}, x_{2s}] + T.$$

It follows that $C(G_k)$ is generated as a T-space by the polynomials (2), (3) and $x_0^p \ , \ x_0^p \, q_1 \ , \ \ldots \ , \ x_0^p \, q_{n-2}$. The proof of Theorem 5 is completed.

Finally, we prove Proposition 6. Here we assume $\operatorname{char} F = 0$. By Proposition 7, $C(G_k) = C(G) + T_{n-1}$ where $n = [\frac{k}{2}] + 1$. By Proposition 4, the T-space $C(G)$ is generated by 1 and by the polynomials (2) and $[x_1, x_2]$. Since the T-space T_{n-1} is generated by the polynomials (2) and (3), the T-space $C(G_k)$ is generated by 1 and by the polynomials (2), (3) and $[x_1, x_2]$, as required. Proposition 6 is proved.

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